

Circular Waves in Thermoelastic Plates Sandwiched between Liquid Layers

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Abstract: This paper examines in detail the characteristics of the propagation of thermoelastic waves in a homogeneous, transversely isotropic, thermally conducting elastic plate bordered with layers (or half-spaces) of conducting viscous fluid on both sides. Complex secular equations for symmetric and antisymmetric wave motion of the circular plate, in completely separate terms, are derived. The waves of short wavelength and Leaky Lamb waves have also been discussed. The results have been deduced and compared with the existing one in relevant publications available in the literature at various stages of this work. Finally, in order to illustrate the analytical results, the numerical solution is carried out for cobalt plate bordered with water by using the functional iteration method. The analytical and numerical results are found to be in close agreement.

Keywords: Crested waves, viscous, conducting, thermal relaxation, Biot's constant, transversely isotropic.

1. Introduction

Lamb waves are a form of elastic perturbation that can propagate in a solid plate with free boundaries; there are two groups of waves, symmetric and anti-symmetric. The temperature changes due to the elastic deformation cannot be ignored; therefore it is required to determine the thermal and mechanical fields in the body concurrently. The theory to include the effect of temperature change, known as the theory of thermoelasticity, has also been well established. According to the theory, the temperature field is coupled with the elastic strain field. In thermoelasticity, classical heat transfer, Fourier's conduction equation is extensively used in many engineering applications. Ultrasonic nondestructive inspection plays important role in multilayered structures testing, especially in aeroindustry, where composites are replacing metallic parts. Detecting damage in composite materials, several

techniques have been developed; however Lamb wave methods have recently emerged as a consistent way to situate defects in such materials. Every technique implemented in the literature offers their own unique advantages in detecting certain types of defects. This study will enhance the knowledge base of the NDE community as well as the seismologists and geophysicists where anisotropy effects are significant.

Viktorov¹ studied the characteristics of Rayleigh and Lamb waves in elastokinetics. The propagation of plane harmonic waves in a homogeneous anisotropic material has also been studied in coupled theory of Thermoelasticity by Chadwick². Dhaliwal and Sherief³ extended the generalized Thermoelasticity theories to anisotropic elastic bodies. Watkins et al.⁴ calculated the attenuation of Lamb waves in the presence of an inviscid liquid using an acoustic impedance method. Massalas et al.⁵ studied the influence of a constant flux on the static and dynamic response of a simply supported and clamped circular plate with edge immovable and constrained. Graff⁶, Zhu and Wu⁷ brought out detailed analysis of the characteristics of Lamb waves in elastokinetics. The investigations on the propagation of Circular crested waves in anisotropic plates bordered with inviscid liquid maintained at uniform temperature have been carried out by Sharma and Pathania⁸. Sharma and Sharma⁹ investigated the propagation of Lamb wave in thermoviscoelastic plate loaded with viscous fluid layer of varying temperature. Fabien Josse et al.¹⁰ presented an analytical solution for the resonance condition of the piezoelectric quartz resonator with one surface in contact with a viscous conductive liquid. Yang and Shue¹¹ presented theoretical and experimental study on a leaky Lamb wave propagating in a piezoelectric plate loaded by a dielectric/conductive fluid. Lee and Kuo¹² examined in detail the characteristics of a leaky Lamb wave propagating in a piezoelectric plate immersed in a dielectric or conductive fluid.

In this work we consider the problem of circular crested wave propagation in an infinite homogeneous, transversely isotropic, thermoelastic plate bordered with an viscous liquid layers or half spaces on both sides in the context of generalized (Lord-Shulman (LS) and Green-Lindsay (GL)) theories of thermoelasticity. It is noticed that the motion for circular crested waves is also governed by Rayleigh-Lamb type secular equations as in the case of a rectangular plate. More general dispersion equations of Lamb type waves are derived and discussed. The secular equations for leaky Lamb waves have also been deduced. The analytical results so obtained have been verified numerically and are illustrated graphically in case of cobalt material and water.

2. Formulation of the Problem

We consider an infinite homogeneous, transversely isotropic, thermally conducting elastic plate of thickness $2d$, initially at temperature T_0 , having a conducting viscous liquid layer of thickness h on both sides. We take origin of the co-ordinate system (r, θ, z) on the middle surface of the plate. The z -axis is taken along the thickness of the plate. The plate is axi-symmetric with z -axis as axis of symmetry. We take $r-z$ plane as the plane of incidence. The basic governing equations in non-dimensional form for generalized thermoelasticity in the absence of body forces and heat sources and for viscous fluid (homogeneous liquid) medium, are given by Sharma and Pathania⁸, Sharma and Sharma⁹ respectively.

