

## Analysis of the m/m/1 Queue Model with Multiple Vacations and Server Break-Down

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(Received December 22, 2010)

**Abstract:** Analysis of M/M/1 queue model with multiple vacations and server breakdown is the specific queuing model where the server takes a vacation as soon as it becomes idle. The vacation ends, if there are customers arrive in the queue, the server activates; otherwise, another vacation is taken. Once service begins, the service mechanism is subject to breakdowns which occur at a constant rate. The service mechanism then goes through a repair process of random duration, and when repair is completed, the server returns to the customers whose services were interrupted. In this paper we have obtained the queue length for both cases i.e. for vacations and break down. Also for these two cases we have discussed the model for the numerical values and further value of queue length has been obtained for some fix values and variable parameter.

**Keywords:** Queueing model, Multiple Vacations, Sever Breakdown.

**Mathematical subject Classification (AMS 2000):** 68B20

### 1. Introduction

In this paper we consider a queuing model in which the server takes a vacation when the system becomes idle. When the vacation ends, if there are customers in the queue, the server activates; otherwise, another vacation is taken. Once service begins, the service mechanism is subject to breakdowns which occur at a constant rate. The service mechanism then goes through a repair process of random duration, and when repair is completed, the server returns to the customers whose service was interrupted. During vacation, active service, and repair process customers arrive according to Poisson process with different rates  $\lambda$ . Under the assumptions that vacation times, uninterrupted service times, and repair times have exponential distributions,

we first obtain a necessary and sufficient condition for the existence of the stationary queue length distribution. In this paper we obtain the queue length distribution and mean queue length and then the server goes on vacation at the instant when the queue becomes empty, and continues to take vacations of exponential length until, at the end of a vacation, customers are found in the queue. The uninterrupted service times of customers are of exponential length. The service mechanism breakdowns occur only during active service and the breakdowns the following a repaired or vacation is exponential. Repair times are also exponential. The arrival processes during vacation, active service, and breakdown are Poisson with different rates. All inter-arrival, vacation, service inter-breakdown and repair times are independent of each other.

The study of queuing models with interruptions dates back to the 1950s. Some of the earliest papers in the area are in <sup>1-6</sup>. Since that time, a large number of papers have been devoted to subject. One example is Ibe <sup>7</sup> who studies a queuing system in which a single server serves two stations in cyclic order, and where server breakdown occur.

The vacation model terminology seems to have first appeared in the 1970s. Doshi <sup>2</sup> wrote an excellent survey paper on vacation models. Numerous papers on vacation have appeared since that time; worth to maintained a few examples. A considerable body of material on vacation and priority queuing models can be found in <sup>4</sup>. Takagi <sup>5</sup> studies the M/G/1 vacation queuing model with finite waiting room, and Machihara <sup>3</sup> considers a preemptive priority queue as a vacation queuing model. Wang et al <sup>8</sup> elaborate on an interesting approach to estimate the equilibrium distribution for the number of customers in the  $M^{[x]}/M/1$  queuing model with multiple vacations and server breakdowns. Their approach consists of maximizing an entropy function subject to constraints, where the constraints are formed by some known exact results.

## 2. Steady-State Equations for the queue length

We assume that  $\lambda, \lambda_0, \mu, v, b, r > 0$  and  $\lambda_1 \geq 0$  with traffic efficient  $\rho = \lambda/\mu$  and  $\rho_0 = \lambda_0/(\lambda_0 + v)$  are provided that  $\lambda$  is arrival rate during active service,  $\lambda_0$  is arrival rate during vacation time,  $\lambda_1$  is arrival rate during breakdown,  $v$  is vacation rate,  $\mu$  is service rate,  $b$  is breakdown rate and  $r$  is repair rate of queuing system under multiple vacations:

The states for the model are as follows:

- (0, i) is the state in which there are i customers in the queue and the server is on vacation,  $i \geq 0$ . the probability is  $p(0, i)$ .
- (1, i) is the state in which there are i customers in the system during active service  $i \geq 1$ . The probability is  $p(1, i)$ .
- (2, i) is the state in which there are i customers in the system during repair process,  $i \geq 2$ . The probability is  $p(2, i)$ .

