

Related Fixed Point Theorem of Set Valued Mappings in Three ε -Chainable Menger Spaces

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Abstract: In this paper, we prove a related fixed point theorem for set-valued mappings in three ε -chainable Menger spaces using implicit relations. Some examples are also furnished to support the results. Our main result generalizes the result of Jain et al. ¹ and extends the result of Beg and Chauhan ², Fisher and T"urko"glu ³.

Key Words: Fixed Point, Set-Valued Mapping, Menger Space, Three Metric Space, Implicit Relations.

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1. Introduction

There have been a number of generalizations of metric space. One such generalization is Menger space introduced in 1942 by Menger ⁴ who was use distribution functions instead of nonnegative real numbers as valued of the metric. Schweizer and Sklar ⁵ studied this concept and then the important development of Menger space theory was due to Sehgal and Bharucha-Reid ⁶. For more results in this field see ⁷⁻¹⁵.

Fisher in ¹⁶⁻¹⁷ initiated the study of conditions for the existence of a relation connecting the fixed points of two mappings in two different metric spaces. Afterward, Fisher and T"urko"glu ¹⁸ proved a related fixed point theorem for set valued mappings in two metric spaces. Chourasia and Fisher ¹⁹ established a related fixed point theorem for two pairs of set-valued mappings in two metric spaces. Several researchers have extensively developed related fixed point theorems in metric spaces. Pant ²⁰ firstly studied related fixed point theorems for single valued mappings in two complete Menger spaces and generalized the results of Fisher ¹⁶⁻¹⁷. Pant and

Kumar ²¹ extended the results of Namdeo et al. ²² to two pairs of single-valued mappings. For more results ²³⁻²⁵.

2. Preliminaries

Definition 2.1: A triangular norm Δ (shortly t-norm) is a binary operation on the unit interval $[0, 1]$ such that for all $a, b, c, d \in [0, 1]$ following conditions are satisfied

- (i) $a \Delta 1 = a$;
- (ii) $a \Delta b = b \Delta a$;
- (iii) $a \Delta b \leq c \Delta d$ whenever $a \leq c$ and $b \leq d$;
- (iv) $a \Delta b (b \Delta c) = (a \Delta b) \Delta c$.

Examples 2.1: A triangular norm Δ (or t-norm) are

$$a \Delta b = \min \{a, b\}, a \Delta_P b = ab$$

$$\text{and } a \Delta_L b = \max \{a + b - 1, 0\}.$$

Definition 2.2: A mapping $F: R \rightarrow R^+$ is called distribution function if it is non-decreasing, left continuous with $\inf\{F(t): t \in R\} = 0$ and $\sup\{F(t): t \in R\} = 1$, such that $F(-\infty) = 0, F(\infty) = 1$.

We will denote by Δ the family of all distribution functions on $(-\infty, \infty)$. $\mathcal{H}$ (also known as Heaviside function) is a special element of Δ defined by

$$\mathcal{H}(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

If X is a nonempty set, $F: X \times X \rightarrow \Delta$ is called a probabilistic distance on X is usually denoted by $F_{x,y}$.

Definition⁵ 2.3 : The ordered pair (X, F) is called a probabilistic metric space (shortly PM-space) if X is a nonempty set and F is a probabilistic distance satisfying the following conditions

$$(M-0) \quad F_{x,y}(t) = 1 = \mathcal{H}(t) \Leftrightarrow x = y;$$

$$(M-1) \quad F_{x,y}(0) = 0;$$

$$(M-2) \quad F_{x,y}(t) = F_{y,x}(t);$$

(M-3) $F_{x,z}(t) = 1, F_{z,y}(s) = 1 \Rightarrow F_{x,y}(t+s) = 1$, for all $x, y, z \in X$ and $t, s > 0$.

The ordered triple (X, F, Δ) is called Menger space if (X, F) is a PM-space, Δ is a t -norm and the following condition is also satisfies

(M-4) $F_{x,y}(t+s) \geq F_{x,z}(t) \Delta F_{z,y}(s)$, or $F_{x,y}(t+s) \geq \Delta(F_{x,z}(t), F_{z,y}(s))$, for all $x, y, z \in X$ and $t, s > 0$.

Proposition⁶ 2.1: Let (X, d) be a metric space. Then the metric d induces a distribution function F defined by $F_{x,y}(t) = \mathcal{H}(t - d(x, y))$ for all $x, y \in X$ and $t > 0$. If t -norm Δ is a $\Delta b = \min \{a, b\}$ for all $a, b, \in [0, 1]$ then (X, F, Δ) is a Menger space. Further, (X, F, Δ) is a complete Menger space if (X, d) is complete.

Definition¹² 2.4: Let (X, F, Δ) be a Menger space and Δ be a continuous t -norm

- (a) A sequence $\{x_n\}$ in X is said to be converge to a point x in X (written $x_n \rightarrow x$) iff for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n, x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.
- (b) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n, x_{n+p}}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$ and $p > 0$.
- (c) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Remark 2.1: If Δ is a continuous t -norm, it follows from (FM-4) that the limit of sequence in Menger space is uniquely determined.

Lemma¹⁵ 2.1: Let $\{x_n\}$ be a sequence in a Menger space (X, F, Δ) with continuous t -norm Δ and $t \Delta t \geq t$. If there exists a constant $k \in (0, 1)$ such that

$$F_{x_n, x_{n+1}}(kt) \geq F_{x_{n-1}, x_n}(t)$$

for all $t > 0$ and $n = 1, 2, \dots$, then $\{x_n\}$ is a Cauchy sequence in X .

Lemma 2.2: Let (X, F, Δ) be a Menger space. Then for all $x, y \in X$, $F_{x,y}(\cdot)$ is non decreasing.

Lemma¹⁵ 2.3: Let (X, F, Δ) be a Menger space. If there exists a constant $k \in (0, 1)$ such that

$$F_{x,y}(kt) \geq F_{x,y}(t),$$

for all $t > 0$ with fixed $x, y \in X$, then $x = y$.

Definition 2.5: Let (X, F, Δ) be a Menger space and $\varepsilon > 0$. A finite sequence $x = x_0, x_1, \dots, x_n = y$ is called ε -chain from x to y if

$$F_{x_i, x_{i+1}}(t) > 1 - \varepsilon, \quad \text{for all } t > 0 \text{ and } i = 1, 2, 3, \dots, n.$$

A Menger space (X, F, Δ) is called ε -chainable if for any $x, y \in X$, there exist an ε -chain from x to y .