$$(1) \quad u_{,rr} + \frac{1}{r}u_{,r} - \frac{1}{r^2}u + c_2u_{,zz} + c_3w_{,rz} - (T + t_1\delta_{2k}\dot{T})_{,r} = \ddot{u},$$

$$(2) \quad c_3\left(u_{,rz} + \frac{1}{r}u_{,z}\right) + c_2w_{,rr} + \frac{1}{w_{,r}} + c_1w_{,zz} - \bar{\beta}(T + t_1\delta_{2k}\dot{T})_{,z} = \ddot{w},$$

$$(3) \quad T_{,rr} + \frac{1}{r}T_{,r} + \bar{K}T_{,zz} - (\dot{T} + t_0\ddot{T}) = \varepsilon\left(\frac{\partial}{\partial t} + t_0\delta_{1k}\frac{\partial^2}{\partial t^2}\right)\left(u_{,r} + \frac{1}{r}u + \bar{\beta}w_{,z}\right),$$

$$(4) \quad \nu_L \frac{\partial}{\partial t} \nabla^2 \vec{u}_{Lij} + \left(\delta_L^2 + \frac{1}{3}\nu_L \frac{\partial}{\partial t}\right) \nabla \nabla \cdot \vec{u}_{Lij} - \frac{\bar{\beta}_L}{\bar{\rho}} \nabla T_{Lj} = \ddot{u}_{Lij},$$

$$a^* \nabla^2 T_{Lij} - \dot{T}_{Lij} = \frac{\varepsilon_L \rho_L \delta_L^2}{\bar{\beta}_L \rho} \nabla \cdot \dot{u}_{Lij}, \quad (i, j = 1, 2),$$

where $\beta_1 = (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_3$, $\beta_3 = 2c_{13}\alpha_1 + c_{33}\alpha_3$, $\bar{\beta} = \frac{\beta_3}{\beta_1}$, $\bar{K} = \frac{K_3}{K_1}$

$$\varepsilon = \frac{\beta_1^2 T_0}{\rho C_e c_{11}}, \quad \bar{\beta}_L = \frac{\beta_L^*}{\beta_1}, \quad \delta_L^2 = \frac{c_L^2}{\nu_1^2}, \quad c_L^2 = \frac{\lambda_L}{\rho_L}, \quad \beta_L^* = 3\lambda_L \alpha^*,$$

$$\varepsilon_L = \frac{\beta_L^{*2} T_0^*}{\rho_L C_v^* \lambda_L}, \quad \nu_L = \frac{\mu_L \omega^*}{\rho_L \nu_1^2}, \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad a^* = \frac{\rho C_e \bar{k}}{\rho_L C^*}, \quad \bar{k} = \frac{k_L^*}{k_s}$$

Here t_0 and t_1 are the thermal relaxation times. δ_{ik} is the Kronecker's with $k=1$ for Lord-Shulman (LS) theory and $k=2$ for Green-Lindsay (GL) theory of generalized thermoelasticity. ε is the thermomechanical coupling constant and v_1 is the longitudinal wave velocity. $\vec{u}(r, z, t) = (u, 0, w)$ is the displacement vector, $T(r, z, t)$ is the temperature change, c_L is the velocity of sound in the liquid, α^* is the coefficient of volume thermal expansion. ε_L is the thermomechanical coupling in the liquid. Along the z direction for the top liquid layer ($j=1$) and for the bottom liquid layer ($j=2$). c_L , ρ_L are velocity of sound, the density of the liquid respectively, λ_L is the bulk modulus, C_V^* is the specific heat at constant volume of fluid. T_{L_j} is the temperature deviation of liquid medium from ambient temperature T_0^* . $\vec{u}_{L_j}(r, z, t) = (u_{L_j}, 0, w_{L_j})$ ($j=1, 2$) is the velocity vector. u_{L_j} and w_{L_j} are respectively. Along the z direction for the top liquid layer ($j=1$) and for the bottom liquid layer ($j=2$). Moreover liquid is assumed to be thermally conducting hypothetical solid. Here the dot notation is used for time differentiation and comma denotes spatial derivatives.

3. Boundary Conditions

The boundary conditions at the solid-liquid interfaces $z = \pm d$ to be satisfied are:

Mechanical conditions

$$(5) \quad \sigma_{zz} = (\sigma_{zz})_L, \quad \sigma_{xz} = (\sigma_{xz})_L, \quad u = u_L, \quad w = w_L.$$

Thermal conditions

The conduction-convection boundary condition is given by

$$(6) \quad T = T_{L_{ij}}, T_{,z} + \bar{k} T_{L_{ij},z} = 0,$$

where H is Biot's heat transfer coefficient.

