

A Bound on Weak Domination Number Using Strong (Weak) Degree Concepts in Graphs

R. S. Bhat

Manipal Institute of Technology, Manipal, India
E-mail: ravishankar.bhats@gmail.com

S. S. Kamath

Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal, India.

Surekha

Milagres College, Kallianpur, Udupi, India

(Received December 22, 2010)

Abstract. For a vertex v of a graph $G = (V, X)$, $N(v) = \{u \in V \mid u \text{ is adjacent to } v\}$. Then degree of the vertex v , $d(v) = |N(v)|$. We define $N_s(v) = \{u \in N(v) \mid d(v) \geq d(u)\}$. Similarly, $N_w(v) = \{u \in N(v) \mid d(v) \leq d(u)\}$. Then *Strong degree* of a vertex v is $d_s(v) = |N_s(v)|$ and *weak degree* of a vertex v is $d_w(v) = |N_w(v)|$. Consequently, we have the following graph parameters – maximum strong degree $\Delta_s(G)$, minimum strong degree $\delta_s(G)$, maximum weak degree $\Delta_w(G)$ and minimum weak degree $\delta_w(G)$. For any two adjacent vertices u and v in a graph $G = (V, X)$, u strongly [weakly] dominates v if $d(u) \geq d(v)$ [$d(u) \leq d(v)$]. A set $D \subseteq V$ is a *dominating set* (*strong dominating set* [sd-set], *weak dominating set* [wd-set] respectively) of G if every $v \in V - D$ is dominated (strongly dominated, weakly dominated respectively) by some $u \in D$. The *domination number* (*strong domination number*, *weak domination number* respectively) $\gamma = \gamma(G)$ ($\gamma_s = \gamma_s(G)$, $\gamma_w = \gamma_w(G)$ respectively) of G is the minimum cardinality of a dominating set (sd-set, wd-set respectively) of G .

In this paper, we obtain a new lower bound and also an improved upper bound for the weak domination number using the new parameters. A new version of the “First Theorem in Graph Theory” is presented. We define a new matrix called *strong weak adjacency matrix* of a graph and give an algorithm to obtain the strong weak adjacency matrix from the adjacency matrix and vice versa. Using the properties of the new matrix, we can compute all the information about the new parameters defined above.

Keywords: Strong degree, weak degree, strong vertex, weak vertex, weak domination.

AMS Subject Classification No.: 05C69

1. Introduction

The motivation to this paper comes from the strong weak domination parameters defined by Sampathkumar and Pushpalatha¹². For any two adjacent vertices u and v in a graph $G = (V, X)$, u strongly [weakly] dominates v if $d(u) \geq d(v)$ [$d(u) \leq d(v)$]. A set $D \subseteq V$ is a *dominating set* (*strong dominating set* [sd-set], *weak dominating set* [wd-set] respectively) of G if every $v \in V - D$ is dominated (strongly dominated, weakly dominated respectively) by some $u \in D$. The *domination number* (*strong domination number*, *weak domination number* respectively) $\gamma = \gamma(G)$ ($\gamma_s = \gamma_s(G)$, $\gamma_w = \gamma_w(G)$ respectively) of G is the minimum cardinality of a dominating set (sd-set, wd-set respectively) of G . This concept has been further studied by Domke, Hattingh, Rautenbach et al in¹⁻²⁻⁴⁻⁵⁻⁸⁻¹¹. For undefined terminologies, the reader is referred to³⁻⁶⁻¹³.

We obtain some sharp bounds to the weak domination number by introducing some new degree concepts. In this paper, we define three new parameters, viz., strong degree, weak degree and regular degree.