The following are partial generating functions for the model:

$$F_0(z) = \sum_{i=0}^{\infty} p(0,i) z^i, \quad F_1(z) = \sum_{i=1}^{\infty} p(1,i) z^i,$$

$$\& \quad F_2(z) = \sum_{i=1}^{\infty} p(2,i) z^i,$$

Finally,  $F(z) = F_0(z) + F_1(z) + F_2(z)$  is the generating function for the queue length distribution, the relations between the steady-state probabilities are as given below:

$$(3.1) \quad \lambda_0 p(0,0) = \mu p(1,1),$$

$$(3.2) \quad (\lambda_0 + v) p(0,i) = \lambda_0 p(0,i-1), \quad i \geq 1,$$

$$(3.3) \quad (\lambda + \mu + b) p(1,1) = v p(0,1) + \mu p(1,2) + r p(2,1),$$

$$(3.4) \quad (\lambda + \mu + b) p(1,i) = \lambda p(1,i-1) + v p(0,i) + \mu p(1,i+1) + r p(2,i), \quad i \geq 2,$$

$$(3.5) \quad (\lambda_1 + r) p(2,1) = b p(1,1),$$

$$(3.6) \quad (\lambda_1 + r) p(2,i) = b p(1,i) + \lambda_1 p(2,i-1), \quad i \geq 2.$$

From equation (3.1) we get

$$(3.7) \quad p(1,1) = \frac{\lambda_0}{\mu} p(0,0).$$

From Equation (3.5) and (3.7) we get

$$(3.8) \quad p(2,1) = \frac{b}{\lambda_1 + r} p(1,1) = \frac{b\lambda_0}{\mu(\lambda_1 + r)} p(0,0).$$

From equation (3.2) we get

$$(3.9) \quad p(0, i) = \rho_0^i p(0, 0).$$

### 3. Described the Probability Generating Functions

Probability generating function (pgf) may be used to obtain analytic solution  $P_0(0)$  in neat closed - form expression as there is no way of solving (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6) in a recursive manner. Define the following probability generating function (p g f) of  $P_i(n)$  as follows:

$$F_i(z) = \sum_{n=0}^{\infty} z^n P_i(n) \quad i = 0, 1, 2, \dots$$

where  $|z| \leq 1$ . From Equation (3.9) we get

$$(4.0) \quad F_0(z) = \sum_{i=0}^{\infty} p(0,i) z^i = \frac{p(0,0)}{1 - \rho_0 z}.$$

Multiply (3.4) by  $z^i$  and sum for  $i = 2, 3, \dots$

Then using (3.3) we obtain

$$(4.1) \quad \begin{aligned} (\lambda + \mu + b)[F_1(z) - zp(1,1)] &= \lambda z F_1(z) + \nu[F_0(z) - zp(0,1) - p(0,0)] \\ &+ \frac{\mu}{z}[F_1(z) - zp(1,1) - z^2 p(1,2)] \\ &+ r[F_2(z) - zp(2,1)]. \end{aligned}$$

Since  $F_0(z)$  is given by (4.0), we need to find  $F_2(z)$  in terms of  $F_1(z)$ , then use (4.1) to find  $F_1(z)$ . Multiply (3.6) by  $z^i$  and sum for  $i = 2, \dots$

$$(4.2) \quad (\lambda_1 + r - \lambda_1 z) F_2(z) = b F_1(z) + (\lambda_1 + r) zp(2,1) - bz p(1,1).$$

Using (3.5) we have

$$(4.3) \quad F_2(z) = \frac{b}{\lambda_1 + r - \lambda_1 z} F_1(z).$$

Now substitute (4.3) into (4.1); after some algebraic manipulations we find that

$$(4.4) \quad \frac{(z-1)Q(z)}{z(\lambda_1 + r - \lambda_1 z)} F_1(z) = z[(\lambda + \mu + b)p(1,1) - \nu p(0,1) - \mu p(1,2) - r p(2,1)] \\ - [\nu p(0,0) + \mu p(1,1)] + \frac{\nu p(0,0)}{1 - \rho_0 z}.$$

where

$$(4.5) \quad Q(z) = \lambda \lambda_1 z^2 - (\lambda_1 \mu + \lambda_1 b + \lambda \lambda_1 + \lambda r)z + \mu(r + \lambda_1).$$

