

Tripled Coincidence Point Theorem for Compatible Maps in Partially Ordered Fuzzy Metric Spaces

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Abstract: In this paper, we introduce the concept of compatible mapping for trivariate self mapping F and self mapping g having mixed g -monotone property in GV-fuzzy metric space and establish tripled coincidence point theorem for such nonlinear contractive mappings. More to the point together with several recent developments, our result fuzzifies and extends recent tripled fixed point theorem by Borcut and Berinde [Borcut M, Berinde V. Tripled fixed point theorem for contractive type mappings in partially ordered metric spaces. *Nonlinear Anal.* 75(2011) 4889-4897]. Finally, we construct an example to support our result.

Keywords and Phrases: Fuzzy metric, Fixed (coincidence) point theorem, Nonlinear contraction, Partially ordered set, Tripled fixed (coincidence) point, mixed g -monotone.

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1. Introduction and Preliminaries

Fuzzy metric space is an important notion in fuzzy analysis. The concept of fuzzy metric space was introduced by different authors (see^{1,2,3}) in different ways and further using these different concepts various authors⁴⁻¹³ proved theorem which assures the existence of fixed point. Here, we use notion of fuzzy metric space established by George and Veeramani¹.

Specifically, Bhaskar and Lakshmikanthan¹⁵ established coupled fixed point for mixed monotone operator in partially ordered metric spaces. Afterward, Lakshmikanthan and Ćirić¹⁶ extended the results¹⁵ by furnishing coupled coincidence and coupled fixed point theorem for two commuting mappings having mixed g -monotone property. In a subsequent series, B. S. Choudhary and A. Kundu¹⁷ introduced the concept of compatibility and proved the result¹⁶ under different set of condition. Very recently Borcut

and Berinde¹⁸ introduce tripled fixed point theorem for contractive type mapping in partially ordered metric spaces.

The main reason of our being interested in such type of spaces is that of important applications in engineering problems, such as for access history prediction, to achieve accesses optimization in information systems, in quantum particle physics particularly in concern with both string and E-infinity theory which were given and studied by El-Naschie¹⁹⁻²⁴. It seems that E-infinity may be first journal theory which can calculate electromagnetic fine structure constant from first principle, rather than just except it as an experimental quantity, determine in laboratory.

Recently, fuzzy metrics have been applied to color image filtering, improving some filters when replacing some classical metrics²⁵⁻²⁷. Also fixed point theorems play central role in proof of existence of general equilibrium in market economics as developed in the. 1950's by economics noble prize winner Gerard Debreu and Keneth Arrow, a financial equilibria and incomplete market. Equilibrium price is fixed point in stable market

Motivated by Bhaskar and Lakshmikantham¹⁵ and Borcut and Berinde¹⁸, the purpose of present study is to investigate tripled coincidence point theorem for mappings that possess monotonicity type properties, in the context of GV-fuzzy metric space which combine method of contraction principle with method of monotone iterations.

In what follows, we collect some relevant definitions, results, examples for our further use.

Definition²⁸ 1.1. A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition 1.2 A continuous t-norm (in sense of Schweizer and Sklar¹² is a binary operation T on $[0, 1]$ satisfying the following conditions:

- (i) T is a commutative and associative;
- (ii) $T(a, 1) = a$ for all $a \in [0, 1]$;
- (iii) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$, $(a, b, c, d \in [0, 1])$
- (iv) The mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous.

Remark 1.3 The following are classical example of continuous t-norm

- (i) $T_M(a, b) = \min \{a, b\}$, minimum t-norm.

- (ii) $T_H(a, b) = \begin{cases} 0 & \text{if } a = b = 0, \\ \frac{ab}{a + b + ab} & \text{otherwise,} \end{cases}$ Hamacher product.
- (iii) $T_P(a, b) = ab$, product t-norm.
- (iv) $T_N(a, b) = \begin{cases} \min\{a, b\} & \text{if } a + b > 1, \\ 0 & \text{otherwise,} \end{cases}$ Nilpotent minimum.
- (v) $T_L(a, b) = \max\{a + b - 1, 0\}$, Lukasiewicz t-norm.
- (vi) $T_D(a, b) = \begin{cases} b & \text{if } a = 1, \\ a & \text{if } b = 1, \\ 0 & \text{otherwise,} \end{cases}$ Drastic t-norm.

The minimum t-norm is point wise largest t-norm and the drastic t-norm is point wise smallest t-norm; that is, $T_M(a, b) \geq T(a, b) \geq T_D(a, b)$ for any t-norm t with $a, b \in [0, 1]$.

Kramosil and Michalek² generalized the concept of probabilistic metric space given by Menger to the fuzzy framework as follows.

