

Well Posed Cauchy Problems with Delays

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Abstract: In this paper we consider parabolic differential equation with delay:

$$p(x) \frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) + au(x, t) + bu(x, (t - \tau)), t > 0, x \in (0, 1),$$

$$u(x, t) = \phi(x, t), \quad t \in [-\tau, 0], \quad x \in (0, 1),$$

$$u(0, t) = u(1, t) = 0, \quad t \geq 0,$$

where the function $p : (0, 1) \rightarrow \mathbb{R}$ is positive and a, b and τ are positive constants. The above equation appears in the modeling of internal migration of a species, the constants a and b are related to the death and birth rates and the number τ is related to the pregnancy period.

We reformulated the above problem as an evolution equation in a Hilbert Space H :

$$B \frac{du}{dt}(t) = Au(t) + u(t - \tau), \quad t > 0,$$

$$u(t) = \phi(t), \quad t \in [-\tau, 0],$$

where $B : D(B) \subset H \rightarrow H$ is one-one with $R(A) \subset R(B)$, the operator $AD(A) \subset H \rightarrow H$ is such that $B^{-1}A$ generates a semigroup $S(t)$ of bounded linear operators in H . We first establish that the above evolution equation is well-posed on $[-\tau, \infty]$. It is well-posed on a finite length interval $[-\tau, T]$, $0 < T < \infty$. Then, under certain assumptions we prove that the evolution equation is well-posed on $[-\tau, T]$, for some

positive T which ensures that it is well-posed on the non-negative part of the real line $[0, \infty]$.

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1. Introduction

Consider the following type of a partial differential equation

$$(1) \quad p(x) \frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) + au(x, t) + bu(x, (t - \tau)), t > 0, x \in (0, 1),$$

$$(2) \quad u(x, t) = \phi(x, t), \quad t \in [-\tau, 0], x \in (0, 1),$$

$$(3) \quad u(0, t) = u(1, t) = 0, \quad t \geq 0,$$

where $p(x) > 0$ a.e. on $(0, 1)$, $\tau > 0$, $a > 0$, $b > 0$ and $b > 0$ are constants. We shall rewrite (1) as the following evolution equation in the Banach space $X = L^p(0, 1)$,

$$(4) \quad \begin{aligned} B \frac{du}{dt}(t) &= Au(t) + u(t - \tau), \quad t > 0, \\ u(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned}$$

where $\phi \in C([-\tau, 0]; X)$. We note that $A = \Delta + al$ with $D(A) = H^2(0, 1) \cap H_0^1(0, 1)$, generates a semigroup $S(t)$ of operators on X (cf. Pazy¹ for the definition and related results on group of operators in a Banach space). Also, $D(B) = \{\psi \in L^2(0, 1) : p.\psi \in L^2(0, 1)\}$ with $B\psi = p.\psi$ for $\psi \in D(B)$.

The problem (1) appears in the study of the population dynamics of a species involving internal migration. The delay is due to pregnancy period represented by the number τ (cf. Engel and Nagel²).

Our aim is to establish well-posedness of (4). In this paper, we extend various results established by Zaidman³⁻⁴ for the equation

$$(5) \quad \frac{du}{dt}(t) = Au(t), \quad t > 0, \quad u(0) = u_0.$$

2. Main Results

We use the following definition of well-posedness of (4) similar to the one considered in the papers³⁻⁴.

Definition 1. By a solution u of (4) on $[-\tau, T]$, $0 < T < \infty$, we mean a function $u: [-\tau, T] \rightarrow X$, such that $u(t) = \phi(t)$ on $[-\tau, 0]$, the strong derivative $u'(t)$ exists and $u'(t) \in D(B)$ on $[0, T]$, $u(t) \in D(A)$ for $0 \leq t \leq T$ and $Bu'(t) = Au(t) + u(t - \tau)$, for $0 \leq t \leq T$.