In order for the queue length distribution to exist, the right-hand side of (4.4) must vanish when  $z = 1$ . Since  $p(1, 1)$ ,  $p(2, 1)$  and  $p(0, 1)$  are given by (3.7) – (3.9), this allows us to find  $\mu p(1, 2)$  as

$$(4.6) \quad \mu p(1, 2) = \frac{\lambda \lambda_1 + b \lambda_1 + \lambda r}{\lambda_1 + r} p(1, 1) + \lambda_0 \rho_0 p(0, 0).$$

We substitute (4.5) into (4.4). After simplification, we find that the right-hand side of (4.4) reduces to

$$(z-1) = \frac{\lambda_0}{1 - \rho_0 z} p(0, 0) = \frac{(z-1)Q(z)}{z(\lambda_1 + r - \lambda_1 z)} F_1 = \frac{\lambda_0(z-1)}{(1 - \rho_0 z)} p(0, 0)$$

So

$$(4.7) \quad F_1(z) = \frac{\lambda_0 z(\lambda_1 + r - \lambda_1 z)}{Q(z)(1 - \rho_0 z)} p(0, 0).$$

Similar

$$(4.8) \quad F_2(z) = \frac{\lambda_0 z b}{Q(z)(1 - \rho_0 z)} p(0, 0).$$

We now determine the nature of the roots of  $Q(z)$  for positive  $\lambda_1$ . Referring to (4.5),  $Q(z)$  is a quadratic expression whose discriminate,  $\Delta$ , satisfies

$$\Delta \geq \lambda_1^2 b^2 + \lambda_1^2 \mu^2 + \lambda_1^2 \lambda^2 + \lambda^2 r^2 - 2\lambda_1^2 \lambda \mu - 2\lambda_1 \lambda \mu r + 2\lambda_1 \lambda^2 r \\ = \lambda_1^2 b^2 + (\lambda \lambda_1 + \lambda r - \lambda_1 \mu)^2 > 0,$$

So that the equation  $Q(z) = 0$  has two distinct real roots. In order for the steady-state queue length distribution to exist, both roots of the equation

$Q(z) = 0$  must be greater than 1. Since in  $Q(z)$ , the coefficient of  $z^2$  is positive, the two roots of  $Q(z) = 0$  will be greater than 1 if and only if  $Q(1) > 0$  and  $Q'(1) < 0$ . Since  $Q(z) = \mu r - \lambda_1 b - \lambda r$ , we must assume that

$$(4.9) \quad \mu r > \lambda_1 b + \lambda r \quad \text{or} \quad \frac{\lambda_1 b}{r\mu} + \frac{\lambda}{\mu} < 1.$$

Now (4.9) implies that  $\mu > \lambda$ , so if (4.9) holds then

$$Q'(1) = \lambda_1 (\lambda - \mu) - \lambda_1 b - \lambda r < 0.$$

Thus if we assume that (4.9) holds, then the roots  $Z_1$  and  $Z_2$  of  $Q(z) = 0$  will be greater than 1.

Now, here we describe vacation and server breakdown and after we discuss two cases (1) when  $\lambda_1 > 0$  for vacation time and case (2) when the  $\lambda_1 = 0$  for server has breakdown. First we discuss for vacation time and create to mean queue length distribution. Returning to the queue length distribution generating functions, from (4.0), (4.3) and (4.7), we get

$$(5.0) \quad F(z) = \frac{Q(z) + \lambda_0 (\lambda_1 + b + r - \lambda_1 z)z}{Q(z)(1 - \rho_0 z)} p(0, 0).$$

From (5.0) and the normalizing condition  $F(1) = 1$ , we obtain

$$(5.1) \quad p(0, 0) = \frac{(\mu r - \lambda_1 b - \lambda r)(1 - \rho_0)}{\mu r - \lambda_1 b - \lambda r + \lambda_0 (b + r)}.$$

Under the condition (4.9), note that we have  $0 < p(0, 0) < 1$ .