Definition⁸ 2.6: Let (X, F, Δ) be a Menger space and U a nonempty subset of X . Then U is said to be probabilistically bounded if

$$\sup_{t>0} \inf_{x,y \in U} F_{x,y}(t) = 1.$$

If X is itself probabilistically bounded, then X is said to be a probabilistically bounded space.

Let $CB(X)$ be a family of nonempty closed and bounded subsets of a Menger space (X, F, Δ) . For all $U, V \in CB(X)$ and for every $t > 0$, we define

$$D(F_{u,v}(t)) = \sup \{ F_{a,b}(t); a \in U, b \in V \}$$

$$\text{and} \quad \delta(F_{u,v}(t)) = \inf \{ F_{a,b}(t); a \in U, b \in V \}.$$

If the set U consists of a single point a , we write $\delta(F_{u,v}(t)) = \delta(F_{a,v}(t))$. Also if the set V consists of a single point b , we write $\delta(F_{u,v}(t)) = \delta(F_{a,b}(t))$. It follows immediately from the definition that

- (i) $\delta(F_{u,v}(t)) = \delta(F_{v,u}(t)) \geq 0$,
- (ii) $\delta(F_{u,v}(t)) = 1$, if and only if $U = V = \{a\}$, for all $U, V \in CB(X)$.

Proposition 2.2: In a Menger space (X, F, Δ) if $a \Delta a \geq a$ for all $a \in [0, 1]$, then $a \Delta a = \min\{a, b\}$, for all $a, b \in [0, 1]$.

3. Implicit Relation

Let Γ^* be the set of all real continuous functions $\Gamma(t_1, t_2, t_3, t_4, t_5): (\mathbb{R}^+)^5 \rightarrow \mathbb{R}$ which is non-decreasing in the first and last argument satisfying the following conditions

(Γ_a) For $u, v \geq 0$, $\Gamma(u, v, v, u, v) \geq 0$, implies that $u \geq v$.

(Γ_b) $\Gamma(u, 1, 1, u, 1) \geq 0$, or $\Gamma(u, u, 1, 1, u) \geq 0$ implies $u \geq 1$.

Example 3.1: If $\Gamma(t_1, t_2, t_3, t_4, t_5) = t_1 - at_2 - bt_3 - ct_4 + t_5$, when $a, b, c \geq 0$ and $a + b + c \geq 2$. Here $u, v \geq 0$, we have in (Γ_a)

$$\Gamma(u, v, v, u, v) = u - av - bv - cu + v \geq 0 \Rightarrow u \geq v,$$

$$\text{if } a + b + c \geq 2 \text{ or } \frac{a+b-1}{1-c} \geq 1.$$

$$\text{Also in } (\Gamma_b), \quad \Gamma(u, 1, 1, u, 1) = u - a.1 - b.1 - cu + 1 \geq 0 \Rightarrow u \geq 1,$$

$$\text{if } a + b + c \geq 2,$$

$$\text{or } \Gamma(u, u, 1, 1, u) = u - au - b.1 - c.1 + u \geq 0 \Rightarrow u \geq 1, \text{ if } a + b + c \geq 2,$$

Example 3.2: Define $\Gamma(t_1, t_2, t_3, t_4, t_5) = 30t_1 - 18t_2 - 7t_3 - 6t_4 + t_5$ then from (Γ_a)

$$\Gamma(u, v, v, u, v) = (24u - 24v) \geq 0 \Rightarrow u \geq v,$$

and from (Γ_b)

$$\Gamma(u, 1, 1, u, 1) = 30u - 18.1 - 7.1 - 6.u + 1 \geq 0 \Rightarrow u \geq 1$$

$$\text{or } \Gamma(u, u, 1, 1, u) = 30u - 18u - 7.1 - 6.1 + u \geq 0 \Rightarrow u \geq 1.$$

Example 3.3: Let $\Gamma(t_1, t_2, t_3, t_4, t_5) = pt_1 - q \max\{t_2, t_3\} - rt_4 + t_5$, $p, q, r \geq 0$ and $(q + r - p) \geq 1$.

From (Γ_a),

$$\Gamma(u, v, v, u, v) = pu - q \max\{v, v\} + ru + v = (q + r)u - (p - 1)v \geq 0$$

$$\Rightarrow u \geq v, \text{ if } (q + r - p) \geq 1,$$

from (Γ_b),

$$\Gamma(u, 1, 1, u, 1) = pu - q \max\{1, 1\} - ru + 1 = (p - r)u - (q - 1) \geq 0 \\ \Rightarrow u \geq 1, \text{ if } (q + r - p) \geq 1,$$

$$\text{or } \Gamma(u, u, 1, 1, u) = pu - q \max\{u, 1\} - r.1 + u$$

$$\Rightarrow (p + 1)u - (q + r) \geq 0 \Rightarrow u \geq 1, \text{ if } (q + r - p) \geq 1.$$

Example 3.4: Let $\Gamma(t_1, t_2, t_3, t_4, t_5) = t_1^3 - \frac{2t_3^2 t_4^2}{t_2 + t_5}$

$$\text{from } (\Gamma_a) \quad \Gamma(u, v, v, u, v) = u^3 - \frac{2v^2 u^2}{v + v} = u^2(u - v) \geq 0 \Rightarrow u \geq v,$$

$$\text{from } (\Gamma_b) \quad \Gamma(u, 1, 1, u, 1) = u^3 - \frac{2.1^2 u^2}{1 + 1} = u^2(u - 1) \geq 0 \Rightarrow u \geq 1, \Rightarrow u \geq 1,$$

$$\text{or } \Gamma(u, u, 1, 1, u) = u^3 - \frac{2.1^2 1^2}{u + u} = (u^5 - 1) \geq 0 \Rightarrow u \geq 1.$$