Definition 1.4. A fuzzy metric space (in sense of Kramosil and Michalek²) is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ such that the following axioms holds:

$$(FM-1) \quad M(x, y, 0) = 0 \quad (x, y \in X);$$

$$(FM-2) \quad M(x, y, t) = 1, \text{ for all } t > 0 \text{ iff } x = y;$$

$$(FM-3) \quad M(x, y, t) = M(y, x, t) \quad (x, y \in X, t > 0);$$

$$(FM-4) \quad M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1] \text{ is left continuous for all } x, y \in X;$$

$$(FM-5) \quad M(x, z, t + s) \geq M(x, y, t) * M(y, z, s) \text{ for all } x, y, z \in X \text{ and } s, t > 0.$$

We will refer to these spaces as KM-fuzzy metric spaces.

Lemma⁷ 1.5. For every $x, y \in X$, the mapping $M(x, y, \cdot)$ is non-decreasing on $(0, \infty)$.

In order to introduce an Hausdorff topology on the fuzzy metric spaces, George and Veeramani¹ modified in a slight but appealing way the notion of fuzzy metric spaces of Kramosil and Michalek.

Definition 1.6. A fuzzy metric space (in sense of George and Veeramani¹) is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ such that the following axioms holds:

$$(GV-1) \quad M(x, y, t) > 0 \quad (x, y \in X);$$

$$(GV-2) \quad M(x, y, t) = 1, \text{ for all } t > 0 \text{ iff } x = y;$$

$$(GV-3) \quad M(x, y, t) = M(y, x, t) \quad (x, y \in X, t > 0);$$

$$(GV-4) \quad M(x, y, \cdot): (0, \infty) \rightarrow (0, 1] \text{ is continuous for all } x, y \in X;$$

$$(GV-5) \quad M(x, z, t + s) \geq M(x, y, t) * M(y, z, s) \text{ for all } x, y, z \in X \text{ and } s, t > 0.$$

We will refer to these spaces as GV-fuzzy metric spaces.

Notice that condition (GV-5) is a fuzzy version of triangular inequality. The value $M(x, y, t)$ can be thought of as degree of nearness between x and y with respect to t and from axiom (GV-2) we can relate the value 0 and 1 of a fuzzy metric to the notions of ∞ and 0 of classical metric respectively.

Definition⁵ 1.7. A fuzzy metric M on X is said to be stationary if M does not depend on t , i.e., the function $M_{x,y}(t) = M(x, y, t)$ is constant.

We can fuzzify example of metric space into fuzzy metric spaces in a normal way:

Example 1.8. Let (X, d) be metric space and $g: R^+ \rightarrow R^+$ is an increasing continuous function, for $m > 0$, we define the function M by

$$(1.1) \quad M(x, y, t) = \frac{g(t)}{g(t) + m \cdot d(x, y)}.$$

Then for $a * b = a \cdot b$, $(X, M, *)$ is a GV-fuzzy metric space on X .

As a particular case if we take $g(t) = t^n$ where $n \in N$ and $m = 1$. Then (1.1) becomes

$$(1.2) \quad M(x, y, t) = \frac{t^n}{t^n + d(x, y)}.$$

Then for $a * b = T_M(a, b)$, $(X, M, *)$ is a GV-fuzzy metric space on X .

If we take $n = 1$, in (1.2), the well-known fuzzy metric space is obtained.

On the other hand, if we take g as a constant function in (1.1) i.e., $g(t) = k > 0$ and $m = 1$, we obtain

$$M(x, y, t) = \frac{k}{k + d(x, y)}$$

and so $(X, M, *)$ is a stationary GV-fuzzy metric space for $a * b = a \cdot b$ but, in general, (X, M, T_M) is not.

Definition 1.9. Let $(X, M, *)$ be a (KM or GV) fuzzy metric space. A sequence $\{x_n\}$ is said to be

(i) convergent to a point $x \in X$ (denoted by $\lim_{n \rightarrow \infty} x_n = x$), if

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = 1, \text{ for all } t > 0.$$

(ii) a Cauchy sequence if

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1, \text{ for all } t > 0 \text{ and } n, p \in N.$$

(iii) a complete if every Cauchy sequence in X is convergent to some point in X .

A map $f : X \rightarrow X$ is called continuous at x_0 if $\{f(x_n)\}$ converges to $f(x_0)$ for each $\{x_n\}$ converges to x_0 .

In 1994, Mishra et. Al⁹ introduced the concept of compatible mapping in FM-space akin to concept of compatible mapping in metric space as follows:

Definition 1.10. Let f and g be mapping from a fuzzy metric space $(X, M, *)$ into itself. A pair of maps $\{f, g\}$ is said to be compatible if $\lim_{n \rightarrow \infty} M(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.

Definition¹⁵ 1.11. An element $(x, y) \in X \times X$, is called a coupled fixed point of mapping $F: X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition¹⁸ 1.12. An element $(x, y) \in X \times X$, is called a coupled coincident point of mapping $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Definition¹⁷ 1.13. The mappings F and g where $F: X \times X \rightarrow X$ and $g: X \rightarrow X$, are said to be compatible if

$$\lim_{n \rightarrow \infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0 \quad \text{and}$$

$$\lim_{n \rightarrow \infty} (g(F(y_n, x_n)), F(gy_n, gx_n)) = 0.$$

Whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X , such that $\lim_{n \rightarrow \infty} F(x_n, y_n) = \lim_{n \rightarrow \infty} gx_n = x$ and $\lim_{n \rightarrow \infty} F(y_n, x_n) = gy_n = y$, for all $x, y \in X$ are satisfied.