4. Solution of the Problem

We assume solutions of the form

$$(7) \quad \{u, w, T\} = \{J_1(\xi r), VJ_0(\xi r), WJ_0(\xi r)\} Se^{i\xi(x\sin\theta + mz - ct)},$$

$$(8) \quad \{\varphi_{L_{ij}}, \psi_{L_j}\} = \{\Phi_{L_{ij}} J_0(\xi r), \Psi_{L_j} J_0(\xi r)\} Se^{i\xi(mz - ct)},$$

where $c = \frac{\omega}{\xi}$ is the non-dimensional phase velocity, ω is the frequency and ξ is the wave number. J_0, J_1 are respectively, the Bessel functions of order zero and one. θ is the angle of inclination of wave normal with axis of symmetry. m is still unknown parameter. V, W are respectively, the amplitude ratios of displacement w temperature T and $\Phi_{L_{ij}}, \Psi_{L_j}$ are the amplitude ratios of $\varphi_{L_{ij}}, \psi_{L_j}$ respectively.

On seeking the solution of (1)-(3) in the form (7), displacements, temperature change can be expressed as

$$u = \sum_{q=1}^5 (A_q \cos \xi m_q z + B_q \sin \xi m_q z) J_1(\xi r) e^{-i\omega t},$$

$$w = \sum_{q=1}^5 V_q (-A_q \sin \xi m_q z + B_q \cos \xi m_q z) J_0(\xi r) e^{-i\omega t},$$

$$(9) \quad T = \sum_{q=1}^5 W_q (A_q \cos \xi m_q z + B_q \sin \xi m_q z) J_0(\xi r) e^{-i\omega t}.$$

where A_q, B_q are amplitudes. V_q, W_q are amplitude ratios and m_q are given as

$$\sum m_1^2 = \frac{P - Jc^2}{c_1 c_2} + \frac{1 - c^2 \tau_0}{\bar{K}} + \frac{i\varepsilon \xi c^3 \bar{\beta}^2 \tau_0' \tau_1}{c_1 \bar{K}},$$

$$\sum m_1^2 m_2^2 = \frac{(c_2 - c^2)(1 - c^2)}{c_1 c_2} + \frac{(1 - c^2 \tau_0)(P - Jc^2)}{c_1 c_2 \bar{K}} + \frac{i\varepsilon \xi c^3 \tau_0' \tau_1}{c_1 c_2 \bar{K}} [c_1 - 2c_3 \bar{\beta} + (1 - c^2) \bar{\beta}^2],$$

$$(10) \quad \sum m_1^2 m_2^2 m_3^2 = \frac{c_2 - c^2}{c_1 c_2 \bar{K}} [(1 - c^2 \tau_0)(1 - c^2) + i\varepsilon \xi c^3 \tau_0' \tau_1],$$

$$P = c_1 + c_2^2 - c_3^2, J = c_1 + c_2, \tau_0 = t_0 + i\omega^{-1}, \tau'_0 = t_0\delta_{1k} + i\omega^{-1}, \tau_1 = t_1\delta_{2k} + i\omega^{-1},$$

$$V_q = \begin{cases} -im_q a_q, & q = 1, 3 \\ \frac{ib_q}{m_q} & q = 5 \end{cases},$$

$$W_q = \begin{cases} \frac{i(c_2 + c_3 a_q)m_q^2 + 1 - c^2}{c\tau_1}, & q = 1, 3 \\ i \frac{[c_1 c_2 m_q^4 + (P - Jc^2)m_q^2 + (c_2 - c^2)(1 - c^2)]a_q}{c\bar{\beta}\tau_1(c_2 m_q^2 + 1 - c_3/\bar{\beta} - c^2)}, & q = 5 \end{cases}$$

$$a_q = \bar{\beta} \frac{c_2 m_q^2 + (1 - c_3/\bar{\beta}) - c^2}{(c_1 - c_3 \bar{\beta})m_q^2 + c_2 - c^2}, \quad b_q = (c_2 m_q^2 + 1 - c^2 + ic\tau_1 W_q) \frac{1}{c_3}.$$

On using stress strain relation

$$(11) \quad \sigma_{zz} = (c_3 - c_2) \left(u_{,r} + \frac{1}{r} u \right) + c_1 w_{,z} + i\omega \tau_1 \bar{\beta} T, \quad \sigma_{rz} = u_{,z} + w_{,r}$$

axial stress and shear stress in rz plane and temperature change are obtained as

$$\sigma_{zz} = -\xi \sum_{q=1}^5 P_q (A_q \cos \xi m_q z + B_q \sin \xi m_q z) J_0(\xi r) e^{-i\omega t},$$

$$\sigma_{rz} = i\xi c_2 \sum_{q=1}^5 f_q (-A_q \sin \xi m_q z + B_q \cos \xi m_q z) J_1(\xi r) e^{-i\omega t},$$

$$(12) \quad T_{,z} = \sum_{q=1}^5 N_q (A_q \cos \xi m_q z + B_q \sin \xi m_q z) J_0(\xi r) e^{-i\omega t},$$