4. Main Result

Theorem 4.1: Let (X, F, Δ) , (X, G, Δ) , and (Z, H, Δ) be three complete Menger spaces, where Δ is a continuous t -norm. If P is a continuous mapping of X into $CB(Y)$, Q is a continuous mapping of Y into $CB(Z)$, and R is a mapping of Z into $CB(X)$ satisfying the inequalities

$$(4.1) \quad \Gamma\{\delta(F_{RQP_x, RQP_{x'}}(kt)), F_{x, x'}(t), \delta(F_{x, RQP_x}(t)), \delta(F_{x', RQP_{x'}}(t)), \delta(G_{P_x, P_{x'}}(t)), \delta(H_{QP_x, QP_{x'}}(t))\} \geq 0,$$

$$(4.2) \quad \Gamma\{\delta(G_{PRQ_x, PRQ_{x'}}(kt)), G_{y, y'}(t), \delta(G_{y, PRQ_y}(t)), \delta(G_{y', PRQ_{y'}}(t)), \delta(H_{Q_y, Q_{y'}}(t)), \delta(F_{RQ_y, RQ_{y'}}(t))\} \geq 0.$$

$$(4.3) \quad \Gamma\{\delta(H_{QPR_z, QPR_{z'}}(kt)), H_{z, z'}(t), \delta(H_{z, QPR_z}(t)), \delta(H_{z', QPR_{z'}}(t)), \delta(F_{Rz, Rz'}(t)), \delta(G_{PR_z, PR_{z'}}(t))\} \geq 0,$$

for all $x, x' \in X$, $y, y' \in Y$, $z, z' \in Z$, $k \in (0, 1)$ and $t > 0$, then RQP has a unique fixed point u in X , PRQ has a unique fixed point v in Y , and QPR has a unique fixed point w in Z . Further $Pu = \{v\}$, $Qv = \{w\}$, and $Rw = \{u\}$.

Proof: Let $x = x'$ be an arbitrary point in X . We define the sequences $\{x_n\}$ in X , $\{y_n\}$ in Y , and $\{z_n\}$ in Z inductively as follows. We choose a point y_1 in Px_1 , then a point z_1 in Qz_1 , and then a point x_2 in Rz_1 . In general, having chosen x_n in X , y_n in Y , and z_n in Z , choose a point x_{n+1} in Rz_n , then a point y_{n+1} in Px_{n+1} , and then a point z_{n+1} in Qy_{n+1} for $n = 1, 2, \dots$. Applying inequality (4.1), we get

$$\begin{aligned}
& \Gamma\{\delta(F_{RQPX_n, RQPX_{n+1}}(kt)), F_{X_n, X_{n+1}}(t), \delta(F_{X_n, RQPX_n}(t)), \delta(F_{X_{n+1}, RQPX_{n+1}}(t)), \\
& \delta(G_{PX_n, PX_{n+1}}(t)), \delta(H_{QPX_n, QPX_{n+1}}(t))\} \geq 0, \\
& \text{or, } \Gamma\{\delta(F_{RQPX_n, RQPX_{n+1}}(kt)), 1, \delta(F_{RQPX_{n-1}, RQPX_n}(t)), \delta(F_{RQPX_n, RQPX_{n+1}}(t)), \\
& \delta(G_{PRQY_{n-1}, PRQY_n}(t)), \delta(H_{QPRZ_{n-1}, QPRZ_n}(t))\} \geq 0, \\
& \text{or, } \Gamma\{\delta(F_{RQPX_n, RQPX_{n+1}}(kt)), 1, \delta(F_{RQPX_{n-1}, RQPX_n}(t)), \\
& \delta(G_{PRQY_{n-1}, PRQY_n}(t)), \delta(H_{QPRZ_{n-1}, QPRZ_n}(t))\} \geq 0, \quad (a)
\end{aligned}$$

Using inequality (4.2), we get

$$\begin{aligned}
& \Gamma\{\delta(G_{PRQY_n, PRQY_{n+1}}(kt)), G_{Y_n, Y_{n+1}}(t), \delta(G_{Y_n, PRQY_n}(t)), \delta(G_{Y_{n+1}, PRQY_{n+1}}(t)), \\
& \delta(H_{QY_n, QY_{n+1}}(t)), \delta(F_{RQY_n, RQY_{n+1}}(t))\} \geq 0, \\
& \Gamma\{\delta(G_{PRQY_n, PRQY_{n+1}}(kt)), 1, \delta(G_{PRQY_{n-1}, PRQY_n}(t)), \delta(G_{PRQY_n, PRQY_{n+1}}(t)), \\
& \delta(H_{QPRZ_{n-1}, QPRZ_n}(t)), \delta(F_{RQPX_{n-1}, RQPX_n}(t))\} \geq 0, \\
& \Gamma\{\delta(G_{PRQY_n, PRQY_{n+1}}(kt)), 1, \delta(G_{PRQY_{n-1}, PRQY_n}(t)), \\
& \delta(H_{QPRZ_{n-1}, QPRZ_n}(t)), \delta(F_{RQPX_{n-1}, RQPX_n}(t))\} \geq 0, \quad (b)
\end{aligned}$$

on using inequality (a).

Further using (4.3), we get

$$\begin{aligned}
& \Gamma\{\delta(H_{QPRZ_n, QPRZ_{n+1}}(kt)), H_{Z_n, Z_{n+1}}(t), \delta(H_{Z_n, QPRZ_n}(t)), \\
& \delta(H_{Z_{n+1}, QPRZ_{n+1}}(t)), \delta(F_{RZ_n, RZ_{n+1}}(t)), \delta(G_{PRZ_n, PRZ_{n+1}}(t))\} \geq 0, \\
& \text{or, } \Gamma\{\delta(H_{QPRZ_n, QPRZ_{n+1}}(kt)), 1, \delta(H_{QPRZ_{n-1}, QPRZ_n}(t)), \\
& \delta(H_{QPRZ_n, QPRZ_{n+1}}(t)), \delta(F_{RQPX_{n-1}, RQPX_n}(t)), \delta(G_{PRQY_{n-1}, PRQY_n}(t))\} \geq 0, \\
& \text{or, } \Gamma\{\delta(H_{QPRZ_n, QPRZ_{n+1}}(kt)), 1, \delta(H_{QPRZ_{n-1}, QPRZ_n}(t)), \\
& \delta(F_{RQPX_{n-1}, RQPX_n}(t)), \delta(G_{PRQY_{n-1}, PRQY_n}(t))\} \geq 0, \quad (c)
\end{aligned}$$

on using inequality (a) and (b). By induction using inequality (a), (b) and (c), we get

using condition (Γ_a) , we have

$$\delta(F_{RQP x_n, RQP x_{n+1}}(kt)) \geq \delta(F_{RQP x_{n-1}, RQP x_n}(t))$$

this implies $F_{RQP x_n, RQP x_{n+1}}(kt) \geq F_{RQP x_{n-1}, RQP x_n}(t)$

or

$$(4.4) \quad F_{x_n, x_{n+1}}(kt) \geq F_{x_{n-1}, x_n}(t).$$

Similarly we also have

$$(4.5) \quad F_{x_{n-1}, x_n}(kt) \geq F_{x_{n-2}, x_{n-1}}(t).$$

From (4.4) and (4.5), we have

$$F_{x_n, x_{n+1}}(kt) \geq F_{x_{n-1}, x_n}(t) \geq F_{x_{n-2}, x_{n-1}}(t/k) \dots \geq F_{x_1, x_2}(t/k^n) \rightarrow 1$$

as $n \rightarrow \infty$, for any, $t > 0$.