2. Main Results

Let $(X, \leq)$ be a partially ordered set and M be a fuzzy metric on X such that $(X, M, *)$ is a complete fuzzy metric space. Consider on the product $X \times X \times X$ the following partial order: for $(x, y, z), (u, v, w) \in X \times X \times X$,

$$(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w.$$

Definition¹⁵ 2.1. Let $(X, \leq)$ be a partially ordered set and $F: X \times X \times X \rightarrow X$. We say that F has the mixed monotone property if $F(x, y, z)$ is monotone non decreasing in x and z , and is monotone non increasing in y , that is, for any $x, y, z \in X$,

$$\begin{aligned} x_1, x_2 \in X, x_1 \leq x_2 &\Rightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, y_1 \leq y_2 &\Rightarrow F(x, y_1, z) \geq F(x, y_2, z), \end{aligned}$$

$$\text{and } z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Definition¹⁶ 2.2. Let $(X, \leq)$ be a partially ordered set and $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$. We say that F has the g-mixed monotone property if $F(x, y, z)$ is monotone non decreasing in x and z , and is monotone non increasing in y , that is, for any $x, y, z \in X$,

$$\begin{aligned} x_1, x_2 \in X, g(x_1) \leq g(x_2) &\Rightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, g(y_1) \leq g(y_2) &\Rightarrow F(x, y_1, z) \geq F(x, y_2, z), \end{aligned}$$

$$\text{and } z_1, z_2 \in X, g(z_1) \leq g(z_2) \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Definition¹⁸ 2.3. An element $(x, y, z) \in X$ is called tripled fixed point of $F: X \times X \times X \rightarrow X$ if $F(x, y, z) = x$, $F(y, x, y) = y$, and $F(z, y, x) = z$.

Definition 2.4 An element $(x, y, z) \in X$ is called tripled coincidence fixed point of $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ if

$$F(x, y, z) = gx, F(y, x, y) = gy, \text{ and } F(z, y, x) = gz.$$

Now, we introduce the notion of compatibility of mappings for g-mixed monotone mapping in fuzzy metric spaces.

Definition 2.5 The mapping F and g where $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ are said to be compatible if for all $t > 0$

$$\lim_{n \rightarrow \infty} M(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n), t) = 1$$

$$\lim_{n \rightarrow \infty} M(g(F(y_n, x_n, y_n)), F(gy_n, gx_n, gy_n), t) = 1, \text{ and}$$

$$\lim_{n \rightarrow \infty} M(g(F(z_n, y_n, x_n)), F(gz_n, gy_n, gx_n), t) = 1.$$

whenever $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are sequences in X , such that

$$\lim_{n \rightarrow \infty} F(x_n, y_n, z_n) = gx_n = x, F(y_n, x_n, z_n) = gy_n = y$$

and $F(z_n, y_n, x_n) = gz_n = z$ for all $x, y, z \in X$ are satisfied.

Now we are ready to establish the main result:

Theorem 2.6. Let $(X, \leq)$ be partially ordered set and let there be a fuzzy metric M on X such that $(X, M, *)$ be a complete fuzzy metric space. Let $\varphi: [0, +\infty) \rightarrow [0, +\infty)$ be such that $\varphi(t) > t$ and $\lim_{r \rightarrow t^+} \varphi(r) > t$ for all $t > 0$. Also let $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ are such that F has mixed g -monotone property and satisfy

$$(2.1) \quad M(F(x, y, z), F(u, v, w), t) \geq \varphi(M(gx, gu, t) * M(gy, gv, t) * M(gz, gw, t)),$$

for all $x, y, z, u, v, w \in X$ and $t > 0$, with $gx \leq gu$, $gy \geq gv$ and $gz \leq gw$. Let $F(X \times X \times X) \subseteq g(X)$, g be continuous and monotone increasing and F and g be compatible mappings. Also suppose

(a) F is continuous or

(b) X has following properties

(i) if a non-decreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$ for all $n \geq 0$,

(ii) if a non-increasing sequence $\{y_n\} \rightarrow y$, then $y_n \geq y$ for all $n \geq 0$,

(iii) if a non-decreasing sequence $\{z_n\} \rightarrow z$, then $z_n \leq z$ for all $n \geq 0$.

If there exist $x_0, y_0, z_0 \in X$, such that $g(x_0) \leq F(x_0, y_0, z_0)$, $g(y_0) \geq F(y_0, x_0, z_0)$ and $g(z_0) \leq F(z_0, y_0, x_0)$, there exist $x, y, z \in X$ such that $g(x) = F(x, y, z)$, $g(y) = F(y, x, z)$ and $g(z) = F(z, y, x)$, that is, F and g have tripled coincidence point in X .