where

$$P_q = \begin{cases} c_2(1+a_q) - c^2 a_q, & q=1, 3 \\ \frac{c_2 m_q^2 - (c_2 - c^2) b_q}{m_q^2}, & q=5 \end{cases}, \quad f_q = \begin{cases} (1+a_q) m_q, & q=1, 3 \\ \frac{m_q^2 - b_q}{m_q}, & q=5 \end{cases},$$

$$N_q = i\xi W_q m_q, q=1, 3, 5$$

In the liquid layers, in addition to introducing non-dimensional quantities, we take

$$(13) \quad u_{L_j} = \frac{\partial \phi_{L_{ij}}}{\partial r} + \frac{\partial \psi_{L_j}}{\partial z}, \quad w_{L_j} = \frac{\partial \phi_{L_{ij}}}{\partial z} - \frac{\partial \psi_{L_j}}{\partial r} - \frac{\psi_{L_j}}{r},$$

where $\phi_{L_{ij}}$ and ψ_{L_j} , ($j=1, 2$) are respectively, the scalar velocity potential and vector velocity potential. We obtain

$$(14) \quad \nabla^2 \phi_{L_{ij}} - \frac{1}{\left(\delta_L^2 (1 + \varepsilon_L) + \frac{4}{3} \nu_L \frac{\partial}{\partial t} \right)} \ddot{\phi}_{L_{ij}} = 0,$$

$$(15) \quad \nabla^2 \psi_{L_j} - \frac{1}{\nu_L} \dot{\psi}_{L_j} = 0$$

$$(16) \quad a^* \nabla^2 T_{L_{ij}} - \dot{T}_{L_{ij}} = \frac{\varepsilon_L \rho_L \delta_L^2}{\bar{\beta}_L \rho} \nabla^2 \phi_{L_{ij}}, \quad (i, j=1, 2)$$

On seeking solution of form (8) for $d < z < d+h$ and $-(d+h) < z < -d$, such that the acoustical pressure is zero at $z = \pm(d+h)$, obtained as,

$$(\Phi_{L_{i1}}, T_{L_{i1}}) = (1, S_L \xi^2) A_{7i} \sin \xi m_{7i} [z - (d+h)] e^{-i\omega t},$$

$$(\Phi_{L_{i2}}, T_{L_{i2}}) = -(1, S_L \xi^2) A_{8i} \sin \xi m_{7i} [z + (d+h)] e^{-i\omega t},$$

$$\Psi_{L_1} = A_9 \sin \xi m_9 [z - (d + h)] e^{-i\omega t} \quad \Psi_{L_2} = -A_{10} \sin \xi m_9 [z + (d + h)] e^{-i\omega t},$$

$$(17) \quad S_{Li} = -\frac{i\omega \varepsilon_L \rho_L \delta_L^2 c^2 (a^* \nabla^2 + i\omega)}{\bar{\beta}_L \rho (a^* \nabla^2 + i\omega) \left(\left(\delta_L^2 - \frac{4}{3} \nu_L \frac{\partial}{\partial t} \right) (a^* \nabla^2 + i\omega) + i\omega \varepsilon_L \delta_L^2 \right)},$$

$$m_{7i,8i} = (b_i^2 c^2 - 1), i = 1, 2, \quad m_{9,10} = \pm \sqrt{\frac{i c^2}{\nu_L \omega} - 1}$$

$$b_1^2 + b_2^2 = \frac{a^* + i\omega^{-1} \left[(1 + \varepsilon_L) \delta_L^2 - \frac{4}{3} i\omega \nu_L \right]}{a^* \left(\delta_L^2 - \frac{4}{3} i\omega \nu_L \right)}, \quad b_1^2 * b_2^2 = \frac{i\omega^{-1}}{\left(\delta_L^2 - \frac{4}{3} i\omega \nu_L \right)}$$

5. Derivation of the Secular Equations

By invoking the interface conditions (5) to (6) at plate surfaces $z = \pm d$, we obtain a system of ten simultaneous linear equations in amplitudes. System of equations have a non-trivial solution if the determinant of the coefficients of amplitudes vanishes, which leads to a characteristic equation for the propagation of modified guided thermoelastic waves in the plate. The characteristic equation for the thermoelastic plate waves in this case, after applying lengthy algebraic reductions of the determinant along with conditions $\gamma^* \neq 0$ and $\gamma^* \neq (2n-1)\pi/2, n=1,2,3,\dots$ leads to the following secular equations

$$(18) \quad \left[\frac{T_1}{T_5} \right]^{\pm 1} - \frac{P_{39} G_3}{P_{19} G_1} \left[\frac{T_3}{T_5} \right]^{\pm 1} + \frac{P_{719} G_{71}}{P_{19} G_1} \frac{T_{71}}{[T_5]^{\pm 1}} - \frac{P_{729} G_{72}}{P_{19} G_1} \frac{T_{72}}{[T_5]^{\pm 1}} + \frac{\bar{F}_9}{P_{19} G_1} \frac{T_9}{[T_5]^{\pm 1}} = -\frac{P_{59} G_5}{P_{19} G_1},$$