For each $\varepsilon > 0$ and each $t > 0$, we can choose $n_0 \in \mathbb{N}$ such that

$$F_{x_n, x_{n+1}}(t) > 1 - \varepsilon, \quad \text{for all } n > n_0.$$

For $m, n \in \mathbb{N}$, we suppose $m \geq n$. Then we have

$$\Gamma\{F_{x_m, x_n}(t); F_{x_n, x_{n+1}}(t/(m-n)); F_{x_{n+1}, x_{n+2}}(t/(m-n)); \\ \dots F_{x_{m-1}, x_m}(t/(m-n))\} \geq 0,$$

$$\text{or} \quad \Gamma\{F_{x_m, x_n}(t); (1-\varepsilon); (1-\varepsilon) \dots (1-\varepsilon)\} \geq 0,$$

i.e.

$$(4.6) \quad F_{x_n, x_m}(t) \geq 1 - \varepsilon.$$

From definition 2.4 and (Γ_b) , it is shown that $\{x_n\}$ is a Cauchy sequence in complete Mengar space X .

Using inequality (4.2), we get

$$\Gamma\{\delta(G_{\text{PRQY}_n, \text{PRQY}_{n+1}}(kt)); G_{y_n, y_{n+1}}(t); \delta(G_{y_n, \text{PRQY}_n}(t)); \\ \delta(G_{y_{n+1}, \text{PRQY}_{n+1}}(t)); \delta(H_{QY_n, QY_{n+1}}(t/2))\Delta\delta(F_{\text{RQY}_n, \text{RQY}_{n+1}}(t/2))\} \geq 0,$$

Using lemma 2.4(b), we have

$$\Gamma\{\delta(G_{\text{PRQY}_n, \text{PRQY}_{n+1}}(kt)); \delta(G_{\text{PRQY}_{n-1}, \text{PRQY}_n}(t)); \\ \delta(G_{\text{PRQY}_{n-1}, \text{PRQY}_n}(t)); \delta(G_{\text{PRQY}_n, \text{PRQY}_{n+1}}(kt)); \delta(G_{\text{PRQY}_{n-1}, \text{PRQY}_n}(t))\} \geq 0,$$

Using condition (Γ_a) , we have

$$\delta(G_{\text{PRQY}_n, \text{PRQY}_{n+1}}(kt)) \geq \delta(G_{\text{PRQY}_{n-1}, \text{PRQY}_n}(t))$$

this implies

$$(4.7) \quad G_{y_{n+1}, y_{n+2}}(kt) \geq G_{y_n, y_{n+1}}(t).$$

Similarly we also have

$$(4.8) \quad G_{y_n, y_{n+1}}(kt) \geq G_{y_{n-1}, y_n}(t).$$

From (4.7) and (4.8), we have

$$G_{y_n, y_{n+1}}(kt) \geq G_{y_{n-1}, y_n}(t) \geq G_{y_{n-2}, y_{n-1}}(t/k) \geq \dots \geq G_{y_1, y_2}(t/k^n) \rightarrow 1,$$

as $n \rightarrow \infty$, for any, $t > 0$. For each $\varepsilon' > 0$ and each $t > 0$, we can choose $n_{\varepsilon'} \in \mathbb{N}$ such that

$$G_{y_n, y_{n+1}}(t) > 1 - \varepsilon', \quad \text{for all } n > n_{\varepsilon'}$$

For $m', n' \in \mathbb{N}$, we suppose $m' \geq n'$. Then we have

$$\Gamma\{G_{x_{m'}, x_{n'}}(t); G_{x_{n'}, x_{n'+1}}(t/(m' - n')); G_{x_{n'+1}, x_{n'+2}}(t/(m' - n')); \\ \dots G_{x_{m'-1}, x_{m'}}(t/(m' - n'))\} \geq 0,$$

$$\text{or } \Gamma\{G_{x_{m'}, x_{n'}}(t); (1 - \varepsilon'); (1 - \varepsilon') \dots (1 - \varepsilon')\} \geq 0,$$

$$(4.9) \quad G_{x_{n'}, x_{m'}}(t) \geq 1 - \varepsilon'.$$

From definition 2.4 and (Γ_b) , it is shown that $\{y_n\}$ is a Cauchy sequence in complete Mengar space Y . Using condition (Γ_a) , we have

$$\delta(H_{QPRZ_n, QPRZ_{n+1}}(kt)) \geq \delta(H_{QPRZ_{n-1}, QPRZ_n}(t)),$$

this implies

$$(4.10) \quad H_{QPRZ_n, QPRZ_{n+1}}(kt) \geq H_{QPRZ_{n-1}, QPRZ_n}(t) \Rightarrow H_{Z_n, Z_{n+1}}(kt) \geq H_{Z_{n-1}, Z_n}(t).$$

Similarly we also have

$$(4.11) \quad H_{Z_{n-1}, Z_n}(kt) \geq H_{Z_{n-2}, Z_{n-1}}(t),$$

From (4.10) and (4.11), we have

$$H_{Z_n, Z_{n+1}}(kt) \geq H_{Z_{n-1}, Z_n}(t) \geq H_{Z_{n-2}, Z_{n-1}}(t/k) \geq \dots \geq H_{Z_1, Z_2}(t/k^n) \rightarrow 1$$

as $n \rightarrow \infty$, for any, $t > 0$.