Now assume that  $\lambda_1 > 0$ , and let  $\alpha = 1/Z_1$ ,  $\beta = 1/Z_2$ , where  $Z_1$  and  $Z_2$  are the roots of  $Q(z) = 0$ . Then from

$$(5.2) \quad p(0, 0) = \frac{\mu(r + \lambda_1)(1 - \alpha)(1 - \beta)(1 - \rho_0)}{\mu r - \lambda_1 b - \lambda r + \lambda_0 (b + r)}.$$

Then from (5.0) and (5.2) we obtain the pgf for vacation period

$$(5.3) \quad F(z) = R(z) \frac{(1 - \alpha)(1 - \beta)(1 - \rho_0)}{(1 - \alpha z)(1 - \beta z)(1 - \rho_0 z)},$$

where

$$(5.4) \quad R(z) = \frac{Q(z) + \lambda_0(\lambda_1 + b + r - \lambda_1 z)z}{\mu r - \lambda_1 b - \lambda r + \lambda_0(b+r)}.$$

**Case I:** When,  $\lambda_1 > 0$ , the mean queue length,  $L_q$ , can be found by computing  $F^1(1)$  from (5.3) and (5.4)

$$(5.5) \quad L_q = \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} + \frac{\rho_0}{1-\rho_0} + \frac{\lambda_1(\lambda - \mu - b) + \lambda_0(b+r-\lambda_1) - \lambda r}{\mu r - \lambda_1 b - \lambda r + \lambda_0(b+r)}.$$

Now, we take the normalizing condition  $R(1) = 1$  for server Breakdown. The case in which  $\lambda_1 = 0$  is also of interest. In this case no customers are admitted to the queue during a repair process, if  $\lambda_1 = 0$ , then from (4.5)

$$(5.6) \quad Q(z) = \mu r(1 - \rho z).$$

So  $Q(z) = 0$  has the single root  $1/\rho$ .

Now from (5.0)

$$(5.7) \quad F(z) = \frac{\mu r(1 - \rho z) + \lambda_0(b+r)z}{\mu r(1 - \rho z)(1 - \rho_0 z)} p(0,0).$$

Then

$$(5.8) \quad p(0,0) = \frac{\mu r(1 - \rho)(1 - \rho_0)}{\mu r(1 - \rho) + \lambda_0(b+r)},$$

and

$$(5.9) \quad F(z) = R(z) \frac{(1 - \rho)(1 - \rho_0)}{(1 - \rho z)(1 - \rho_0 z)},$$

where

$$(6.0) \quad R(z) = \frac{\mu r(1 - \rho z) + \lambda_0(b+r)z}{\mu r(1 - \rho) + \lambda_0(b+r)}.$$

Here, we describe the second condition for server breakdown and create queue length distribution.

**Case II:** When,  $\lambda_1 = 0$ , the mean queue length  $L_q$  is found from (5.9) and (6.0)

$$(6.1) \quad Lq = \frac{\rho}{1-\rho} + \frac{\rho_0}{1-\rho_0} + \frac{\lambda_0(b+r) - \mu r \rho}{\mu r(1-\rho) + \lambda_0(b+r)}.$$

Now the complete queue length distribution can be determined from (5.3) when  $\lambda_1 > 0$  and from (5.9) when  $\lambda_1 = 0$ . The condition (4.9) is necessary and sufficient for the queue length distribution to exist.

#### 4. Numerical Illustration

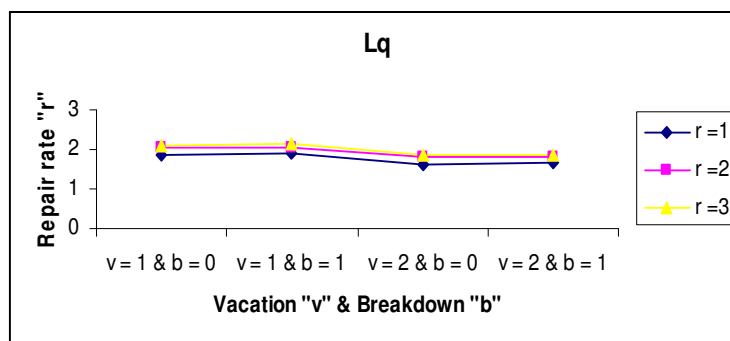
Now we apply  $\lambda_1 = 0$  when the system is breakdowns simply have the effect of suspending the operation of the M/M/1 vacation queue for an exponential period of the time. Equation (4.9) says that the operation of the queue length distribution is independent of the nature of these suspensions.