where

$$P_{71} = -\bar{\rho} \nu_L^{-1} i \xi \left[i \xi c^2 - 4 \nu_L \xi^2 + c \xi^2 m_{71}^2 \right],$$

$$P_{72} = -\bar{\rho} \nu_L^{-1} i \xi \left[i \xi c^2 - 4 \nu_L \xi^2 + c \xi^2 m_{72}^2 \right], \quad P_9 = -2 \bar{\rho} i \xi^3 c m_9,$$

$$\begin{aligned}
f_{71} &= -2i\bar{\rho}\xi^3 c m_{71}, \quad f_{72} = -2i\bar{\rho}\xi^3 c m_{72}, \quad f_9 = -\bar{\rho}i\xi^3 c [m_9^2 - 1], \quad V_{71} = \xi m_{71}, \\
V_{72} &= \xi m_{72}, \quad V_9 = -\xi, \quad U_{71} = -\xi = U_{72}, \quad U_9 = \xi m_9, \quad W_{71} = -S_{L1}\xi^2, \quad W_{72} = -S_{L2}\xi^2 \\
N_{71} &= \bar{K}S_{L1}\xi^2 \xi m_{71}, \quad N_{72} = \bar{K}S_{L2}\xi^2 \xi m_{72}, \quad \bar{F}_9 = f_{19}G'_1 - f_{39}G'_3 + f_{59}G'_5 + f_{79}G'_7, \\
(18) \text{ (a)} \quad P_{q9} &= (P_q U_9 - U_q P_9), \quad f_{q9} = (f_q V_9 - V_q f_9), \quad q = 1, 3, 5, 71, 72
\end{aligned}$$

$$\begin{aligned}
G_1 &= \begin{vmatrix} f_3 & f_5 & -f_{71} & -f_{72} \\ V_3 & V_5 & -V_{71} & -V_{72} \\ W_3 T_3^{\pm 1} & W_5 T_5^{\pm 1} & -W_{71} T_{71} & -W_{72} T_{72} \\ N_3 & N_5 & -2N_{71} & -2N_{72} \end{vmatrix} \\
G'_1 &= \begin{vmatrix} P_3 T_3^{\pm 1} & P_5 T_5^{\pm 1} & -P_{71} T_{71} & -P_{72} T_{72} \\ U_3 T_3^{\pm 1} & U_5 T_5^{\pm 1} & -U_{71} T_{71} & -U_{72} T_{72} \\ W_3 T_3^{\pm 1} & W_5 T_5^{\pm 1} & -W_{71} T_{71} & -W_{72} T_{72} \\ N_3 & N_5 & -2N_{71} & -2N_{72} \end{vmatrix}.
\end{aligned}
\tag{18} \text{ (b)}$$

Here $T_n = \tan \gamma m_n$, $n=1,3,5$, $T_n = \tan \gamma^* m_n$, $n=71,72,9$, $\gamma = \xi d$, G_3, G_5, G_{71}, G_{72} and $G'_3, G'_5, G'_{71}, G'_{72}$ can be written from G_1 and G'_1 by replacing $(3,5,71,72)$ in column index elements in G_1 and G'_1 with $(1,5,71,72), (1,3,71,72), (1,3,5,72), (1,3,5,71)$ elements respectively. The superscript +1 corresponds to skew-symmetric and -1 refers to symmetric modes.

The Rayleigh-Lamb type equation also governs circular crested thermoelastic waves in a plate bordered with layers of viscous liquid or half space viscous liquid on both sides. Although the frequency wave number relationship holds whether the waves are straight or circularly crested, the displacement and stress vary according to Bessel functions rather than trigonometric functions as far as the radial coordinate is concerned. For large values of r , we have

$$J_0(\xi r) \rightarrow \frac{\sin \xi r + \cos \xi r}{\sqrt{\pi \xi r}}, J_1(\xi r) \rightarrow \frac{\sin \xi r - \cos \xi r}{\sqrt{\pi \xi r}}$$

Thus, far from the origin the motion becomes periodic in r . Actually, “far” occurs rather rapidly, within four to five zeros of the Bessel function. As r becomes very large, the straight crested behavior is the limit of circular crested waves.

6. Waves of Short Wavelength

The waves of short wavelength correspond to the case when the transverse wavelength of the plate is quite small as compared to the thickness of the plate.

