

## **A Mathematical Model on Effect of Stenosis in Two Phase Blood Flow in Arteries Remote from the Heart**

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**Abstract:** Keeping in view the nature of human circulatory system, a mathematical model of two phase blood flow in arteries remote from the heart has been presented herein. The previous researchers assumed the blood flow to be single phased. The Campbell-Pitcher two phase model has been transformed into biofluid mechanical set up. For the purpose, blood has been assumed to be constituted by plasma and blood cells which is realistic so far. The pressure drop along the length of stenosis has been calculated. The role of Hematocrit is explicit in determination of blood pressure drop.

### **1. Introduction**

The circulatory system is a blood-filled dynamic stem that permits the community of cells of specific tissues to perform their individual and interconnected roles and provides a stable environmental medium for cells of body. The circulatory system is made of three major components – the heart, the blood vessels and blood – a tissue that flows. The muscular heart pumps the blood into aorta. The aorta branches into arteries which further branch into smaller and smaller arteries. The arteries are tubes with relatively thick walls containing elastic and muscle tissues<sup>1</sup>. The present model is confined on these arteries remote from heart. The pattern of blood flow in the arteries remote from heart may be considered to be steady laminar due to negligible effect of pumping of heart. L. Debnath<sup>2</sup> and R.

Girija Devi and R. Devnath<sup>3</sup> have considered the blood as a single phase. The blood is supposed to be constituted by two phases – the first one is plasma while the other one is that of blood cells which are semi-permeable packets of a liquid of higher density than that of plasma in the studies of V. Upadhyay and P.N. Pandey<sup>4</sup>. Whenever the percentage of red blood cells which constitute major part of blood cells in blood i.e. Hematocrit (H) increases, the effective viscosity begins to depend upon the strain rate tensor. In other words there is a non-linear relation between the stress tensor and strain rate tensor. IN this way, the blood may be treated as non-Newtonian<sup>5</sup>. If the strain rate varies from 5 to 200 per second, then the power law  $\tau^{ij} = \eta_m (e^{ij})^n$ , where  $0.68 \leq n \leq 0.80$  holds good for blood flow. In this situation, the constitutive equation of blood flow<sup>6</sup> may be taken as

$$(1) \quad \tau^{ij} = p g^{ij} + \eta_m (e^{ij})^n$$

where  $\tau^{ij}$  is stress tensor and  $\tau^{ij}$  is shearing stress tensor while  $e^{ij}$  is strain rate tensor,  $g^{ij}$  is conjugate metric tensor and  $\eta_m$  is viscosity of blood which is composed of a homogeneous mixture of plasma and blood cells.

Stenosed



Remote Arterial Stenosis

Sometimes the narrowing of arteries due to the development of arteriosclerotic plaques or other types abnormal tissue developments which is called as stensosis in medical term takes place on vessel walls. As the growth is projected into the lumen (cavity) of the arteries, the blood flow is obstructed upto some considerable extent<sup>7</sup>. That's why fluctuation in blood pressure is recorded along the length of the stenosis. The first phase of stenosis has been taken into consideration in the present model.

## 2. The Mathematical Model (The formulation of the problem)

The human circulatory system behaves just like a closed circuit where there is neither source nor sink. The heart behaves merely like a pumping

station. If a given volume is pumped from the heart, then the same will return to the heart after passing through the subsequent subdivisions of blood vessels e.g. aorta, arteries, arterioles, capillaries, venules, veins and vena-cava. Thus, the law of conservation of mass of blood can be applied in biofluid mechanical setup. As a result of this law the equations of continuity can be written as follows:

$$(2) \quad \frac{\partial [(1-X)\rho_p]}{\partial t} + [(1-X)\rho_p v^i]_{,i} = 0.$$

and

$$(3) \quad \frac{\partial (X\rho_c)}{\partial t} + (X\rho_c v^i)_{,i} = 0.$$

for the plasma phase and phase of blood cells respectively, where  $(1-X)$  is the volume portion covered by plasma in unit volume of blood while  $X$  is the volume portion covered by blood cells. As the red cells cover most of the volume fractions of blood cells since white cells and platelets are very small in comparison to red cells;  $X$  may be expressed as  $H/100$  where  $H$  is Hematocrit.  $\rho_p$  and  $\rho_c$  stand for densities of plasma phase and that of blood cells respectively.