Proof: Let x_0, y_0, z_0 be such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, z_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$. Since $F(X \times X \times X) \subseteq g(X)$, we can choose $x_1, y_1, z_1 \in X$ such that $gx_1 = F(x_0, y_0, z_0)$, $gy_1 = F(y_0, x_0, z_0)$ and $gz_1 = F(z_0, y_0, x_0)$. Again we can choose $x_2, y_2, z_2 \in X$ such that $gx_2 = F(x_1, y_1, z_1)$,

$gy_2 = F(y_1, x_1, y_1)$ and $gz_2 = F(z_1, y_1, x_1)$. Continuing like this we can construct three sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ such that for all $n \geq 0$.

$$(2.2) \quad gx_n = F(x_{n-1}, y_{n-1}, z_{n-1}), \quad gy_n = F(y_{n-1}, x_{n-1}, y_{n-1})$$

$$\text{and } gz_n = F(z_{n-1}, y_{n-1}, x_{n-1})$$

We shall show that

$$(2.3) \quad gx_n \leq gx_{n+1}, gy_n \geq gy_{n+1} \text{ and } gz_n \leq gz_{n+1}$$

Since $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$ and $gx_1 = F(x_0, y_0, z_0)$, $gy_1 = F(y_0, x_0, y_0)$ and $gz_1 = F(z_0, y_0, x_0)$,

we have $gx_0 \leq gx_1$, $gy_0 \geq gy_1$ and $gz_0 \leq gz_1$; that is, (2.3) hold for $n = 0$.

We presume that (2.3) hold for some $n > 0$. As F has mixed g -monotone property and $gx_n \leq gx_{n+1}$, $gy_n \geq gy_{n+1}$ and $gz_n \leq gz_{n+1}$, from (2.2), we have

$$(2.4) \quad \begin{aligned} gx_{n+1} &= F(x_n, y_n, z_n) \leq F(x_{n+1}, y_n, z_n), \\ gy_{n+1} &= F(y_n, x_n, y_n) \geq F(y_{n+1}, x_n, y_n) \end{aligned}$$

$$\text{and } gz_{n+1} = F(z_n, y_n, x_n) \leq F(z_{n+1}, y_n, x_n).$$

Also for the same reason, we have

$$(2.5) \quad \begin{aligned} F(x_{n+1}, y_n, z_n) &\leq F(x_{n+1}, y_{n+1}, z_{n+1}) = gx_{n+2}, \quad F(y_{n+1}, x_n, y_n) \\ &\geq F(y_{n+1}, x_{n+1}, y_{n+1}) = gy_{n+2} \end{aligned}$$

and

$$F(z_{n+1}, y_n, x_n) \leq F(z_{n+1}, y_{n+1}, x_{n+1}) = gz_{n+2}.$$

From (2.3), we have that $gx_{n+1} \leq gx_{n+2}$, $gy_{n+1} \geq gy_{n+2}$ and $gz_{n+1} \leq gz_{n+2}$.

Then by mathematical induction it follows that (2.3) hold for $n \geq 0$.

Therefore

$$(2.6) \quad gx_0 \leq gx_1 \leq gx_2 \leq gx_3 \leq \dots \leq gx_n \leq gx_{n+1} \leq \dots$$

$$(2.7) \quad gy_0 \geq gy_1 \geq gy_2 \geq gy_3 \geq \dots \geq gy_n \geq gy_{n+1} \geq \dots$$

and

$$(2.8) \quad gz_0 \leq gz_1 \leq gz_2 \leq gz_3 \leq \dots gz_n \leq gz_{n+1} \leq \dots$$

If for some n , we have $(gx_{n+1}, gy_{n+1}, gz_{n+1}) = (gx_n, gy_n, gz_n)$, then $gx_n = F(x_n, y_n, z_n)$, $gy_n = F(y_n, x_n, y_n)$ and $gz_n = F(z_n, y_n, x_n)$ that is, F and g have a coincidence point. So from now we assume $(gx_{n+1}, gy_{n+1}, gz_{n+1}) \neq (gx_n, gy_n, gz_n)$ for all $n \in N$, that is, we assume that either $gx_{n+1} = F(x_n, y_n, z_n) \neq gx_n$ or $gy_{n+1} = F(y_n, x_n, y_n) \neq gy_n$

$$\text{or } gz_{n+1} = F(z_n, y_n, x_n) \neq gz_n.$$

$$\text{Let, } \delta_n = M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t) * M(gz_n, gz_{n+1}, t).$$

Now, we prove that

$$(2.9) \quad \delta_n \geq \varphi(\delta_{n-1}).$$

Since for all $n \geq 0$, $g(x_{n-1}) \leq g(x_n)$, $g(y_{n-1}) \geq g(y_n)$ and $g(z_{n-1}) \leq g(z_n)$, then by (2.1) and (2.2),

$$(2.10) \quad \begin{aligned} M(gx_n, gx_{n+1}, t) &= M(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n), t) \\ &\geq \varphi(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t) * M(gz_{n-1}, gz_n, t)) \\ &= \varphi(\delta_{n-1}). \end{aligned}$$

Similarly, we have from (2.1) and (2.2),

$$(2.11) \quad M(gy_n, gy_{n+1}, t) \geq \varphi(\delta_{n-1}), \quad M(gz_n, gz_{n+1}, t) \geq \varphi(\delta_{n-1}).$$

Combining (2.10) and (2.11), we obtain (2.9).