For each $\varepsilon > 0$ and each $t > 0$, we can choose $n_\varepsilon \in \mathbb{N}$ such that

$$H_{Z_n, Z_{n+1}}(t) > 1 - \varepsilon, \quad \text{for all } n > n_\varepsilon.$$

For $m, n \in \mathbb{N}$, we suppose $m \geq n$. Then we have

$$\Gamma\{F_{x_m, x_n}(t); F_{x_n, x_{n+1}}(t/(m-n)); F_{x_{n+1}, x_{n+2}}(t/(m-n)); \dots F_{x_{m-1}, x_m}(t/(m-n))\} \geq 0,$$

$$\text{or} \quad \Gamma\{F_{x_m, x_n}(t); (1-\varepsilon); (1-\varepsilon); \dots (1-\varepsilon)\} \geq 0,$$

$$(4.12) \quad F_{x_n, x_m}(t) \geq 1 - \varepsilon.$$

From definition 2.4 and (Γ_b) , it is shown that $\{z_n\}$ is a Cauchy sequence in complete Mengar space Z . Since X is complete Mengar space and $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequence so one has a limit u in X . Similarly, the sequences $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences with limits v and w in the complete Menger spaces Y and Z , respectively. Then we have

$$\lim_{n \rightarrow \infty} RQP x_n = \{u\} = \lim_{n \rightarrow \infty} RQ y_n$$

$$(4.13) \quad \lim_{n \rightarrow \infty} PRQ y_n = \{v\} = \lim_{n \rightarrow \infty} PR z_n$$

$$(4.14) \quad \lim_{n \rightarrow \infty} QPR z_n = \{w\} = \lim_{n \rightarrow \infty} QP x_n$$

Using the continuity of P and Q , we have

$$(4.15) \quad \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} P x_n = P u = \{v\}$$

$$(4.16) \quad \lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} Q y_n = Q v = \{w\}$$

and then we see that $Q P u = \{w\}$.

Now we show that u is a fixed point of RQP . Applying inequality (4.1), we have

$$\begin{aligned} & \Gamma \{ \delta(F_{RQP u, RQP x_{n+1}}(kt)); \delta(F_{u, x_{n+1}}(t)); \delta(F_{u, RQP u}(t)) \\ & \delta(F_{x_{n+1}, RQP x_{n+1}}(kt)); \delta(G_{P u, P x_{n+1}}(t/2)) \Delta \delta(H_{QP u, QP x_{n+1}}(t/2)) \} \geq 0. \end{aligned}$$

Using lemma 2.4(a), we have

$$\begin{aligned} & \Gamma \{ \delta(F_{RQP u, RQP x_{n+1}}(kt)); \delta(F_{RQP u, RQP x_n}(t)); \delta(F_{RQP u, RQP x_n}(t)); \\ & \delta(F_{RQP u, RQP x_{n+1}}(kt)); \delta(F_{RQP u, RQP x_n}(t)) \} \geq 0, \end{aligned}$$

Using condition (Γ_a) , we have

$$\begin{aligned} & \delta(F_{RQP u, RQP x_{n+1}}(kt)) \geq \delta(F_{RQP u, RQP x_n}(t)) \\ & \Rightarrow F_{RQP u, u}(kt) \geq F_{RQP u, u}(t), \end{aligned}$$

which implies that $RQP u = \{u\}$ and so u is a fixed point of RQP . We now have

$$(4.17) \quad RQP u = R w = \{u\},$$

on using (4.17) and then

$$(4.18) \quad PRQ v = PRQP u = \{v\}, \quad QPR w = QPRQ v = \{w\},$$

Hence v and w are fixed points of PRQ and QPR , respectively. Further, we see that

$$RQ v = \{u\}, \quad PR w = \{v\}.$$

To prove the uniqueness of u , suppose that RQP has another fixed point u' . Then, using inequality (4.1), we have

$$\Gamma\{\delta(F_{RQPu', RQPu'}(kt)); \delta(F_{u', u'}(t)); \delta(F_{u', RQPu'}(t)); \\ \delta(F_{RQPu', u'}(kt)); \delta(G_{Pu', Pu'}(t/2))\Delta\delta(H_{QPu', QPu'}(t/2))\} \geq 0,$$

using lemma 2.4(a), we have

$$\Gamma\{\delta(F_{RQPu', RQPu'}(kt)); 1; 1; \delta(F_{RQPu', RQPu'}(kt)); 1\} \geq 0$$

using condition (Γ_b) , we have

$$\delta(F_{RQPu', RQPu'}(kt)) \geq 1,$$

$$(4.19) \quad \Rightarrow \delta(F_{RQPu', RQPu'}(kt)) \geq \delta(F_{RQPu', RQPu'}(t)).$$

Next, using inequality (4.2), we have

$$\delta(G_{Pu', Pu'}(kt)) \geq \delta(G_{PRQPu', PRQPu'}(kt))$$

$$\text{and } \Gamma\{\delta(G_{PRQPu', PRQPu'}(kt)); \delta(G_{Pu', Pu'}(t)); \delta(G_{Pu', PRQPu'}(t)); \\ \delta(G_{PRQPu', PRQPu'}(kt)); \delta(H_{QPu', QPu'}(t/2))\Delta\delta(F_{RQPu', RQPu'}(t/2))\} \geq 0,$$

using lemma 2.4(b), we have

$$\Gamma\{\delta(G_{PRQPu', PRQPu'}(kt)); 1; 1; \delta(G_{PRQPu', PRQPu'}(kt)); 1\} \geq 0$$

using condition (Γ_b) , we have

$$\delta(G_{PRQPu', PRQPu'}(kt)) \geq 1$$

$$(4.20) \quad \delta(G_{PRQPu', PRQPu'}(kt)) \geq \delta(G_{PRQPu', PRQPu'}(t)).$$

Again using inequality (4.3), we obtain

$$\delta(H_{QPu', QPu'}(kt)) \geq \delta(H_{QPRQPu', QPRQPu'}(kt))$$

$$\text{or } \Gamma\{\delta(H_{QPRQPu', QPRQPu'}(kt)); \delta(H_{QPu', QPu'}(t)); \delta(H_{QPu', QPRQPu'}(t)); \\ \delta(H_{QPRQPu', QPRQPu'}(kt)); \delta(F_{RQPu', RQPu'}(t/2))\Delta\delta(G_{PRQPu', PRQPu'}(t/2))\} \geq 0,$$

using lemma 2.4(c), we have

$$\Gamma\{\delta(H_{QPRQPu',QPRQPu'}(kt)); 1; 1; \delta(H_{QPRQPu',QPRQPu'}(kt); 1\} \geq 1$$

using condition (Γ_b) , we have

$$\delta(H_{QPRQPu',QPRQPu'}(kt)) \geq 1,$$

$$(4.21) \quad \delta(H_{QPRQPu',QPRQPu'}(kt)) \geq \delta(H_{QPRQPu',QPRQPu'}(t)).$$