2. Strong Degree, Weak Degree, S-Silent Number and W-Silent Number

For a vertex v of a graph $G = (V, X)$, $N(v) = \{u \in V \mid u \text{ is adjacent to } v\}$. Then degree of v is $d(v) = |N(v)|$. Similarly, let $N_s(v) = \{u \in N(v) \mid d(v) \geq d(u)\}$ and $N_w(v) = \{u \in N(v) \mid d(v) \leq d(u)\}$. $N_r(v) = \{u \in N(v) \mid d(v) = d(u)\}$. Then we define *strong degree* of v as $d_s(v) = |N_s(v)|$, *weak degree* of v as $d_w(v) = |N_w(v)|$ and *regular degree* of v as $d_r(v) = |N_r(v)|$. As a consequence, we have the following six graph parameters – maximum strong degree of a graph $\Delta_s(G)$, minimum strong degree $\delta_s(G)$, maximum weak degree $\Delta_w(G)$, minimum weak degree $\delta_w(G)$, maximum regular degree $\Delta_r(G)$, and minimum regular degree $\delta_r(G)$. Further, a vertex v is called *strong - silent* (s- silent), if $d_s(v) = 0$ and *weak - silent* (w- silent), if $d_w(v) = 0$. A set $S \subset V$ is said to be *strong-silent set* (*weak-silent set*) if every vertex in S is s-silent (w-silent). The *s-silent* (*w-silent*) *number* $\eta_s = \eta_s(G)$ ($\eta_w = \eta_w(G)$) is the maximum

cardinality of a s -silent (w-silent) set of G . We shall illustrate the above concepts with an example.

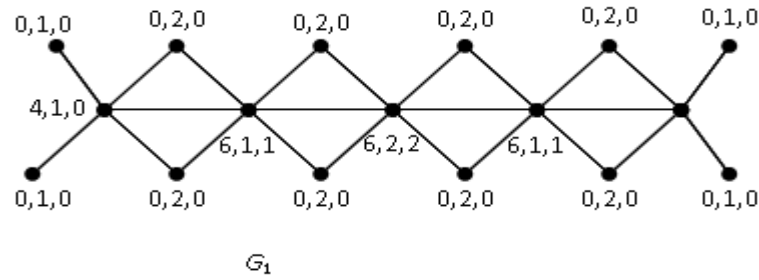
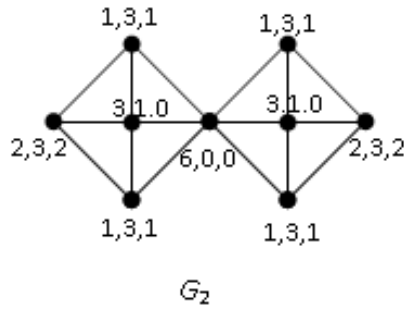


Fig.2.1



In the graphs G_1 and G_2 shown in Fig.2.1, the first, second and third elements of a vertex label represent the strong degree, weak degree and regular degree of the corresponding vertices respectively. $\Delta_s(G_1) = \Delta(G_1) = 6$, $\delta_s(G_1) = 0$, $\Delta_w(G_1) = 2$, $\delta_w(G_1) = 1$, $\Delta_r(G_1) = 2$, $\delta_r(G_1) = 0$, $\eta_s(G_1) = 12$ and the η_s -set is the set of all vertices with first coordinate zero in G_1 . $\eta_w(G_1) = 0$. Similarly, $\Delta_s(G_2) = \Delta(G_2) = 6$, $\delta_s(G_2) = 1$, $\Delta_w(G_2) = 3$, $\delta_w(G_2) = 0$, $\Delta_r(G_2) = 2$, $\delta_r(G_2) = 0$, $\eta_s(G_2) = 0$ and $\eta_w(G_2) = 1$ and the η_w -set is the set with maximum degree vertex in G_2 . These two examples also exhibit that δ_s and δ_w are not comparable. Also η_s and η_w are not comparable. One can find few more varieties of degree concepts in ⁹.

Proposition 2.1. Let $G = (V, E)$ be any graph. Then for any vertex $v \in V$, $d(v) = d_s(v) + d_w(v) - d_r(v)$.

Proof. For any vertex $v \in V$, let $A = N(v)$, $S = N_s(v)$, $W = N_w(v)$ and $R = N_r(v)$. We observe that $A = S \cup W$ and $R = S \cap W$. Therefore $d(v) = |A| = |S \cup W| = |S| + |W| - |S \cap W| = |S| + |W| - |R| = d_s(v) + d_w(v) - d_r(v)$.

An edge $x = uv$ is called a *beam* if $d(u) = d(v)$. The number of beams in a graph is denoted as q_r . With this definition we give a simple but elegant result.

Proposition 2.2. For any (p, q) -graph G ,

$$(2.1) \quad \sum_{v \in V} d_s(v) = \sum_{v \in V} d_w(v) = q + q_r,$$

$$(2.2) \quad \sum_{v \in V} d_r(v) = 2q_r,$$

where q_r is the number of beams in G .