Since $\varphi(t) > t$ for $t > 0$, it follows from (2.9) that the sequence $\{\delta_n\}$ is a monotone increasing sequence of non-negative real numbers. Hence there exist $\delta \geq 1$ such that $\lim_{n \rightarrow \infty} \delta_n = \delta$. If possible, let $\delta > 1$, Taking the limit as

$n \rightarrow \infty$ in (2.9) and using $\lim_{r \rightarrow t^+} \varphi(r) > t$ for all $t > 0$.

$$\delta = \lim_{n \rightarrow \infty} \delta_n \geq \lim_{n \rightarrow \infty} \varphi(\delta_{n-1}) = \lim_{\delta_{n-1} \rightarrow \delta^+} \varphi(\delta_{n-1}) > \delta,$$

which is a contradiction. Thus $\delta = 1$. Hence we have

$$(2.12) \quad \lim_{n \rightarrow \infty} [M(x_n, x_{n+1}, t) * M(y_n, y_{n+1}, t) * M(z_n, z_{n+1}, t)] = \lim_{n \rightarrow \infty} \delta_n = 1.$$

Next, we show that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences. Let at least one of $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and the sequence of natural numbers $\{m(k)\}$ and $\{l(k)\}$ such that for every natural number k with $m(k) > l(k) \geq k$

$$(2.13) \quad m_k = M(gx_{l(k)}, gx_{m(k)}, t) * M(gy_{l(k)}, gy_{m(k)}, t) * M(gz_{l(k)}, gz_{m(k)}, t) \leq 1 - \varepsilon.$$

Now corresponding to $l(k)$ we can choose $m(k)$ to be smallest positive integer for which (2.13) holds. Then,

$$(2.14) \quad M(gx_{l(k)}, gx_{m(k)-1}, t) * M(gy_{l(k)}, gy_{m(k)-1}, t) * M(gz_{l(k)}, gz_{m(k)-1}, t) > 1 - \varepsilon$$

Further from (2.9), (2.10) and using GV-5, for all $k \geq 0$, we have

$$\begin{aligned} 1 - \varepsilon &\geq m_k \geq M(gx_{l(k)}, gx_{m(k)-1}, \frac{t}{2}) * M(gx_{m(k)-1}, gx_{m(k)}, \frac{t}{2}) \\ &\quad * M(gy_{l(k)}, gy_{m(k)-1}, \frac{t}{2}) * M(gy_{m(k)-1}, gy_{m(k)}, \frac{t}{2}) \\ &\quad * M(gz_{l(k)}, gz_{m(k)-1}, \frac{t}{2}) * M(gz_{m(k)-1}, gz_{m(k)}, \frac{t}{2}) \\ &= M(gx_{l(k)}, gx_{m(k)-1}, \frac{t}{2}) * M(gy_{l(k)}, gy_{m(k)-1}, \frac{t}{2}) \\ &\quad * M(gz_{l(k)}, gz_{m(k)-1}, \frac{t}{2}) * m_{m(k)-1} \\ &> 1 - \varepsilon + m_{m(k)-1}. \end{aligned}$$

Proceeding with limit as $k \rightarrow \infty$, we have by (2.12)

$$(2.15) \quad \lim_{k \rightarrow \infty} m_k = 1 - \varepsilon.$$

Again, by GV-5, for all $k \geq 0$, we have

$$m_k = M(gx_{l(k)}, gx_{m(k)}, t) * M(gy_{l(k)}, gy_{m(k)}, t) * M(gz_{l(k)}, gz_{m(k)}, t)$$