Case1: the dispersion equation for thermoelastic Rayleigh waves of an infinite half space solid bordered with an infinite half space homogeneous liquid can be obtained with conditions i.e Some information on the asymptotic behavior is obtainable by letting $R \rightarrow \infty$. If we take $R > \omega/\sqrt{c_2}$, it follows that $V < \sqrt{c_2}$ and the roots of characteristic equation lies in region I in this case. Then we replace m_1 , m_3 and m_5 in the secular equation by im'_1 , im'_3 and im'_5 .

$$\text{Herefor } R \rightarrow \infty, \frac{\tanh(\gamma m'_1)}{\tanh(\gamma m'_5)} \rightarrow 1, \quad \frac{\tanh(\gamma m'_3)}{\tanh(\gamma m'_5)} \rightarrow 1$$

$$\frac{\tan(\gamma^* m_{71})}{\tanh(\gamma m'_5)} \rightarrow -i, \frac{\tan(\gamma^* m_{72})}{\tanh(\gamma m'_5)} \rightarrow -i, \frac{\tan(\gamma^* m_9)}{\tanh(\gamma m'_5)} \rightarrow -i, \text{ so that secular equation (18) reduces to}$$

$$(19) \quad \begin{aligned} P'_{19} G''_1 - P'_{39} G''_3 + P'_{59} G''_5 = & \mp P'_{719} G''_{71} \pm P'_{729} G''_{72} \\ & \mp [f'_{19} G''_1 - f'_{39} G''_3 + f'_{59} G''_5 + f'_{719} G''_{71} - f'_{729} G''_{72}] \end{aligned}$$

Case2: The dispersion equation for thermoelastic Rayleigh waves of an infinite half space solid bordered with a homogeneous liquid layer of thickness h is given as

$$(20) \quad \begin{aligned} P'_{19} G''_1 - P'_{39} G''_3 + P'_{59} G''_5 = & \mp i P'_{719} G''_{71} \tan \gamma^* m_{71} \pm i P'_{729} G''_{72} \tan \gamma^* m_{72} \\ & \mp i [f'_{19} G''_1 - f'_{39} G''_3 + f'_{59} G''_5 + f'_{719} G''_{71} - f'_{729} G''_{72}] \tan \gamma^* m_9, \end{aligned}$$

where $P_{q9}, f_{q9}, q=1,3,5,7,1,72$ are given by equation (18) (a) G_j'' and G_j''' , ($j=1,3,5,7,1,72$) are given by equations (18) (b) by replacing m_1, m_3 and m_5 in the secular equation by im_1', im_3' and im_5' .

7. Leaky Lamb Waves

When a plate of finite thickness is bordered with a half space homogeneous liquid on both sides, part of Lamb wave energy in the plate is coupled into the liquid as radiation; most of the energy is still in the solid. This type of disturbance is called the leaky Lamb wave.

Case1: The dispersion equations for leaky Lamb waves i.e. Lamb waves in a transversely isotropic plate bordered with an infinite half space homogeneous liquid ($h \rightarrow \infty$) at both sides are as follows.

$$(21) \quad \left[\frac{T_1}{T_5} \right]^{\pm 1} - \frac{P_{39}G_3}{P_{19}G_1} \left[\frac{T_3}{T_5} \right]^{\pm 1} - \frac{iP_{719}G_{71}}{P_{19}G_1 [T_5]^{\pm 1}} + \frac{iP_{729}G_{72}}{P_{19}G_1 [T_5]^{\pm 1}} - \frac{i\bar{F}_9}{P_{19}G_1 [T_5]^{\pm 1}} \\ = - \frac{P_{59}G_5}{P_{19}G_1}$$

where $P_{q9}, f_{q9}, q=1,3,5,7$ and $\bar{F}_9$ is given by equation (18.1). G_1, G_3, G_5, G_7 and G_1', G_3', G_5', G_7' are given by equations (18.2).

Case2: If we multiply a factor of $i \tan \gamma^* m_{71}, i \tan \gamma^* m_{72}$ and $i \tan \gamma^* m_9$ to the terms $(-P_{39})$ and $(-P_{59})$ respectively in equation (21), the dispersion equation (18) of Lamb waves in a transversely isotropic thermoelastic plate bordered with a homogeneous layer of thickness h on both sides can be obtained.