Proof. The result (2.1) follows from the fact that in counting the strong degree or weak degree of a vertex every edge which is not a beam is counted exactly once and the beams are counted twice. The result (2.2) follows from the fact that in counting the regular degree of a vertex the beams are counted twice and the edge which is not a beam is not at all counted. Note that for the graph G_2 in Fig.1, $q = 16$ and the number of beams = $q_r = 4$. $\sum_{v \in V} d_s(v)$ = sum of all first coordinates of vertex labels in,

$$G_2 = \sum_{v \in V} d_w(v) = \text{sum of all second coordinates of vertex labels in,}$$

$$G_2 = 20 = 16 + 4 = q + q_r. \text{ Thus, Proposition 2.2 is verified.}$$

The following theorem is an aliter of first theorem in graph theory using strong (weak) degree concepts.

Theorem 2.3. For any (p, q) -graph G ,

$$\sum_{v \in V} (2d_s(v) - d_r(v)) = \sum_{v \in V} (2d_w(v) - d_r(v)) = 2q,$$

$$\begin{aligned} \text{Proof. } \sum_{v \in V} d(v) &= \sum_{v \in V} (d_s(v) + d_w(v) - d_r(v)) \quad (\text{using Proposition 1}) \\ &= \sum_{v \in V} (2d_s(v) - d_r(v)) \quad (\text{using Proposition 2}) \\ &= 2(q + q_r) - 2q_r = 2q. \end{aligned}$$

Corollary 2.3.1. *For any graph G of order p and size q ,*

$$\sum_{v \in V} [d_s(v) + d_w(v)] \text{ is an even number.}$$

Proof. From Proposition 2.1, for any vertex v ,

$$d(v) = d_s(v) + d_w(v) - d_r(v).$$

Hence $\sum_{v \in V} d_s(v) + d_w(v) = \sum_{v \in V} (d(v) + d_r(v)) = 2q + 2q_r$, which is an even number. We recall the following definitions given in ⁷. A vertex $v \in V$ is *strong* [*weak*] if $d(v) \geq d(u)$ [$d(v) \leq d(u)$] for every $u \in N(v)$ in G . A vertex v which is neither strong nor weak is called a *balanced* vertex. A vertex v is *regular* if $d(v) = d(u)$ for every $u \in N(v)$ in G . A regular vertex is both strong and weak. A vertex v is *strictly strong* [*strictly weak*] if $d(v) > d(u)$ [$d(v) < d(u)$] for every $u \in N(v)$. A set $S \subseteq V$ is said to be *strong* (*weak*, *balanced*, *regular* respectively) if every vertex in S is a strong (weak, balanced, regular respectively) vertex in G . The *strong number* (*weak number*, *balanced number*, *regular number* respectively) $s = s(G)$ ($w = w(G)$, $b = b(G)$, $r = r(G)$ respectively) is the maximum cardinality of a strong (weak, balanced, regular respectively) set of G . The maximum strong set is denoted as s -set. Similarly we have w -set, b -set and r -set. Further it is proved that $p = s + w + b - r$.

Proposition 2.4. *Let $G = (V, E)$ be any graph. Then*

- (i) *A vertex $v \in V$ is strong if and only if $d(v) = d_s(v)$.*
- (ii) *A vertex $v \in V$ is weak if and only if $d(v) = d_w(v)$.*
- (iii) *A vertex $v \in V$ is balanced if and only if $d(v) = d(u)$ for some $u \in N(v)$ and $d(v) = d(w)$ for some $w \in N(v)$.*
- (iv) *A vertex $v \in V$ is regular if and only if $d(v) = d_s(v) = d_w(v)$.*
- (v) *A vertex $v \in V$ is strictly strong if and only if $d_w(v) = 0$.*
- (vi) *A vertex $v \in V$ is strictly weak if and only if $d_s(v) = 0$.*
- (vii) $(\delta_s(G), \delta_w(G)) \leq \delta(G) \leq \Delta_w(G) \leq \Delta_s(G) = \Delta(G)$.