$$\begin{aligned}
&\geq M(gx_{l(k)}, gx_{l(k)+1}, \frac{t}{2}) * M(gx_{l(k)+1}, gx_{m(k)+1}, \frac{t}{4}) * \\
&\quad M(gx_{m(k)+1}, gx_{m(k)}, \frac{t}{4}) * M(gy_{l(k)}, gy_{l(k)+1}, \frac{t}{2}) \\
&\quad * M(gy_{l(k)+1}, gy_{m(k)+1}, \frac{t}{4}) * M(gy_{m(k)+1}, gy_{m(k)}, \frac{t}{4}) \\
&\quad * M(gz_{l(k)}, gz_{l(k)+1}, \frac{t}{2}) * M(gz_{l(k)+1}, gz_{m(k)+1}, \frac{t}{4}) * \\
&\quad M(gz_{m(k)+1}, gz_{m(k)}, \frac{t}{4}) \\
&= M(gx_{l(k)}, gx_{l(k)+1}, \frac{t}{2}) * M(gy_{l(k)}, gy_{l(k)+1}, \frac{t}{2}) * M(gz_{l(k)}, gz_{l(k)+1}, \frac{t}{2}) \\
&\quad * M(gx_{l(k)+1}, gx_{m(k)+1}, \frac{t}{4}) * M(gy_{l(k)+1}, gy_{m(k)+1}, \frac{t}{4}) * \\
&\quad M(gz_{l(k)+1}, gz_{m(k)+1}, \frac{t}{4}) * M(gx_{m(k)+1}, gx_{m(k)}, \frac{t}{4}) * \\
&\quad M(gx_{m(k)+1}, gx_{m(k)}, \frac{t}{4}) * M(gz_{m(k)+1}, gz_{m(k)}, \frac{t}{4}).
\end{aligned}$$

Hence, for all $k \geq 0$

$$\begin{aligned}
(2.16) \quad m_k &= \delta_{l(k)} * \delta_{m(k)} * M(gx_{l(k)+1}, gx_{m(k)+1}, \frac{t}{4}) \\
&\quad * M(gy_{l(k)+1}, gy_{m(k)+1}, \frac{t}{4}) * M(gz_{l(k)+1}, gz_{m(k)+1}, \frac{t}{4}).
\end{aligned}$$

From (2.1)-(2.3) and (2.13), for all $k \geq 0$, we obtain

$$\begin{aligned}
(2.17) \quad M(gx_{l(k)+1}, gx_{m(k)+1}, \frac{t}{2}) &= M(F(x_{l(k)}, y_{l(k)}, z_{l(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)}), \frac{t}{4}) \\
&\leq \varphi(M(gx_{l(k)}, gx_{m(k)}, \frac{t}{4}) * M(gx_{l(k)}, gx_{m(k)}, \frac{t}{4}) * M(gx_{l(k)}, gx_{m(k)}, \frac{t}{4})
\end{aligned}$$

$$= \varphi(m_k)$$

Similarly, from (2.1) - (2.3) and (2.13), for all $k \geq 0$, we get

$$(2.18) \quad M(gy_{l(k)+1}, gy_{m(k)+1}, \frac{t}{2}) = \varphi(m_k)$$

and $M(gz_{l(k)+1}, gz_{m(k)+1}, \frac{t}{2}) = \varphi(m_k)$. Using (2.17) and (2.18) in (2.16), for all $k \geq 0$, we have $m_k = \delta_{l(k)} * \delta_{m(k)} * \varphi(m_k)$. Letting $n \rightarrow \infty$ in the above inequality and using (2.8)-(2.11), we have

$$1 - \varepsilon = \lim_{k \rightarrow \infty} m_k = \lim_{k \rightarrow \infty} \varphi(m_k) = \lim_{m \rightarrow \varepsilon^+} \varphi(m_k) > \lim_{m \rightarrow \varepsilon^+} m_k > 1 - \varepsilon,$$

which is a contradiction. Therefore, $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequence in X and hence they are convergent in the complete fuzzy metric space $(X, M, *)$. Let

$$(2.19) \quad \begin{aligned} F(x_{n+1}, y_{n+1}, z_{n+1}) &= gx_n = x, \lim_{k \rightarrow \infty} F(y_{n+1}, x_{n+1}, y_{n+1}) = gy_n \\ &= y, \lim_{k \rightarrow \infty} F(z_{n+1}, y_{n+1}, x_{n+1}) = gz_n = z. \end{aligned}$$

Since F and g are compatible mappings, we have by (2.15)

$$(2.20) \quad \lim_{k \rightarrow \infty} M(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n, t)) = 1,$$

$$(2.21) \quad \lim_{k \rightarrow \infty} M(g(F(y_n, x_n, y_n)), F(gy_n, gx_n, gy_n, t)) = 1,$$

$$(2.22) \quad \lim_{k \rightarrow \infty} M(g(F(z_n, y_n, x_n)), F(gz_n, gy_n, gx_n, t)) = 1.$$

Finally, we prove that $gx = F(x, y, z)$, $gy = F(y, x, y)$ and $gz = F(z, y, x)$.

Let (a) holds. For all $n \geq 0$, using GV-5, we have

$$M(gx, F(gx_n, gy_n, gz_n), t) \geq M(gx, gF(x_n, y_n, z_n), \frac{t}{2}) \square$$

$$M(g(F(x_n, y_n, z_n), F(gx_n, gy_n, gz_n), \frac{t}{2})).$$

Proceeding with the limit as $n \rightarrow \infty$, using (2.3), (2.19) and (2.20) and the fact that F and g are continuous, we have $M(gx, F(x, y, z), \frac{t}{2}) = 1$.