8. Numerical Results and Discussion

In order to illustrate the theoretical results obtained in the previous section, some numerical results are presented. The materials chosen for this purpose is cobalt and liquid chosen for the purpose of numerical calculations is water. The physical data is given by Sharma and Pathania⁸, Sharma and Sharma⁹.

$$\rho = 8.836 \times 10^3 \text{ Kg m}^{-3}, \quad c_{11} = 3.071 \times 10^{11} \text{ Nm}^{-2},$$

$$c_{12} = 1.650 \times 10^{11} \text{ Nm}^{-2} \quad c_{13} = 1.027 \times 10^{11} \text{ Nm}^{-2}$$

$$c_{33} = 3.581 \times 10^{11} \text{ Nm}^{-2}, \quad c_{44} = 1.510 \times 10^{11} \text{ Nm}^{-2}$$

$$T_0 = 298 \text{ K} \quad \beta_1 = 7.04 \times 10^6 \text{ Nm}^{-2} \text{ deg}^{-1}$$

$$\beta_3 = 6.90 \times 10^6 \text{ Nm}^{-2} \text{ deg}^{-1}, \quad C_e = 4.27 \times 10^2 \text{ JKgm}^{-1} \text{ deg} \quad K_1 = 0.690 \times 10^2 \text{ Wm}^{-1} \text{ deg}^{-1}$$

$$K_3 = 0.690 \times 10^2 \text{ Wm}^{-1} \text{ deg}^{-1}, \quad \omega^* = 1.88 \times 10^{11} \text{ s}^{-1}, \quad \varepsilon = 1.29 \times 10^{-2}$$

$$d = 1, \quad h = 0.25 \text{ H} = 0.1 \quad c_L = 1.5 \times 10^3 \text{ m/s}, \quad \rho_L = 1000 \text{ Kg m}^{-3}$$

The secular equation (18) is solved by using functional iteration method and using $c^{-1} = V^{-1} + i\omega^{-1}Q$, phase speed V and attenuation coefficient Q have been obtained. The phase velocity and attenuation coefficient of first few symmetric and skew symmetric modes has been computed and corresponding dispersion curves for Rayleigh-Lamb type modes are represented graphically. In Figure 1 and 2, the computed simulated results corresponding to three situations of liquid loading namely, inviscid (ideal) liquid ($\rho_L \neq 0.0$, $\mu_L = 0.0$), low viscous liquid ($\rho_L \neq 0.0$, $\mu_L = 0.1$) and high viscous liquid ($\rho_L \neq 0.0$, $\mu_L = 1.0$). From Figure 1, the phase velocity of acoustic asymmetric mode is observed to increase from zero value at vanishing wave number and become closer to Rayleigh wave velocity. The velocity of the acoustic symmetric mode becomes dispersion less. The phase velocity of optical modes of wave propagation, symmetric and asymmetric; attain quite large values at vanishing wave number to become asymptotically closer to the shear wave speed. These behaviors of optical modes attributed to the fact that the viscosity gives rise to dissipative forces at supersonic speed and hence phase velocity of these modes decreases. In Figure 2, it is observed that the magnitude of attenuation coefficient of low viscous liquid for (n=1) mode become high and then slashes down sharply as compare to ideal and high viscous liquid. From Figure 3, it is observed that the phase velocity function in case of symmetric and asymmetric modes show constant and stabilized behavior with thermal conductivity ratio. The velocity of the acoustic symmetric mode remains non-dispersive. The magnitude of phase velocity of symmetric and asymmetric (n=2) mode is observed to be greater than that of the other modes. From Figure 4, it is noticed that the profiles of attenuation

coefficient in other modes have almost negligible variation as compare to the symmetric and asymmetric ($n=1$) mode. For symmetric and asymmetric ($n=1$) mode, the attenuation coefficient is observed to increase monotonically and then slashes down. The magnitude of the attenuation coefficient of symmetric modes is greater than the asymmetric modes.

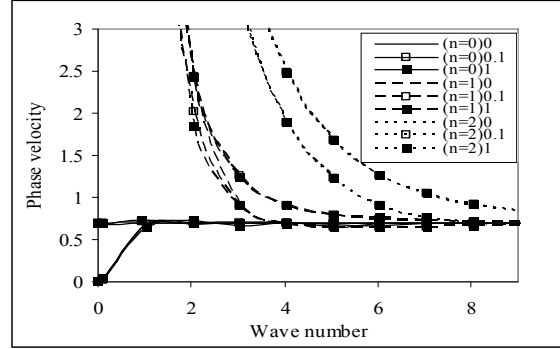


Fig. 1. Variation of phase velocity with wave number for angle of inclination $\theta = 45^\circ$

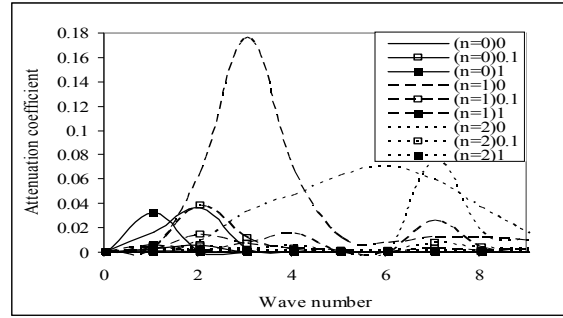


Fig. 2. Variation of attenuation coefficient with wave number for angle of inclination $\theta = 45^\circ$