Proof. (i) Let $v \in V$ be a strong vertex. Then $d(v) \geq d(u)$ for every $u \in V$. Hence $N_w(v) = \{u \in N(v) \mid d(v) = d(u)\} = N_r(v)$. Therefore $d_w(v) = d_r(v)$. Then by Proposition 1, we have $d(v) = d_s(v)$. Converse follows from the definition.

(ii) can be proved with the similar argument and hence we omit the proof.

(iii) Let v be a balanced vertex. Suppose both the conditions do not hold. Then $d(v) = d(u)$ for every $u \in N(v)$. Hence v is a regular vertex – a contradiction to our assumption. Conversely, suppose both the conditions hold. Since $d(v) < d(u)$ for some $u \in N(v)$, we conclude that v is not a strong vertex. Similarly, v cannot be a weak vertex as $d(v) > d(w)$ for some $w \in N(v)$. Further, v is not a regular vertex as $d(v) \neq d(u)$ for every $u \in N(v)$. Then v must be a balanced vertex.

(iv), (v) and (vi) follow from the definitions.

(vi) Since the strong degree is maximum at a vertex of maximum degree Δ , we have $\Delta_s(G) = \Delta(G)$. Since $d_w(v) \leq d(v)$, we have $\Delta_w(G) \leq \Delta(G)$. Let v be a vertex of minimum degree δ . As every minimum degree vertex is a weak vertex, from (ii) we have $d(v) = d_w(v)$. Then $\delta = d(v) = d_w(v) \leq \Delta_w(G)$. Further $d_w(v) \leq d(v)$ implies $\delta_w(G) \leq \delta(G)$ and $d_s(v) \leq d(v)$ implies $\delta_s(G) \leq \delta(G)$. Therefore we have $(\delta_w(G), \delta_s(G)) \leq \delta(G)$. Hence the proof.

Corollary 2.4.1 *Let $G = (V, X)$ be any graph without beams. Then*

(i) $d(v) = d_s(v) + d_w(v)$.

(ii) Further, if the balance number $b = 0$, then for any $v \in V$ either $d(v) = d_s(v)$ or $d(v) = d_w(v)$ holds.

Proof. Since G is a graph without beams we have $q_r = 0$. This implies for any two adjacent vertices u and v , $d(u) \neq d(v)$. Then we have $d_r(v) = 0$ for every $v \in V$. Then the result (i) follows from the fact that $d(v) = d_s(v) + d_w(v) - d_r(v)$.

To prove (ii). Since G is a graph without beams, we have $d_r(v) = 0$. Further, if $b = 0$, we have G has no balanced vertex. Thus for any two

adjacent vertices u and v , either $d(u) > d(v)$ or $d(u) < d(v)$ holds. This implies for any vertex $v \in V$ either v is a strictly strong vertex or v is a strictly weak vertex in G . Then from Proposition 2.4, we have either $d(v) = d_s(v)$ or $d(v) = d_w(v)$ holds for any $v \in V$.

3. Some New Bounds on Covering & Strong (Weak) Covering Numbers

We give the following definitions and a couple of results which we use in sequel. S. S. Kamath and R. S. Bhat ⁷ defined strong (weak) vertex coverings, strong (weak) independent sets. A strong (weak) set which is independent is called a *strong independent set* [SIS] (*weak independent set* [WIS]). The *strong (weak) independence number* $s\beta_0(G)$ ($w\beta_0(G)$) of G is the maximum cardinality of a SIS (WIS). For an edge $x = uv$, v strongly covers the edge x if $d(v) \geq d(u)$ in G . Then u weakly covers x . A set $S \subseteq V$ is a *Vertex Cover* if every edge in G is covered by some vertex in S . A set $S \subseteq V$ is a *Strong Vertex Cover* [SVC] (*Weak Vertex Cover* [WVC]) if every edge in G is strongly (weakly) covered by some vertex in S . The *vertex covering number* $\alpha_0(G)$ (*strong vertex covering number* $s\alpha_0(G)$, *weak vertex covering number* $w\alpha_0(G)$ respectively) is the minimum cardinality of a vertex cover (SVC, WVC respectively). They also studied the edge analogue of these parameters in ¹⁰.

Theorem 3.1 [Gallai]. For any graph $G = (V, E)$ with p vertices, $\alpha_0 + \beta_0 = p$.

Theorem 3.2 [6]. For any graph $G = (V, E)$ with p vertices, $s\alpha_0 + w\beta_0 = p$ and $w\alpha_0 + s\beta_0 = p$.