Definition 2. The problem (4) is said to be well-posed on $[0, T]$, if for any $\phi \in C([-\tau, 0]; X)$ with $\phi(0) \in D(A)$ for $[-\tau, 0]$, has a unique solution $u: [-\tau, T] \rightarrow X$ of (4) on $[-\tau, t]$, and depends continuously on initial data in the sense that for $\phi_n(t)$ with $\phi_n(0) \in D(A)$ with $\phi_n(t) \rightarrow 0$ then the corresponding solutions $u_n(t)$ of (4) with initial value $\phi_n(0)$, satisfy that $u_n(t) \rightarrow 0$ as $n \rightarrow \infty$ on $[-\tau, T]$.

Theorem 1. Suppose that B is one-one and $D(B^{-1}) = R(B) \supset R(A)$. If (4) is well-posed on $[-\tau, T]$ then it is well-posed on $[-\tau, 2T]$.

Proof: Let $\phi \in C([-\tau, 0]; X)$ with $\phi(0) \in D(A)$. Suppose that u is a solution of (4) on $[-\tau, T]$ for this ϕ . Then $u(T) \in D(A)$. Consider the problem $Bv'(t) = Av(t) + v(t - \tau)$ for $0 < t \leq 2T$ and $v(t) = u(t + T)$ for $[-\tau, 0]$. Since $v(0) = u(T) \in D(A)$, this problem by our assumption has a solution v on $[-\tau, T]$. Define

$$W(t) = \begin{cases} u(t) & t \in [-\tau, T], \\ v(t - T), & t \in [T, 2T]. \end{cases}$$

Clearly, $w(t) \in D(A)$ for $0 \leq t \leq 2T$. Strong derivative $w'(t)$ exists for $t \in (0, T) \cup (T, 2T)$. At $t = T$, for $\delta > 0$, we have

$$\frac{w(T + \delta) - w(T)}{\delta} = \frac{v(\delta) - v(0)}{\delta} \rightarrow v'(0), \text{ as } \delta \downarrow 0.$$

Also, for $\delta < 0$, we have

$$\frac{w(T + \delta) - w(T)}{\delta} = \frac{u(T + \delta) - u(T)}{\delta} \rightarrow u'(T), \text{ as } \delta \uparrow 0.$$

Furthermore, we have

$$Bv'(0) = Av(0) + v(-\tau) = Au(T) + u(T - \tau),$$

$$Bu'(T) = Au(T) + u(T - \tau).$$

Thus, we obtain $Bv'(0) = Bu'(T)$. Invertibility of B implies that $v'(0) = u'(T)$.

Therefore $w'(t)$ exists on $[0, 2T]$ and also, $w'(t) \in D(B)$ on $[0, 2T]$. Now, we show that $Bw'(t) = Aw(t) + w(t - \tau)$, for $0 \leq t \leq 2T$. For $[0, T]$, it is obvious. For $T \leq t \leq 2T$, we have

$$Bw'(t) = Bv'(t - T) = Av(t - T) + v(t - T - \tau).$$

Now, for $T \leq t \leq 2T$, if $t - T - \tau \in [-\tau, 0]$, then

$$v(t - T - \tau) = u(t - \tau) = w(t - \tau),$$

since $t - \tau \leq T$. If $t - T - \tau \geq T$, then $t - \tau \geq T$ and hence $w(t - \tau) = v(t - T - \tau)$.

This proves the existence.

Now, we show the uniqueness.

Let $w_1(t)$ be another solution of (4) on $[-\tau, 2T]$. Then by our assumption of well-posedness of (4) on $[-\tau, T]$, we have $w_1(t) = w(t) = u(t)$ on $[-\tau, T]$. Let $v_1: [-\tau, T] \rightarrow X$ be given by $v_1(t) = w_1(t + T)$. Then for $0 \leq t \leq T$,

$$Bv_1'(t) = Bw_1'(t + T) = Aw_1(t + T) + w_1(t + T - \tau) = Av_1(t) + v_1(t - \tau).$$

Also, $v_1(t) = u(t)$ for $t \in [-\tau, 0]$. Hence by well-posedness of (4) on $[-\tau, T]$, we have $v_1(t) = v(t)$ on $[-\tau, T]$. But then $w_1(t + T) = w(t + T)$ for $t \in [0, T]$. Hence $w_1(t) = w(t)$ on $[-\tau, 2T]$.