Note that in both (5.3) and (5.9), the factor  $(1-\rho_0)/(1-\rho_0 z)$  appears in the generating function. The other terms are the generating function of breakdown model without vacations and in which the first customer of a busy period arrives with rate  $\lambda_0$ . thus, we obtain a decomposition theorem for this model: the number, N, of customers in the system is the sum of two independent random variables  $N_1$  and  $N_2$ , where  $N_1$  is the number of customers in the system due to server vacation, and  $N_2$  is the number in the breakdown model without vacations and in which the first customer of a busy period arrives with rate  $\lambda_0$ . The primary objective of this section is to present specific numerical analysis and obtain the queue length with fix and variable parameters. We assume take some fix values as  $\lambda, \lambda_0, \lambda_1, \mu, \alpha$  &  $\beta$  and some variable parameters such as  $r, v$ , &  $b$ .

**Case I:** Vacation ( $\lambda_1 > 0$ ) we obtain mean queue length distribution for different values repair rate ( $r = 1, 2, 3$ ) with vacation rate ( $v = 1, 1, 2, 2$ ) and server breakdown ( $b = 0, 1, 0, 1$ ).

Table (1)

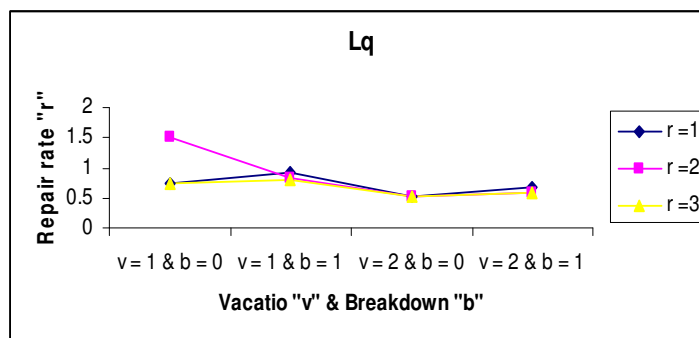
|               | <b>Lq(<math>\lambda_1 &gt; 0</math>)</b> |      |      |
|---------------|------------------------------------------|------|------|
|               | r =1                                     | r =2 | r =3 |
| v = 1 & b = 0 | 1.84                                     | 2.04 | 2.11 |
| v = 1 & b = 1 | 1.89                                     | 2.06 | 2.12 |
| v = 2 & b = 0 | 1.61                                     | 1.8  | 1.87 |
| v = 2 & b = 1 | 1.65                                     | 1.83 | 1.88 |

Fig. 1: Vacation "v" vs Breakdown "b" for  $L_q(\lambda_1 > 0)$ 

**Case II:** Server Breakdown ( $\lambda_1 = 0$ ) we obtain mean queue length distribution for different values repair rate ( $r = 1, 2, 3$ ) with server breakdown ( $b = 0, 1, 0, 1$ ) and vacation rate ( $v = 1, 1, 2, 2$ ).

Table (2)

|           | $L_q(\lambda_1 = 0)$ |      |      |
|-----------|----------------------|------|------|
|           | r=1                  | r=2  | r=3  |
| v=1 & b=0 | 0.75                 | 1.51 | 0.75 |
| v=1 & b=1 | 0.91                 | 0.84 | 0.81 |
| v=2 & b=0 | 0.51                 | 0.51 | 0.51 |
| v=2 & b=1 | 0.67                 | 0.6  | 0.57 |