Similarly, from (2.3), (2.19) and (2.20) and continuity of F and g , we have

$$M(gy, F(y, x, y), t) = 0 \text{ and } M(gz, F(z, y, x), t) = 0.$$

Combining the above three results we get $g(x) = F(x, y, z)$, $g(y) = F(y, x, y)$ and $g(z) = F(z, y, x)$.

Next we suppose that (b) holds.

By (2.3), (2.19) and (2.20), we have $\{gx_n\}$, $\{gz_n\}$ are non-decreasing sequence, $gx_n \rightarrow x$, $gz_n \rightarrow z$ respectively and $\{gy_n\}$ is non-increasing sequence, $gy_n \rightarrow y$ as $n \rightarrow \infty$. The by (i), (ii) and (iii) of (b), we have for $n \geq 0$,

$$(2.23) \quad gx_n \leq x, gy_n \geq y \text{ and } g(z_n) \leq z.$$

Since, F and g are compatible mappings and g is continuous, by (2.20)-(2.22) we obtain

$$(2.24) \quad \lim_{k \rightarrow \infty} g(gx_{n+1}) = gx = \lim_{k \rightarrow \infty} g(F(x_n, y_n, z_n)) \\ = \lim_{k \rightarrow \infty} F(gx_n, gy_n, gz_n)$$

$$(2.25) \quad \lim_{k \rightarrow \infty} g(gy_{n+1}) = gy = \lim_{k \rightarrow \infty} (g(Fy_n, x_n, y_n)) \\ = \lim_{k \rightarrow \infty} F(gy_n, gx_n, gy_n)$$

$$(2.26) \quad \lim_{k \rightarrow \infty} g(gz_{n+1}) = gz = \lim_{k \rightarrow \infty} (g(Fz_n, y_n, x_n)) \\ = \lim_{k \rightarrow \infty} F(gz_n, gy_n, gx_n).$$

Now we have by GV-5

$$M(gx, F(x, y, z)) \geq M(gx, g(gx_{n+1}), \frac{t}{2}) * M(g(gx_{n+1}), F(x, y, z), \frac{t}{2})$$

Taking the limit as $n \rightarrow \infty$, in the above inequality and using (2.2) and (2.20) we have,

$$\begin{aligned} M(gx, F(x, y, z), t) &\geq \lim_{n \rightarrow \infty} M(gx, g(gx_{n+1}), \frac{t}{2}) * \lim_{n \rightarrow \infty} M(g(gx_{n+1}), F(x, y, z), \frac{t}{2}) \\ &= \lim_{n \rightarrow \infty} M(F(gx_n, gy_n, gz_n), F(x, y, z), \frac{t}{2}) * 1 \\ &= \lim_{n \rightarrow \infty} M(F(gx_n, gy_n, gz_n), F(x, y, z), \frac{t}{2}). \end{aligned}$$

Since the mapping g is monotone increasing, by using (2.1). (2.23) and the above inequality, we have for all $n \geq 0$,

$$\begin{aligned} &M(F(gx_n, gy_n, gz_n), F(x, y, z), t) \\ &\geq \lim_{n \rightarrow \infty} [\varphi(M(g(gx_n), gx, t) * M(g(gy_n), gy, t) \\ &\quad * M(g(gz_n), gz, t))]. \end{aligned}$$

Using (2.24) and the property of φ -function, we obtain, $M(gx, F(x, y, z), t) = 1$, which further implies that

$$gx = F(x, y, z)$$

and similarly by virtue of (2.1), (2.25) and (2.26) we can prove

$$gy = F(y, x, y) \text{ and } gz = F(z, y, x).$$

Thus we have proved that F and g have coupled coincidence point in X .

Remark 2.7. Theorem 2.1 fuzzy and generalize the results^{14,15,17} by setting $F(x, y, z) = F_1(x, y)$ and $F(u, v, w) = F_1(u, v)$. Also by further setting $F_1(x, y) = fx$ and $F_1(y, x) = fy$ it reduces and generalizes the results^{9,10} and references there in.

Corollary 2.8. Let $(X, \leq)$ be partially ordered set and let there be a fuzzy metric M on X such that $(X, M, *)$ be a complete fuzzy metric

space. Let $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ are such that F has mixed g -monotone property and for $p \in [0, 1)$ satisfy

$$(2.27) \quad M(F(x, y, z), F(u, v, w), t) \leq p[M(gx, gu, t) * M(gy, gv, t) * M(gz, gw, t)]$$

for all $x, y, z, u, v, w \in X$, with $gx \leq gu$, $gy \geq gv$ and $gz \leq gw$. Let $F(X \times X \times X) \subseteq g(X)$, g be continuous and monotone increasing and F and g be compatible mappings. Also suppose

(a) F is continuous or

(b) X has following properties

(i) if a non-decreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$ for all $n \geq 0$,

(ii) if a non-increasing sequence $\{y_n\} \rightarrow y$, then $y_n \geq y$ for all $n \geq 0$,

(iii) if a non-decreasing sequence $\{z_n\} \rightarrow z$, then $z_n \leq z$ for all $n \geq 0$.