Again, as no external force has been imposed between the two phases of blood, the principle of conservation of momentum holds good in the biofluid mechanical set up. As a consequence the equations of motion can be expressed as follows:

$$(4) \quad (1-X)_{\rho_p} \frac{\partial v^i}{\partial t} + (1-X)\rho_p v^j v_{,j}^i = -(1-X)p_{,j} g^{ij} + (1-X)\eta_p (e^{ij})_{,ij}^n$$

and

$$(5) \quad X_{\rho_c} \frac{\partial v^i}{\partial t} + X\rho_c v^j v_{,j}^i = -Xp_{,j} g^{ij} + X\eta_c (e^{ij})_{,ij}^n,$$

for the plasma phase and the phase of blood cells respectively, where<sup>8</sup>  $e^{ij} = g^{jk} v_{,k}^i$ , while  $\eta_p$  and  $\eta_c$  stand for viscosities for plasma phase and the phase of blood cells respectively.  $p_{,j}$  stands for covariant derivative of blood

pressure. Note that  $v^i$  is the common velocity of the two phases. The mass ratio  $\frac{X_{\rho_c}}{(1-X)_{\rho_p}} = \alpha$  (say), of the two phases is usually not constant, but in present context, it is supposed to be constant which is realistic so far as blood flow in arteries remote from heart is concerned.

Now if uniform density  $\rho_m = X\rho_c + (1-X)\rho_p$  of blood considered as homogeneous mixture of the two phases is taken into account, then the equations (2) and (3) can be combined into a single equation as follows:

$$\frac{\partial(\rho_m)}{\partial t} + (\rho_m v^i)_{,i} = 0,$$

which can be transformed into the following form:

$$(6) \quad \frac{1}{\sqrt{g}} + (\sqrt{g} v^i)_{,i} = 0.$$

where  $g$  is the determinant of metric tensor  $g_{ij}$ , since the blood may be treated as incompressible fluid.

The equations (4) and (5) can be combined as

$$(7) \quad \rho_m \frac{\partial v^i}{\partial t} + \rho_m v^j v_{,j}^i = -p_{,i} g^{ij} + \eta_m (g^{jk} v_{,k}^i)_{,j},$$

where  $\eta_m = X\eta_c + (1-X)\eta_p$  stands for viscosity coefficient for the homogeneous mixture of the two phases of blood.

Further the mild stenosis can be considered to be axially symmetric whose equation may be taken as follows:

$$(8) \quad R(z) = R_0 \left[ 1 - \frac{\delta}{2R_0} \left( 1 + \cos \frac{\pi z}{z_0} \right) \right]$$

where  $-z_0 \leq z \leq z_0$ ,  $\frac{\delta}{R_0} \ll 1$  and  $R_e \frac{\delta}{z_0} \ll 1$ , where  $R_e$  is Reynold's

number of blood flow.  $Z_0$  is the half of the length of the stenosis and  $\delta$  is the amplitude while  $R_0$  is radius of cylindrical arteries remote from heart (Fig.1).

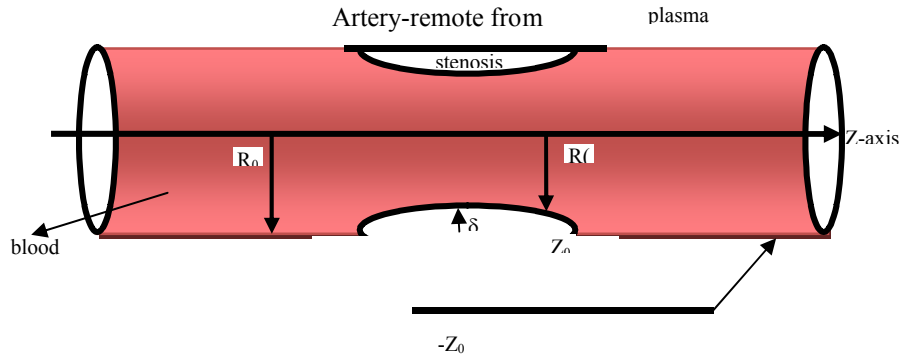


Fig. 1: The mild stenosis

### 3. Analysis

The blood vessels of arteries may be treated as cylindrical though they are not exactly so. The needed transformation from tensorial form to cylindrical co-ordinate system may be effected easily as follows:

The cylindrical polar coordinates are

$$x^1 = r, x^2 = \theta, x^3 = z.$$

The value of determinant of metric tensor is

$$g = |g_{ij}| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = r^2$$

while the matrix of conjugate metric tensor is

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Christoffel symbols of 2<sup>nd</sup> kind are

$$\left\{ \begin{matrix} 1 \\ 2 \end{matrix} \right\} = -r \quad \text{and} \quad \left\{ \begin{matrix} 2 \\ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} = \frac{1}{r}$$

while rest are zero.