Theorem 3.3. Let $G = (V, X)$ be any isolate free graph. Then

(i) $\eta_s(G) = |I_w|$ where $I_w = \{v \in V \mid d(v) > d(w) \text{ for every } w \in N(v)\}$ and I_w is a WIS.

(ii) $\eta_w(G) = |I_s|$ where $I_w = \{v \in V \mid d(v) > d(w) \text{ for every } w \in N(v)\}$ and I_s is a SIS.

Proof. We first note that I_w (I_s) is the set of all strictly weak (strictly strong) vertices in G . Then (i) and (ii) follow from the fact that a vertex v is in a s -silent (w -silent) set if and only if, v is a strictly weak (strictly strong) vertex in G . It is clear that every vertex in I_w is weak. Further, no

two vertices in I_w are not adjacent by the definition of I_w . Thus I_w is a weak independent set. With the similar argument we have I_s is a SIS.

Theorem 3.4. *Let $G = (V, X)$ be any isolate free graph. Let S and W be the s -set and w -set of G respectively with $|S| \geq 2, |W| \geq 2$. Then*

- (i) $\eta_s(G) = 0$ if, and only if, $\langle W \rangle$ the subgraph induced by the w -set has no isolates. Further, the edges of $\langle W \rangle$ are the beams in G .
- (ii) $\eta_w(G) = 0$ if, and only if, $\langle S \rangle$ the subgraph induced by the s -set has no isolates. Further the edges of $\langle S \rangle$ are the beams in G .

Proof. Let $\eta_s(G) = 0$. Then there exists no strictly weak vertex in G . Hence for every weak vertex $w \in W$, there exists at least one $v \in N(w)$ such that $d(w) = d(v)$. But then the vertex v is also a weak vertex in G . Hence $v \in W$. Thus for any $v \in W$ there is some $w \in W$ such that v is adjacent to w . Hence $\langle W \rangle$ has no isolates. Conversely, Let $\langle W \rangle$ the sub graph induced by the w -set has no isolates. Suppose $\eta_s(G) \neq 0$. Then there exists at least one vertex $v \in I_w$ which is s -silent. Since I_w is independent and $I_w \subseteq W$, we have $\langle W \rangle$ has at least one isolated vertex – a contradiction. Any two weak vertices u and v in W are adjacent if and only if $d(u) = d(v)$. But then the edge uv is a beam in G . Thus the edges of $\langle W \rangle$ are the beams in G . The argument in respect of the second result is similar and hence we omit the proof. In the next proposition, we get some bounds on covering numbers in terms of newly defined parameters.

Proposition 3.5. *Let G be a (p, q) graph with maximum degree Δ , maximum weak degree Δ_w , s -silent number η_s and w -silent number η_w . Then*

$$(3.1) \quad \left\lceil \frac{q}{\Delta} \right\rceil \leq \alpha_o \leq s\alpha_o \leq p - \eta_s,$$

$$(3.2) \quad \left\lceil \frac{q}{\Delta_w} \right\rceil \leq w\alpha_o \leq p - \eta_w,$$

Further, these bounds are sharp.

Proof. Since a vertex in G can cover at most Δ edges and we have to exhaust all the q edges, we need at least q/Δ edges to cover all the edges of G . This implies the lower bound in (3.1). A vertex in G can weakly cover at most Δ_w edges and we have to exhaust all the q edges, we need at least $\lceil q/\Delta_w \rceil$ edges to weakly cover all the edges of G . This implies the lower bound in (3.2). Let $S \subset V$ be a η_s -set of G . Since every vertex in S is s -silent, no vertex in S strongly cover any edge in G , we have $V - S$ is a SVC of G . Hence $s\alpha_o \leq |V - S| = p - \eta_s$. With the similar argument we can prove the upper bound in (3.2).

It is easy to verify that any complete bipartite graph $K_{m,n}$ with $m \neq n$ attains the upper bound in (3.1) and (3.2). We further observe that (if $m \leq n$) $\alpha_o(K_{m,n}) = s\alpha_o(K_{m,n}) = m = \frac{mn}{n} \left\lceil \frac{q}{\Delta} \right\rceil$ and $w\alpha_o(K_{m,n}) = n = \frac{mn}{n} \left\lceil \frac{q}{\Delta_w} \right\rceil$. Hence $K_{m,n}$ also attains the lower bound in (3.1) and (3.2). Thus the above bounds in the proposition are sharp.