If there exist $x_0, y_0, z_0 \in X$, such that $g(x_0) \leq F(x_0, y_0, z_0)$, $g(y_0) \geq F(y_0, x_0, z_0)$ and $g(z_0) \leq F(z_0, y_0, x_0)$, there exist $x, y, z \in X$ such that $g(x) = F(x, y, z)$, $g(y) = F(y, x, z)$ and $g(z) = F(z, y, x)$, that is, F and g have tripled coincidence point in X .

Proof: By setting $\varphi(t) = pt$, the proof follows easily from theorem 2.1.

Remark 2.9. If we put $g(x) = x$ for all $x \in X$ in corollary (2.8) it generalizes and fuzzify particular case of theorem¹⁸ 7 and 8 (with setting $i = j = k = p / 3$).

Example 2.10. Let $X = [0, 1]$ be endowed with Euclidean metric, we define $a * b = a.b$ and

$$M(x, y, t) = \frac{d(x, y)}{t + d(x, y)}, \text{ for all } x, y \in X.$$

Then, $(X, \leq)$ is a partial ordered set with natural ordering of real numbers.

Let $F: X \times X \times X \rightarrow X$ and $g: X \rightarrow X$ defined as $g(x) = x$ for all $x \in X$ and

$$F(x, y, z) = \begin{cases} \left(\frac{x+z-y}{3} \right)^2, & \text{if } x, y, z \in [0, 1], x \geq z \geq y, \\ 1, & \text{if } x < y \text{ or } z < y \end{cases}$$

respectively.

Clearly, $F(X \times X) \subseteq g(X)$, also F obeys mixed g -monotone property.

Let $\varphi: [0, 1] \rightarrow [0, 1]$ be defined as $\varphi(t) = \sqrt[t]{t}$ for $t \in [0, 1]$.

We define the sequences $\{x_n\} = \frac{1}{n}$, $\{y_n\} = \frac{1}{n^2}$ and $\{z_n\} = \frac{1}{2n}$.

Obviously the pair $\{F, g\}$ is compatible.

Also, $x_0 = 1, y_0 = 2c, z_0 = c$ ($c < 1$) are fixed point in X such that

$$\begin{aligned} g(x_0) &= g(1) = 1 = F(1, 2c, c) = F(x_0, y_0, z_0), \\ g(y_0) &= g(2c) = 2c \geq c^2 = F(2c, c, 1) = F(y_0, z_0, x_0) \end{aligned}$$

$$\text{and } g(z_0) = g(c) = c \leq 1 = F(z_0, y_0, x_0).$$

We next verify inequality (2.1) of theorem 2.6. We take $x, y, u, v \in X$ such that $x \geq u, z \geq w, y \leq v$ or $(x, y, w) \geq (u, v, w)$. We have the following cases

$$\begin{aligned} \text{Case-I If } (x, y, z) &= (u, v, w) \text{ or } (x, y, z) = (0, 0, 0), (u, v, w) = (0, 1, 1) \text{ or } (x, y, z) \\ &= (1, 1, 1), (u, v, w) = (0, 1, 1), \end{aligned}$$

One can easily see

$$M(F(x, y, z), F(u, v, w), t) \geq \varphi(M(gx, gu, t) * M(gy, gv, t) * M(gz, gw, t)).$$

Hence inequality (2.1) holds.

Case-II If $(x, y, z) = (1, 1, 0), (u, v, w) = (0, 0, 0)$, then

$$M(F(x, y, z), F(u, v, w), t) = M(F(1, 1, 0), F(0, 0, 0), t) = \frac{1}{t+1}$$

$$> \frac{1}{t\sqrt{t+1}} = \varphi(M(g1, g0, t) * M(g1, g0, t) * M(g0, g0, t)).$$

Hence inequality (2.1) holds.

Case-III If $(x, y, z) = (1, 1, 0)$, $(u, v, w) = (0, 1, 1)$, then, we obtain

$$M(F(x, y, z), F(u, v, w), t) = M(F(1, 1, 0), F(0, 1, 1), t) = 1$$

$$> \frac{1}{\sqrt{t+1}} = \varphi(M(g1, g0, t) * M(g1, g1, t) * M(g0, g1, t)).$$

Therefore, all the condition of theorem 2.6 is satisfied. Hence $(0, 0, 0)$ is the coupled coincidence point of F and g .

Remark 2.3 As an application of theorem 2.6, the existence and uniqueness of common solution of periodic boundary value problem can be established^{5,6,9,22,23} and references therein.

3. Conclusion

In this paper, the concept of tripled coincidence point has been introduced in setting of partially ordered GV-fuzzy metric spaces and a theorem for compatible trivariate self mapping F and self mapping g has been proved, which reduces in special case to a (coupled) fixed point theorem and coincidence point theorem in fuzzy metric space.

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