Relations between physical components and contravariant components of various parameters utilized in the model are

$$(\sqrt{g_{11}}v^1 = v_r) \Rightarrow (v_r = v^1), (\sqrt{g_{22}}v^2 = v_\theta) \Rightarrow (v_\theta = rv^2) \\ (\sqrt{g_{33}}v^3 = v_z) \Rightarrow (v_z = v^3) \text{ and } -p_{,j}g^{ij} \Rightarrow -\sqrt{g_{ii}}p_{,j}g^{ij}$$

The matrix of physical components of shearing stress tensor  $\tau'^{ij} = \eta_m (e^{ij})^n = \eta_m (g^{ij}v_k^i)^n$  is

$$\begin{bmatrix} 0 & 0 & \eta_m \left( \frac{\partial v}{\partial r} \right)^n \\ 0 & 0 & 0 \\ \eta_m \left( \frac{\partial v}{\partial r} \right)^n & 0 & 0 \end{bmatrix},$$

while the covariant derivative of  $\tau'^{ij}$  is given by

$$\tau'_{,j}{}^{ij} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g} \tau'^{ij}) + \left\{ \begin{matrix} i \\ j \end{matrix} \right\}^k.$$

Keeping in view the above facts, the governing tensorial equations can be transformed into cylindrical form as follows:  
the equation of continuity (6) reduces to

$$(9) \quad \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{\partial v_z}{\partial z} = 0.$$

By carrying out an order of magnitude analysis on equation (9), it can be observed that the radial velocity  $v_r$  can be neglected in relation to axial velocity  $v_z$  which in turn will be  $v$ , say. Note that even at high Reynold's number blood flow in the arteries remote from heart is irrotational/laminar.

The equation of motion (7) reduces to:

$$\begin{aligned} \frac{\partial p}{\partial r} &= 0 \\ (10) \quad \Rightarrow p &= p(z) \end{aligned}$$

and

$$\begin{aligned}
 & -\frac{\partial p}{\partial z} + \eta_m \frac{1}{r} \frac{\partial}{\partial r} \left[ r \left( \frac{\partial v}{\partial r} \right)^n \right] = 0. \\
 (11) \quad & \Rightarrow \eta_m \frac{1}{r} \frac{\partial}{\partial r} \left[ r \left( \frac{\partial v}{\partial r} \right)^n \right] = -P(z)
 \end{aligned}$$

where pressure gradient  $\frac{\partial p}{\partial z} = -P(z)$

Note that the flow has been chosen steady.

Integrating equation (11), we obtain

$$(12) \quad r \left( \frac{\partial v}{\partial r} \right)^n = -\frac{p(z)r^2}{2\eta_m} + A(z) \quad .$$

Applying the boundary condition  $\frac{\partial v}{\partial r} = 0$  at  $r=0$ , we obtain the value of arbitrary constant  $A(z) = 0$ . Thus equation (12) takes the following form:

$$(13) \quad r \left( \frac{\partial v}{\partial r} \right)^n = -\frac{p(z)r^2}{2\eta_m}$$

Again, integrating equation (13), we get:

$$(14) \quad v = -\left( \frac{p(z)}{2\eta_m} \right)^{\frac{1}{n}} \frac{n}{n+1} r^{\frac{1}{n}+1} + B(z) \quad .$$

Applying the boundary condition i.e. the no slip condition on the stenosis surface of the artery  $v=0$  at  $r = R(z)$  and  $v = 0$  at  $r = R_0$ ,  $-z_0 \leq z \leq z_0$ , we

obtain the value of arbitrary constant  $B(z) = \left( \frac{p(z)}{2\eta_m} \right)^{\frac{1}{n}} \frac{n}{n+1} R(z)^{\frac{1}{n}+1}$ .

Thus the equation (14) takes the following form:

$$(15) \quad v = \frac{n}{n+1} \left( \frac{p(z)}{2\eta_m} \right)^{\frac{1}{n}} (R(z)^{\frac{1}{n}+1} - r^{\frac{1}{n}+1})$$

which is the formula for the velocity of blood flow passing through the stenosis.

#### 4. Result

The flow flux (volume/time) / (ml/second) of blood in arteries remote from heart across the length of the stenosis is obtained as follows:

$$Q = \int_0^{R(z)} v \cdot 2\pi r \, dr.$$

or

$$(16) \quad Q = \int_0^{R(z)} \frac{n}{n+1} \left( \frac{p(z)}{2\eta_m} \right)^{\frac{1}{n}} (R(z)^{\frac{1}{n+1}} - r^{\frac{1}{n+1}}) 2\pi r \, dr = \frac{n\pi}{3n+1} \left( \frac{p(z)}{2\eta_m} \right)^{\frac{1}{n}} R(z)^{\frac{1}{n}+3},$$

where  $R(z)$  is given by the formula (8). If flow flux is measured first by another method, then pressure gradient (Newton per  $\text{ml}^3/\text{Pascal}$  per meter) can be evaluated with the help of the above formula as follows:

$$(17) \quad P(z) = -\frac{\partial p}{\partial z} = \left( \frac{3n+1}{n\pi} \right)^n (2\eta_m Q^n) \frac{1}{R(z)^{3n+1}}$$