From Proposition 3.5, Theorem 3.1 and Theorem 3.2, we have the following bounds for independence numbers.

Corollary 7.1. For any (p, q) graph G ,

$$\eta_s \leq w\beta_o \leq p - \left\lceil \frac{q}{\Delta} \right\rceil,$$

and
$$\eta_w \leq s\beta_o \leq p - \left\lceil \frac{q}{\Delta_w} \right\rceil,$$

4. New bounds on weak domination number

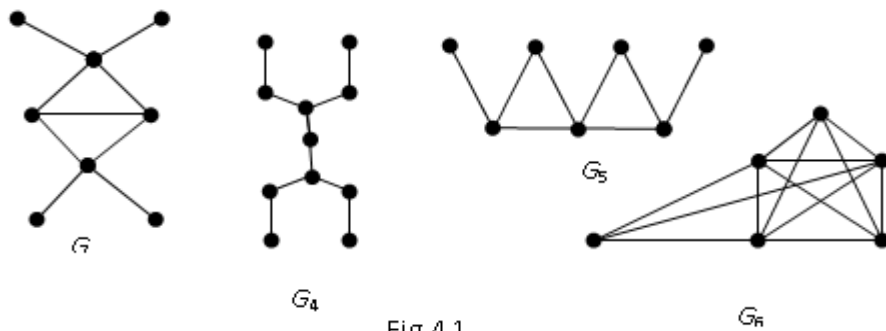


Fig. 4.1

Since every strong (weak) covering is also a strong (weak) dominating set, we have $\gamma_s \leq s\alpha_o$ ($\gamma_w \leq w\alpha_o$). Consequently, we get an upper bound for γ_s and γ_w as a corollary to the Proposition 3.5.

Corollary 3.5.2. *Let G be any graph with p vertices, s – silent number η_s and w – silent number η_w then*

$$(4.1) \quad \gamma_s \leq p - \eta_s,$$

$$(4.2) \quad \gamma_w \leq p - \eta_w.$$

Further these bounds are sharp.

The graphs G_4 and G_5 in Fig.4.1 attain the upper bound in (4.1). The complete bipartite graph $K_{m,n}$, $m \neq n$ attain the upper bounds both in (4.1) and (4.2). Walikar et.al¹⁰ gave the following lower bound for γ .

Theorem 4.1. *For any graph G , $\left\lceil \frac{p}{1+\Delta} \right\rceil \leq \gamma$,*

Sampathkumar and Pushpalatha¹² offered the following upper bounds for strong and weak domination number.

Theorem 4.2. *For any graph G with p vertices, maximum degree Δ , minimum degree δ ,*

$$\gamma_s \leq p - \Delta \quad \text{and} \quad \gamma_w \leq p - \delta.$$

Proof. Since $\gamma \leq \gamma_s$ and $\gamma \leq \gamma_w$, the lower bound given in Theorem C is also a lower bound for γ_s and γ_w . Using the new parameters defined above, we give a similar but an improved upper bound and a new lower bound for γ_w in the next Proposition.

Proposition 4.3. *For any graph G with p vertices and maximum weak degree Δ_w*

$$(4.3) \quad \left\lceil \frac{p}{1+\Delta_w} \right\rceil \leq \gamma_w \leq p - \Delta_w,$$

Proof. Any vertex v can weakly dominate at most Δ_w vertices and it self implies we need at least $\left\lceil \frac{p}{1 + \Delta_w} \right\rceil$ vertices to weakly dominate all the vertices of G . Hence the lower bound in (4.3) follows. Let v be a vertex of maximum weak degree Δ_w . Then $V - N_w(v)$ is a weak dominating set of G . Hence the upper bound follows.

The following examples show that the bound in the above Proposition is sharp. For the graph G_3 in Fig.3, $\gamma_w(G_3) = 5 = 8 - 3 = p - \Delta_w$. Hence G_3 attains the upper bound in (4.3). For the graph G_6 in Fig.4.1, $\Delta_w = 4$ and $\gamma_w = 2$. Hence G_6 attains the upper and lower bounds in (4.3).