Fig. 2: Vacation "v" vs Breakdown "b" for  $L_q(\lambda_1 = 0)$ 

## 5. Conclusion

Analysis of M/M/1 queue model with multiple vacations and server breakdown is the specific queuing model where the server takes a vacation

as soon as it becomes idle. The vacation ends, if there are customers arrive in the queue, the server activates; otherwise, another vacation is taken. Once service begins, the service mechanism is subject to breakdowns which occur at a constant rate. The service mechanism then goes through a repair process of random duration, and when repair is completed, the server returns to the customers whose services were interrupted. Here we obtain the probability generating function for vacation period as given below:

$$F(z) = R(z) \frac{(1-\alpha)(1-\beta)(1-\rho_0)}{(1-\alpha z)(1-\beta z)(1-\rho_0 z)},$$

where

The mean queue length,  $L_q$ , for  $\lambda_1 > 0$ , is computed as given below,

$$L_q = \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} + \frac{\rho_0}{1-\rho_0} + \frac{\lambda_1(\lambda - \mu - b) + \lambda_0(b + r - \lambda_1) - \lambda r}{\mu r - \lambda_1 b - \lambda r + \lambda_0(b + r)}.$$

The mean queue length,  $L_q$ , for  $\lambda_1 = 0$ , is computed as given below,

$$L_q = \frac{\rho}{1-\rho} + \frac{\rho_0}{1-\rho_0} + \frac{\lambda_0(b + r) - \mu r \rho}{\mu r(1-\rho) + \lambda_0(b + r)}.$$

A comparison between server vacation and break down for different values  $r, v, b$ , has been given in the following table-3.

Table 3. Comparison of Vacation Period vs Server Breakdown

|                          | <b>Lq(<math>\lambda_1 &gt; 0</math>)</b> |              |              | <b>Lq(<math>\lambda_1 = 0</math>)</b> |              |              |
|--------------------------|------------------------------------------|--------------|--------------|---------------------------------------|--------------|--------------|
|                          | <b>r = 1</b>                             | <b>r = 2</b> | <b>r = 3</b> | <b>r = 1</b>                          | <b>r = 2</b> | <b>r = 3</b> |
| <b>v = 1 &amp; b = 0</b> | 1.84                                     | 2.04         | 2.11         | 0.75                                  | 1.51         | 0.75         |
| <b>v = 1 &amp; b = 1</b> | 1.89                                     | 2.06         | 2.12         | 0.91                                  | 0.84         | 0.81         |
| <b>v = 2 &amp; b = 0</b> | 1.61                                     | 1.80         | 1.87         | 0.51                                  | 0.51         | 0.51         |
| <b>v = 2 &amp; b = 1</b> | 1.65                                     | 1.83         | 1.88         | 0.67                                  | 0.60         | 0.57         |

The graphical representation of the above data is given below:

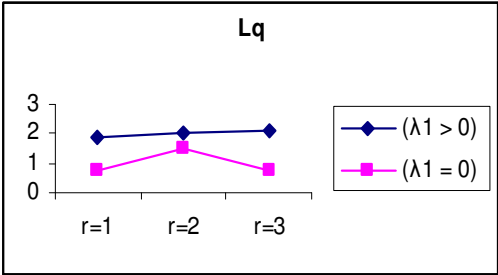


Fig. 3.1

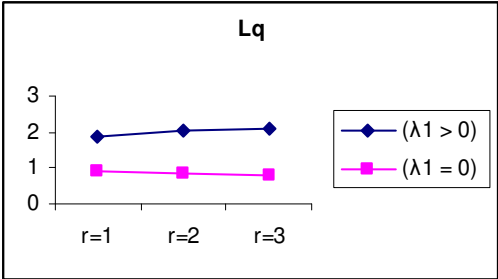


Fig. 3.2

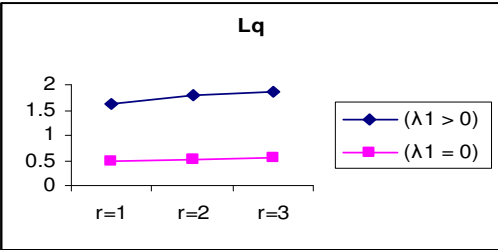


Fig. 3.3

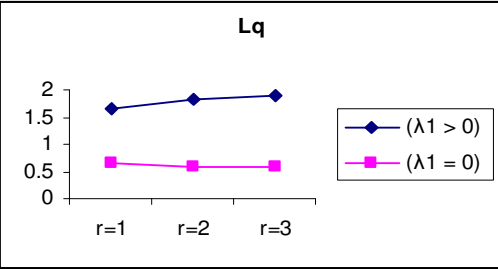


Fig. 3.4

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