Theorem 2. Suppose that B is one-one and $D(B^{-1}) = R(B) \supset R(A)$. If (4) is well-posed on $[-\tau, T]$, then it is well-posed on $[-\tau, \infty]$.

Proof. Let $u_0(t)$ be a unique solution of (4) on $[-\tau, T]$. Then using Theorem 1 we have a unique solution $u_1(t)$ of (4) on $[-\tau, 2T]$ such that $u_1(t) = u_0(t)$ on $[-\tau, T]$. Again applying Theorem 1, we have a unique solution $u_2(t)$ of (4) on $[-\tau, 2^2T]$ with $u_2(t) = u_1(t)$ on $[-\tau, 2T]$. Thus, for

any $n \in \mathbb{N}$, we have unique solutions $u_{n-1}(t)$ and $u_n(t)$ of (4) on $[-\tau, 2^{n-1}T]$ and $[-\tau, 2^n T]$, respectively such that $u_n(t) = u_{n-1}(t)$ on $[-\tau, 2^{n-1}T]$. Now, define $u: [-\tau, \infty) \rightarrow X$ by $u(t) = u_n(t)$ for $t \in [-\tau, 2^n T]$. Then $u(t)$ is the unique solution of (4) on $[-\tau, \infty)$.

Theorem 3. *Let B be invertible on $R(B)$ and $R(B) \supset R(A)$ and let $B^{-1}A$ generates a C_0 semigroup $\{S(t): 0 \leq t < \infty\}$ of bounded linear operators in X . Let $\phi \in C^1([-\tau, 0]; X)$ be such that $\phi(0) \in D(A)$. Then (4) is well-posed on $[-\tau, \infty)$.*

Proof. In view of Theorem 2, we shall prove the existence of a unique solution $u(t)$ of (4) on $[-\tau, \tau]$ and can take $T = \frac{\tau}{2}$.

On $[-\tau, \tau]$, (4) can be written as

$$(6) \quad \begin{aligned} B \frac{du}{dt}(t) &= Au(t) + \phi(t - \tau), \quad t \in [0, \tau], \\ u(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned}$$

if (6) has a solution $u(t)$ on $[-\tau, \tau]$, then the function

$$g(s) = S(t - s)u(s),$$

is differentiable with respect to s for $0 < s < t \leq \tau$ and

$$\begin{aligned} \frac{dg}{ds} &= -B^{-1}AS(t - s)u(s) + S(t - s)u'(s) \\ &= -B^{-1}AS(t - s)u(s) + S(t - s)B^{-1}Au(s) + S(t - s)B^{-1}\phi(s - \tau) \\ &= S(t - s)B^{-1}\phi(s - \tau). \end{aligned}$$

Since the generator $B^{-1}A$ commutes with its semigroup $S(t)$, $u(t)$ is given by

$$u(t) = \begin{cases} \phi(t), & t \in [-\tau, 0], \\ S(t)\phi(0) + \int_0^t S(t - s)B^{-1}\phi(s - \tau)ds, & t \in [0, \tau]. \end{cases}$$

We note that if $\phi \in C^1([-\tau, 0]; X)$, then

$$\int_0^t S(t-s) B^{-1} \phi(s-\tau) ds$$

is continuously differentiable on $(0, \tau)$ (cf. corollary 2.5 on page 107 in Pazy¹). We then use Theorem 2.4 in Pazy¹ on page 107, to conclude

that (6) has a unique solution on $[-\tau, \tau]$. Now, if we take $T = (\tau/2)$, then we have a unique solution $u(t)$ of (6) on $[-\tau, 2T]$. Now we may use Theorem 2. This completes the proof of the theorem.

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