Using inequalities (4.19), (4.20), and (4.21), we obtain

$$\begin{aligned} \delta(F_{RQPu',RQPu'}(kt)) &\geq \delta(G_{PRQPu',PRQPu'}(kt)) \geq \\ \delta(H_{QPRQPu',QPRQPu'}(kt)) &\geq \delta(F_{RQPu',RQPu'}(t)) \geq \\ \delta(G_{PRQPu',PRQPu'}(t)) &\geq \delta(H_{QPRQPu',QPRQPu'}(t)) \geq \\ \delta(F_{RQPu',RQPu'}(\frac{t}{k})) &\geq \delta(G_{PRQPu',PRQPu'}(\frac{t}{k})) \geq \\ \delta(H_{QPRQPu',QPRQPu'}(\frac{t}{k})) &\geq \delta(F_{RQPu',RQPu'}(\frac{t}{k^2})) \geq \\ \delta(G_{PRQPu',PRQPu'}(\frac{t}{k^2})) &\geq \delta(H_{QPRQPu',QPRQPu'}(\frac{t}{k^2})) \geq \\ \geq \dots\dots\dots &\geq \delta(F_{RQPu',RQPu'}(\frac{t}{k^{2n}})) \geq \delta(G_{PRQPu',PRQPu'}(\frac{t}{k^{2n}})) \\ &\geq \delta(H_{QPRQPu',QPRQPu'}(\frac{t}{k^{2n}})). \\ \Rightarrow \delta(F_{RQPu',RQPu'}(kt)) &\geq \delta(F_{RQPu',RQPu'}(t)) \geq \delta(F_{RQPu',RQPu'}(\frac{t}{k})) \\ &\geq \delta(F_{RQPu',RQPu'}(\frac{t}{k^2})) \geq \dots\dots\dots \geq \delta(F_{RQPu',RQPu'}(\frac{t}{k^{2n}})), \\ \delta(G_{PRQPu',PRQPu'}(kt)) &\geq \delta(G_{PRQPu',PRQPu'}(t)) \geq \delta(G_{PRQPu',PRQPu'}(\frac{t}{k})) \geq \\ \delta(G_{PRQPu',PRQPu'}(\frac{t}{k^2})) &\geq \dots\dots\dots \geq \delta(G_{PRQPu',PRQPu'}(\frac{t}{k^{2n}})), \\ \text{and} & \\ \delta(H_{QPRQPu',QPRQPu'}(kt)) &\geq \delta(H_{QPRQPu',QPRQPu'}(t)) \geq \delta(H_{QPRQPu',QPRQPu'}(\frac{t}{k})) \geq \end{aligned}$$

$$\delta(H_{QPRQPu', QPRQPu'}(\frac{t}{k^2})) \geq \dots \geq \delta(H_{QPRQPu', QPRQPu'}(\frac{t}{k^{2n}})).$$

for all $n = 1, 2, \dots$

Since
$$\delta(F_{RQPu', RQPu'}(\frac{t}{k^{2n}})) \rightarrow 1 \text{ as } n \rightarrow \infty,$$

$$\delta(G_{PRQPu', PRQPu'}(\frac{t}{k^{2n}})) \rightarrow 1 \text{ as } n \rightarrow \infty$$

and
$$\delta(H_{QPRQPu', QPRQPu'}(\frac{t}{k^{2n}})) \rightarrow 1 \text{ as } n \rightarrow \infty,$$

it follows that $RQPu' = u'$, $PRQPu' = Pu'$ and $QPRQPu' = QPu'$ are singleton in F , G and H . Again using inequality (4.1), lemma 2.4(a), and lemma 2.3, we have

$$\begin{aligned} & \Gamma\{\delta(F_{RQPu, RQPu'}(kt)); \delta(F_{u, u'}(t)); \delta(F_{u, u}(t)); \\ & \delta(F_{RQPu', RQPu'}(kt)); \delta(G_{Pu, Pu'}(t/2))\Delta\delta(H_{QPu, QPu'}(t/2))\} \geq 0, \\ & \Gamma\{\delta(F_{RQPu, RQPu'}(kt)); \delta(F_{u, u'}(t)); \delta(F_{u, u}(t)); \\ & \delta(F_{RQPu', RQPu'}(kt)); \delta(F_{RQPu, RQPu'}(t))\} \geq 0, \\ (4.22) \quad & \Gamma\{\delta(F_{RQPu, RQPu'}(t)); \delta(F_{RQPu, RQPu'}(t)); 1; 1; \delta(F_{RQPu, RQPu'}(t))\} \geq 0, \end{aligned}$$

using condition (Γ_b) , we have

$$(4.23) \quad \delta(F_{RQPu, RQPu'}(t)) \geq 1 \Rightarrow F_{RQPu, RQPu'}(t) \geq 1.$$

Similaraly, using inequality (4.1), lemma 2.4(b), and lemma 2.3, we have

$$\begin{aligned} & \Gamma\{\delta(G_{Pu, PRQPu'}(kt)); \delta(G_{Pu, Pu'}(t)); \delta(G_{Pu, PRQPu}(t)); \\ & \delta(G_{Pu', PRQPu'}(kt)); \delta(H_{QPu, QPu'}(t/2))\Delta\delta(F_{RQPu, RQPu'}(t/2))\} \geq 0. \\ & \Gamma\{\delta(G_{PRQPu, PRQPu'}(t)); \delta(G_{PRQPu, PRQPu'}(t)); 1; 1; \delta(G_{PRQPu, PRQPu'}(t))\} \geq 0. \end{aligned}$$

Using condition (Γ_b) , we have

$$(4.24) \quad \delta(G_{PRQPu, PRQPu'}(t)) \geq 1 \Rightarrow G_{PRQPu, PRQPu'}(t) \geq 1.$$