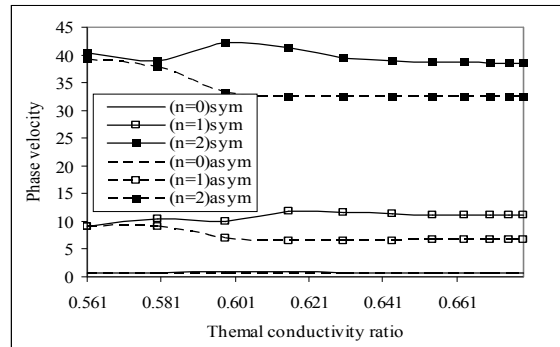


Figure 3: Variation of phase velocity with thermal conductivity ratio for angle of inclination $\theta = 45^\circ$

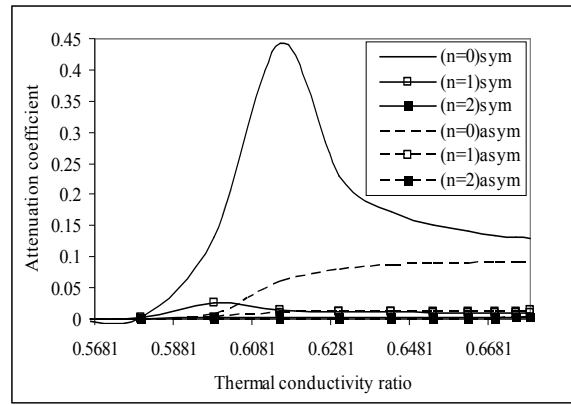


Fig. 4. Variation of attenuation coefficient with thermal conductivity ratio for angle of inclination $\theta = 45^\circ$

9. Conclusions

At higher wave numbers the phase velocities of various higher (optical) symmetric and skew-symmetric modes attains quite high values and significantly decreases to become asymptotically closer to the shear wave velocity in contrast to that of lowest (acoustic) skew-symmetric mode (A_0) which tends to the Rayleigh wave speed. The velocity of the acoustic symmetric mode (S_0) becomes dispersionless. The acoustic skew symmetric (A_0) mode is observed to be most affected and sensitive. The profiles of attenuation coefficient of acoustic modes are noticed to be highly dispersive. It is observed that the phase velocity profile in case of ideal liquid lies between those of high viscous and low viscous liquid at all wave numbers. Significant effect of thermal conductivity ratio has been observed on dispersion curves, attenuation coefficient profiles in the considered material plate.

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References

1. I. A. Viktorov, *Rayleigh and Lamb waves*, Physical theory and application, Plenum Press, New York, 1967.
2. P. Chadwick and L. T. C. Seet, Wave propagation in transversely isotropic heat conducting elastic material, *Mathematica*, **17** (1970) 225-244.

3. R. S. Dhaliwal and H. H. Sherief, Generalized Thermoelasticity for anisotropic media, *Quart. Appl. Math.*, **38** (1980) 355-376.
4. R. D. Watkins, W. H. B. Cooper, A. B. Gillespie and R. B. Pike, The attenuation of Lamb waves in the presence of liquid. *Ultrasonics*, **20** (1982) 257-264.
5. C. Massalas, A. Dalamangas and Leonditsis, Symmetric non-linear response of a circular plate due to heat flux at $z = h/2$, *Rev. Roum. Sci. Tech. Mech. Appl.*, **28** (1983) 643-650.
6. K. F. Graff, *Wave Motion in Elastic Solids*, Dover, New York, 1991.
7. Z. Zhu and J. Wu, The propagation of Lamb waves in a plate bordered with a viscous liquid, *Journal of Acoustical Society of America*, **98** (1995) 1057-1067.
8. J. N. Sharma and V. Pathania, Circular crested waves in anisotropic plates bordered with inviscid liquid, *International Journal of Engineering Science*, **42** (2004) 99-121.
9. J. N. Sharma and R. Sharma, Propagation characteristics of Lamb waves in a thermo-viscoelastic plate loaded with viscous fluid layers, *International Journal of Applied Mathematics and Mechanics*, **6** (2010) 1-20.
10. Fabien Josse, Zack A. Shana, David E. Radtke and Daniel T. Haworth, Analysis of piezoelectric Bulk-Acoustic-Wave Resonators as detectors in viscous conductive liquids, *IEEE Transaction on Ultrasonics Ferroelectrics and Frequency control*, **37**(1990).
11. C. H. Yang and C. J. Shue, Guided waves propagating in a piezoelectric plate immersed in a conductive fluid, *NDT&E International*, **34** (2001) 199-206.
12. Yung-Chun Lee and Shi Hao Kuo, Leaky Lamb wave of a piezoelectric plate subjected to conductive fluid loading: Theoretical analysis and numerical calculation, *Journal of Applied Physics*, 100 073519 (2006).