The independent weak domination number $i_w = i_w(G)$ is the minimum cardinality of an independent weak dominating set (IWDS). Domke et.al.² proved the conjecture $\gamma_w \leq i_w \leq p - \delta$ posed by Sampathkumar and Pushpalatha in ¹². Under certain conditions we get an improved bound for i_w .

Proposition 4.4. *Let G be a (p, q) graph. V_{Δ_w} be the set of all maximum weak degree vertices and D be any IWDS such that $D \cap V_{\Delta_w} \neq \phi$. Then*

$$(4.4) \quad \gamma_w \leq i_w \leq p - \Delta_w,$$

Proof. Let $D \cap V_{\Delta_w} \neq \phi$ and $v \in D \cap V_{\Delta_w}$. Let $N_w(v) = \{u \in N(v) \mid d(v) \leq d(u)\}$. Since D is independent $D \cap N_w(v) = \phi$. Therefore we have $D \subseteq V - N_w(v)$. Hence the result follows.

We strongly believe that if $D \cap V_{\Delta_w} = \phi$ then $i_w \leq p - (\Delta_w + 1)$ and we put it as a conjecture.

Conjecture 4.4.1. *For any graph G with p vertices and maximum weak degree Δ_w , $\gamma_w \leq i_w \leq p - \Delta_w$.*

5. Strong weak adjacency matrix

The adjacency matrix of a graph helps us to study the graph properties like connectivity of a graph, number of components of a graph and maximum and minimum degree of a graph etc. through computer. But this adjacency matrix does not provide information about the new graph parameters defined above like strong degree, weak degree of a vertex etc. This cause compelled us to define a new matrix called strong weak adjacency matrix of a graph. Using the properties of the new matrix we can retrieve all the information about the new parameters defined above through computer.

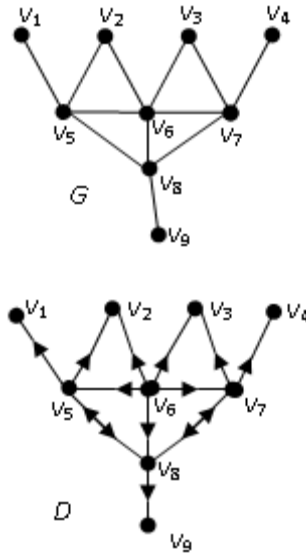


Fig. 5.1

$$A_{sw}(G) = \begin{matrix} & \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Strong-weak adjacency matrix of the graph G

The *strong weak adjacency matrix* $A_{sw} = [a_{ij}]$ of a labeled graph is a $p \times p$ matrix such that

$$\begin{aligned} a_{ij} &= 1 \text{ if } v_i \text{ is adjacent to } v_j \text{ and } d(v_i) > d(v_j) \\ &= -1 \text{ if } v_i \text{ is adjacent to } v_j \text{ and } d(v_i) < d(v_j) \\ &= 0 \text{ if } v_i \text{ is not adjacent to } v_j \text{ or } d(v_i) = d(v_j) \end{aligned}$$

Then *weak strong adjacency matrix* is defined as $A_{ws} = [a_{ij}]$ and

$$\begin{aligned} a_{ij} &= 1 \text{ if } v_i \text{ is adjacent to } v_j \text{ and } d(v_i) < d(v_j) \\ &= -1 \text{ if } v_i \text{ is adjacent to } v_j \text{ and } d(v_i) > d(v_j) \\ &= 0 \text{ if } v_i \text{ is not adjacent to } v_j \text{ or } d(v_i) = d(v_j). \end{aligned}$$

The above strong weak adjacency matrix of a graph G can also be obtained using digraph by orienting the graph G as follows. If v_i is adjacent to v_j and $d(v_i) \geq d(v_j)$ then draw a directed arc from v_i to v_j . The obtained digraph D is called the *strong weak digraph*. By reversing the direction of every edge in D , we get a *weak strong digraph*.

Now define $a_{ij} = 1$ if $v_i v_j$ is a directed edge in D
 $= -1$ if both $v_i v_j$ and $v_j v_i$ are directed edges in D .
 $= 0$ if $v_i v_j$ is not a directed edge in D .

Fig. 5.1 illustrates the construction of strong weak digraph D of a graph G .