#### 5. Discussion

The blood pressure drop along the length of the stenosis with the help of the equation (17) can be calculated as follows:

$$\begin{aligned} \Delta P &= (2\eta_m Q^n) \left( \frac{3n+1}{n\pi} \right)^n \int_{-z_0}^{z_0} \frac{dz}{R(z)^{3n+1}} \\ &= \frac{2\eta_m Q^n}{R_0^{3n+1}} \left( \frac{3n+1}{n\pi} \right)^n \int_{-z_0}^{z_0} \frac{dz}{\left[ 1 - \frac{\delta}{2R_0} \left( 1 + \cos \frac{\pi z}{z_0} \right) \right]^{3n+1}} \\ (18) \quad &= \frac{4\eta_m Q^n}{\pi R_0^{3n+1}} \left( \frac{3n+1}{n\pi} \right)^n \int_0^\pi \frac{du}{[a_1 - a_2 \cos u]^{3n+1}}, \end{aligned}$$

where  $a_1 = 1 - \frac{\delta}{2R_0}$ ,  $a_2 = \frac{\delta}{2R_0}$  and  $u = \frac{\pi z}{z_0}$

When there is no stenosis i.e.  $\delta=0$ , the pressure drop will be as follows:

$$(19) \quad (\Delta p)_p = \frac{4\eta_m Q^n}{R_0^{3n+1}} \left( \frac{3n+1}{n\pi} \right)^n,$$

where subscript P stands for Poisseuille flow.

Now dividing equation (18) by (19), we get the ratio of pressure drop (a dimensionless quality)

$$(20) \quad \frac{\Delta p}{(\Delta p)_p} = \frac{1}{\pi} \int_0^\pi \frac{du}{[a_1 - a_2 \cos u]^{3n+1}}.$$

To sketch a graph, choose  $n = 2/3$ , which lies between 0.68 and 0.80 for power law flow of blood; equation (20) yields

$$(21) \quad \frac{\Delta p}{(\Delta p)_p} = \frac{1}{\pi} \int_0^\pi \frac{du}{[a_1 - a_2 \cos u]^{3n+1}} = \left(1 - \frac{\delta}{R_0} + \frac{3\delta^2}{8R_0^2}\right) \left(1 - \frac{\delta}{R_0}\right)^{-\frac{5}{2}}.$$

Differentiating twice partially with respect to  $a_1$ ,

$$\int_0^\pi \frac{du}{[a_1 - a_2 \cos u]} = \pi (a_1^2 - a_2^2)^{-\frac{1}{2}} = \pi \left(1 - \frac{\delta}{R_0}\right)^{-\frac{1}{2}}.$$

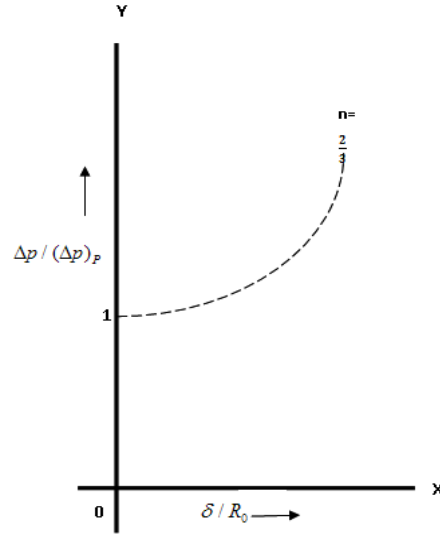


Fig. 2: Ratio of pressure drop versus  $\frac{\delta}{R_0}$  i.e. pressure drop along the length of stenosis

**Remark:** Our choice of  $n$  is the proposed one. If the selection of  $n$  is based on practical observation table for stress and strain rate tensor, then the

exact value of  $n$  can be decided. Then the graph will focus the realistic scenario.

We observe that the value of  $\frac{\Delta p}{(\Delta p)_p}$  increases slowly as  $\frac{\delta}{R_0}$  (a dimensionless quantity) increases as shown in (Fig.2)

## 6. Conclusion

The two phase model of blood flow in the arteries remote from heart across the length of stenosis is useful in the following ways:

1. The blood pressure drop along the length of stenosis can be easily calculated.
2. The role of Hematocrit  $H$  in determination of flow flux or in turn blood pressure drop is explicit in the sense that  $\eta_m = X\eta_c + (1-X)\eta_p$ , where

$$X = \frac{H}{100}.$$

3. In several diseases e.g. intermittent claudication, angina, carotid artery stenosis and heart valves stenosis etc, the red blood cells count is highly increased or decreased and thus Hematocrit is changed sharply. The drop in blood pressure can easily be calculated by the formula presented in the model.

Hence we conclude that the two phase model of blood flow in the arteries remote from heart along the length of stenosis is applicable from medical point of view.

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