Again, using inequality (4.1), lemma 2.4(c), and lemma 2.3, we have

$$\Gamma\{\delta(H_{QPu, QPRQPu'}(kt)); \delta(H_{QPu, QPu'}(t)); \delta(H_{QPu, QPRQPu}(t)); \\ \delta(H_{QPu', QPRQPu'}(kt)); \delta(F_{RQPu, RQPu'}(t/2))\Delta\delta(G_{PRQPu, PRQPu'}(t/2))\} \geq 0.$$

$$\Gamma\{\delta(H_{QPRQPu, QPRQPu'}(t)); \delta(H_{QPRQPu, QPRQPu'}(t)); 1; 1; \delta(H_{QPRQPu, QPRQPu'}(t))\} \geq 0.$$

Using condition (Γ_b) , we have

$$\delta(H_{QPRQPu, QPRQPu'}(t)) \geq 1,$$

$$(4.25) \quad H_{QPRQPu, QPRQPu'}(t) \geq 1.$$

From (4.23), (4.24) and (4.25),

$$\begin{aligned} RQPu &= RQPu' = \{u\}, \\ PRQPu &= PRQPu' = Pu = \{v\}, \\ QPRQPu &= QPRQPu' = QPu = \{w\}, \end{aligned}$$

which shows the uniqueness of u , v and w .

Example 4.1: Consider $X = [0, 2]$, $Y = [1, 5]$, and $Z = [0, 10]$ and define

$$(4.26) \quad F_{x,y}(t) = G_{x,y}(t) = H_{x,y}(t) = \begin{cases} \frac{t}{t + |x - y|}, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Then (X, F, Δ) , (Y, G, Δ) , and (Z, H, Δ) are three complete Menger spaces with continuous t-norm $\Delta(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. Define the set-valued mappings $P: X \rightarrow CB(Y)$, $Q: Y \rightarrow CB(Z)$ and $R: Z \rightarrow CB(X)$ by

$$\begin{aligned} Px &= [1 + x, 2] \text{ for all } x \in X, \\ Qy &= [2y + 1, 5] \text{ for all } y \in Y \end{aligned}$$

and

$$(4.27) \quad Rz = \begin{cases} [\frac{z}{5}, 1], & \text{if } z \in [0, 5], \\ (1, \frac{z}{5}], & \text{if } z \in (5, 10]. \end{cases}$$

We can easily verify the inequalities with $k = 1/2 \in (0, 1)$. Thus all the conditions of Theorem 4.1 are satisfied and RQP has a unique fixed point $\{1\}$ in $CB(X)$, PRQ has a unique fixed point $\{2\}$ in $CB(Y)$, and QPR has a unique fixed point $\{5\}$ in $CB(Z)$. Hence, $P(1) = \{2\}$, $Q(2) = \{5\}$, and $R(5) = \{1\}$, and so $RQP(1) = RQ(2) = R(5) = \{1\}$, $PRQ(2) = PR(5) = P(1) = \{2\}$, and $QPR(5) = QP(1) = Q(2) = \{5\}$.

Corollary 4.1: Let (X, F, Δ) , (Y, G, Δ) and (Z, H, Δ) be three complete Menger spaces, where Δ is a continuous t-norm. If P is a continuous mapping of X into Y , Q is a continuous mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities

$$(4.28) \quad \Gamma\{F_{RQP_x, RQP_{x'}}(kt); F_{x, x'}(t); F_{x, RQP_x}(t); F_{x', RQP_{x'}}(kt); \\ G_{P_x, P_{x'}}(t/2) \Delta H_{QP_x, QP_{x'}}(t/2)\} \geq 0,$$

$$\Gamma\{G_{PRQP_u, PRQP_{u'}}(kt); G_{y, y'}(t); G_{y, PRQ_y}(t); G_{y', PRQ_{y'}}(kt); \\ H_{Q_y, Q_{y'}}(t/2) \Delta F_{RQ_y, RQ_{y'}}(t/2)\} \geq 0,$$

$$\Gamma\{H_{QPR_z, QPR_{z'}}(kt); H_{z, z'}(t); H_{z, QPR_{z'}}(t); H_{z, QPR_{z'}}(kt); \\ F_{R_z, R_{z'}}(t/2) \Delta G_{PR_z, PR_{z'}}(t/2)\} \geq 0,$$

for all $x, x' \in X$, $y, y' \in Y$, $z, z' \in Z$, $k \in (0, 1)$ and $t > 0$. Then RQP has a unique fixed point u in X , PRQ has a unique fixed point v in Y and QPR has a unique fixed point w in Z . Further, $Pu = v$, $Qv = w$ and $Rw = u$.

Example 4.2: Let $X = [0, 2]$, $Y = [1, 3]$, $Z = [3, 5]$ and $F_{x,y}(t)$, $G_{x,y}(t)$ and $H_{x,y}(t)$ be defined as in example 4.1. Then (X, F, Δ) , (Y, G, Δ) , and (Z, H, Δ) be three complete Menger spaces with continuous t-norm $\Delta(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. Define the mappings $P: X \rightarrow Y$, $Q: Y \rightarrow Z$ and $R: Z \rightarrow X$ by

$$Px = x + 1 \text{ for all } x \in X,$$

$Qy = y + 2$ for all $y \in Y$
and

$$(4.29) \quad Rz = \begin{cases} \frac{z}{4}, & \text{if } z \in [3, 4], \\ \frac{z+1}{5}, & \text{if } z \in (4, 5]. \end{cases}$$

Also the inequalities (4.30) can be verified with $k = 1/2 \in (0, 1)$. Thus all the conditions of Corollary 10 are satisfied and RQP has a unique fixed point 1 in X, PRQ has a unique fixed point 2 in Y, and QPR has a unique fixed point 4 in Z. Further, $P(1) = 2$, $Q(2) = 4$, and $R(4) = 1$; that is, $RQP(1) = RQ(2) = R(4) = 1$, $PRQ(2) = PR(4) = P(1) = 2$, and $QPR(4) = QP(1) = Q(2) = 4$.

The following corollaries are the natural results due to theorem 4.1 and Corollary 4.1.