The Strong weak adjacency matrix defined above satisfies the following properties.

1. The number of non-zero entries in i^{th} row gives the strong degree of the vertex v_i .
2. The number of non-zero entries in j^{th} column gives the weak degree of the vertex v_j .
3. The number of -1's in a row or column gives the regular degree of the corresponding vertex and the total number of -1's in the matrix is always even.
4. The sum of number of 1's and half the number of -1's in the matrix is equal to the number of edges in the graph G .
5. $(A_{sw})^T = A_{ws}$.

6. If i^{th} row has at least one 1 and i^{th} column has no 1's, then v_i is a strong vertex. If i^{th} column has at least one 1 and i^{th} row has no 1's, then v_i is a weak vertex. If i^{th} row (column) has only -1's, as non zero entries then v_i is a regular vertex.
7. If G is an isolate free graph, then the number of zero rows (zero columns) gives the s – silent number $\eta_s(G)$ (w-silent number $\eta_w(G)$).
8. Given any strong weak adjacency matrix $A_{sw} = [a_{ij}]$ of an isolate free graph G , we can obtain the adjacency matrix $B = [b_{ij}]$ of G as follows.

$$b_{ij} = \begin{cases} 1 & \text{if } a_{ij} + a_{ji} = 1 \text{ or } -2 \\ 0 & \text{if } a_{ij} + a_{ji} = 0 \end{cases},$$

9. Given any adjacency matrix of an isolate free graph G we can get the strong weak adjacency matrix using the following algorithm.

Let $A = [a_{ij}]$ be the adjacency matrix.

Let $A_{sw} = B_{sw} = [b_{ij}]$.

If $a_{ij} = 0$ then $b_{ij} = 0$.

$$\text{If } a_{ij} = 1 \text{ then } b_{ij} = \begin{cases} 1 & \text{if } \sum_{i=1}^n a_{ij} > \sum_{j=1}^n a_{ij} \\ -1 & \text{if } \sum_{i=1}^n a_{ij} = \sum_{j=1}^n a_{ij} \\ 0 & \text{if } \sum_{i=1}^n a_{ij} < \sum_{j=1}^n a_{ij} \end{cases}.$$

References

1. R. S. Bhat, A study of strong (weak) domination and related parameters of a graph, Ph.D. Thesis, National Institute of Technology Karnataka, Surathkal 2007.
2. G. S. Domke, J. H. Hattingh, L. R. Marcus, Elna Ungerer, On parameters related to strong and weak domination in graphs, *Discrete Mathematics*, **258** (2002) 1-11.
3. F. Harary, *Graph Theory*, Addison Wesley 1969.
4. J. H. Hattingh and M. A. Henning, On strong Domination in graphs, *J. Combin. Math. Combin. Comput.*, **26** (1998) 33-42.
5. J. H. Hattingh and Renu C. Laskar, On Weak Domination in Graphs, *ARS Combinatoria*, **49** (1998) 205-216.
6. T.W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., N.Y., 1999.

7. S. S. Kamath and R. S. Bhat, On strong weak independence and vertex covering number of a graph, *Discrete Mathematics*, **307** (2007) 1136 – 1145.
8. S.S. Kamath and R. S. Bhat, Strong/ Weak Neighbourhood Number of a graph (K3-coverings), *Proc. Int. Conf. on Disc. Math., Narosa Pub., New Delhi, India*, (2004) 142-150.
9. S. S. Kamath and R. S. Bhat, Some new degree concepts in graphs, *Proceedings of the, ICDM* (2006) 237- 243.
10. S. S. Kamath and R. S. Bhat and surekha, Strong (weak) Matchings and Edge coverings of a graph, *J. of Intelligent System Resaerch*, **3 (2)** (2009) 235-243.
11. D. Routenbach, *Domination and Degree*, Ph.D. Thesis, Mathematic, Shaker Verlag, (1998) ISBN 3- 8265-3899-4.
12. E. Sampathkumar and L. Pushpalatha, Strong weak domination and domination balance in a Graph, *Discrete Mathematics*, **161** (1996) 235-242.
13. H. B. Walikar, B. D. Acharya and E. Sampathkumar, *Recent developments in Theory of Domination in Graphs*, MRI Lecture notes **1** The Mehta Research Institute, Allahabad 1979.