Corollary 4.2: Let (X, F, Δ) and (Y, G, Δ) be two complete Menger spaces, where Δ is a continuous t -norm. If P is a continuous mapping of X into $CB(Y)$, Q is a mapping of Y into $CB(X)$ satisfying the inequalities

$$(4.30) \quad \begin{aligned} & \Gamma \{ \delta(F_{QP_X, QP_{X'}}(kt)); \delta(F_{x, x'}(t)); \delta(F_{x, QP_X}(kt)); \\ & \delta(F_{x', QP_X}(t/2)) \Delta \delta(G_{P_X, P_{X'}}(t/2)) \} \geq 0, \\ & \Gamma \{ \delta(G_{PQ_Y, PQ_{Y'}}(kt)); \delta(G_{y, y'}(t)); \delta(G_{y, PQ_Y}(t)); \\ & \delta(G_{y', PQ_Y}(t/2)) \Delta \delta(F_{Q_Y, Q_{Y'}}(t/2)) \} \geq 0, \end{aligned}$$

for all $x, x' \in X$, and $y, y' \in Y$, $k \in (0, 1)$ and $t > 0$. Then QP has a unique fixed point u in X and PQ has a unique fixed point v in Y .

Corollary 4.3: Let (X, F, Δ) and (Y, G, Δ) be two complete Menger spaces, where Δ is a continuous t -norm. If P is a continuous mapping of X into Y , Q is a mapping of Y into X satisfying the inequalities

$$\Gamma \{ F_{QP_X, QP_{X'}}(kt); F_{x, x'}(t); F_{x, QP_X}(t); F_{x', QP_X}(kt); G_{P_X, P_{X'}}(t) \} \geq 0,$$

$$(4.31) \quad \Gamma\{G_{PQY, PQY'}(kt); G_{y, y'}(t); G_{y, PQY}(t); G_{y', PQY'}(kt); F_{QY, QY'}(t)\} \geq 0.$$

for all $x, x' \in X$, and $y, y' \in Y$, $k \in (0, 1)$ and $t > 0$. Then QP has a unique fixed point u in X and PQ has a unique fixed point v in Y .

References

1. S. Jain and B. Fisher, "A related fixed point theorem for three metric spaces," *Hacetatepe Journal of Mathematics and Statistics*, **31**(2002) 19–24.
2. I. Beg and S. Chauhan, Related Fixed Point of Set-Valued Mappings in Three Menger Spaces, *International Journal of Analysis*, **2013**, Article ID 736451, 6 pages.
3. B. Fisher, "Related fixed points on two metric spaces," Kobe University, **10(1)** (1982) 17–26.
4. K. Menger, "Statistical metrics," *Proceedings of the National Academy of Sciences of the United States of America*, **28** (1942) 535–537.
5. B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland Publishing Co., New York, NY, USA, 1983.
6. V. M. Sehgal and A. T. Bharucha Reid, Fixed points of contraction mappings on probabilistic metric spaces, *Math. Systems Theory*, **6** (1972) 97 – 102.
7. G. Constantin, and I. Istratescu, *Elements of probabilistic analysis*, Ed. Acad. Bucuresti and Kluwer Acad. Publ. 1989.
8. O. Hadzic, O., Fixed point theorems for multi-valued mappings in some classes of fuzzy metric spaces, *Fuzzy Sets and Systems*, **29** (1989) 115 – 125.
9. T. L. Hicks, Fixed point theory in probabilistic metric spaces, *Rev. Res. Novi Sad*, **13** (1983) 63 – 72.
10. I. Istratescu, On some fixed point theorems in generalized Menger spaces, *Boll. Un. Mat. Ital.*, **5 (13-A)** (1976) 95 – 100.

11. I. Istratescu, On generalized complete probabilistic metric spaces, *Rev. Roum. Math. Pures Appl.*, **XXV** (1980) 1243 – 1247.
12. S. N. Mishra, “Common fixed points of compatible mappings in PM-spaces,” *Mathematica Japonica*, **36(2)** (1991) 283–289.
13. E. Pap, O. Hadzic and R. Mesiar, A fixed point theorem in probabilistic metric spaces and an application, *J. Math. Anal. Appl.*, **202** (1996) 433 – 449.
14. B. Schweizer, H. Sherwood and R. M. Tardiff, Contractions on PM-space examples and counterexamples, *Stochastica*, **1** (1988) 5 – 17.
15. R. K. Jain, H. K. Sahu, and B. Fisher, “Related fixed point theorems for three metric spaces,” *Novi Sad Journal of Mathematics*, **26(1)** (1996) 11–17.
16. B. Fisher, “Fixed points on two metric spaces,” *Glasnik Matematički*, **16** (36) (1981) 333–337.
17. B. Singh and S. Jain, A fixed point theorem in Menger Space through weak compatibility, *J. Math. Anal. Appl.*, **301** (2005) 439 – 448.
18. B. Fisher and D. Türkoğlu, “Related fixed points for set valued mappings on two metric spaces,” *International Journal of Mathematics and Mathematical Sciences*, **23(3)** (2000) 205–210.
19. V. K. Chourasia and B. Fisher, “Related fixed points for two pairs of set-valued mappings on two metric spaces,” *Hacettepe Journal of Mathematics and Statistics*, **32** (2003) 27–32.
20. B. D. Pant, “Relation between fixed points in Menger spaces,” *Journal of the Indian Academy of Mathematics*, **24(1)** (2002) 135–142.
21. B. D. Pant and S. Kumar, “A related fixed point theorem for two pairs of mappings in two Menger space,” *Varahmihir Journal of Mathematical Sciences*, **6(2)** (2006) 471–476.
22. R. K. Namdeo, S. Jain, and B. Fisher, “A related fixed point theorem for two pairs of mappings on two complete metric spaces,” *Hacettepe Journal of Mathematics and Statistics*, **32** (2003) 7–11.

23. R. K. Saini, Common fixed point theorem in ε -chainable Menger spaces using implicitrelations, *Int. J. Contemp. Math. Sciences*, **5(40)** (2010) 1951 – 1959.
24. S. Kutukcu, D. Turkoglu and C. Yildiz, Common fixed points of compatible maps of type (P) on fuzzy metric spaces, *Commun. Korean Math. Soc.*, in press.
25. R. M. Tardiff, Contraction maps on probabilistic metric spaces, *J. Math. Anal. Appl.*, **165** (1992) 517 – 523.
26. S.S. Chang, Y. J. Cho, and S. M. Kang, *Nonlinear Operator Theory in Probabilistic Metric Spaces*, Nova Science Publishers Inc., Huntington, NY, USA, 2001.