

A Finsler Space with Conservative Normal Projective Curvature Tensor

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Abstract: In this paper, the conservativeness of normal projective curvature tensor of a Finsler Space is considered. The condition of the conservativeness of normal projective curvature tensor is weaker than the condition characterizing a normal projective symmetric space. It is established that Berwald torsion tensor and Berwald deviation tensor of a Finsler Space with conservative normal projective curvature tensor are necessarily conservative. However, the converse is not necessarily true. Apart from other results, the symmetry of the covariant derivative $B_h H_{jk}$ of the Ricci tensor and the tensor $B_h H_k$ in the indices h and k have been proved.

Keywords: Finsler space, Berwald curvature tensor, Berwald deviation tensor, Ricci tensor, conservativeness of Berwald curvature tensor, Berwald deviation tensor and Ricci tensor, normal projective curvature tensor, conservativeness of normal projective curvature tensor.

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1. Introduction

A vector field in a differentiable manifold is said to be conservative if its divergence is zero. If the divergence of curvature is zero in a neighbourhood of any point p of a differentiable manifold M^n then the bending is conservative i.e. rigid around the point p . In 1980, P. N. Pandey¹ established the relation between the normal projective curvature tensor and Berwald curvature tensor. In 1984, P. N. Pandey² discussed projective motion in a symmetric and projectively symmetric manifold. The condition characterizing a Finsler space with conservative Berwald curvature tensor is weaker than the condition characterizing a symmetric Finsler Space. In 1997, U. C. De and A. A. Shaikh³ studied the conservativeness of quasi-

conformal curvature tensor on a K-Contact and Sasakian manifolds. In 2005, C. S. Bagewadi and N. B. Gatti⁴ studied conservative curvature tensor on an Einstein manifold. In 2007, several authors including C. S. Bagewadi and D. G. Prakasha⁵ made significant contributions towards the spaces with conservative curvature tensors.

In 2008, D. G. Prakash, C. S. Bagewadi and Venkatesha⁶ studied the conformally and quasi- conformally conservative curvature tensors on a trans- Sasakian manifold. In 2012, C. S. Bagewadi and M. C. Bharti⁷ introduced the concept of conservative conditions on curvature tensor of a Kaehler manifold. In 2015, S. Kumari and P. N. Pandey⁸ studied a Finsler space with conservative Berwald curvature tensor and obtained some significant results. The aim of the present paper is to study a Finsler Space with conservative normal projective curvature tensor.

2. Preliminaries

Consider an n-dimensional Finsler Space F_n equipped with a metric function $F(x^i, y^j)$ satisfying the requisite conditions (H. Rund⁹ and P. L. Antonelli(ed.)¹⁰). Suppose g_{ij} , G_{jk}^i , H_{jkh}^i , π_{jk}^i & N_{jkh}^i denote the components of the corresponding metric tensor, Berwald connection coefficients, components of Berwald curvature tensor, normal projective connection coefficients and components of normal projective curvature tensor respectively. H_{jkh}^i and N_{jkh}^i are positively homogeneous of degree zero in y^j and are skew symmetric in their last two lower indices.

The Berwald connection coefficients and the normal projective connection coefficients are related as

$$(2.1) \quad \begin{aligned} (a) \quad \pi_{jkh}^i &= \pi_{jkh}^i, & (b) \quad \pi_{jkh}^i y^j &= 0, \\ (c) \quad \pi_{khi}^j &= G_{khi}^j, & (d) \quad \pi_{jkh}^i y^h &= \frac{1}{(n+1)} G_{jkr}^r y^i, \\ (e) \quad \pi_{ikh}^j &= \frac{2}{(n+1)} G_{ikh}^j, & (f) \quad \pi_{jkh}^i - \pi_{hki}^j &= \frac{1}{(n+1)} (\delta_h^i G_{jkr}^r - \delta_j^i G_{hkr}^r). \end{aligned}$$

The components of Berwald- Ricci tensor are given by

$$H_{jk} = H_{jkr}^r.$$

The components of Berwald curvature tensor, deviation tensor, torsion tensor & Ricci tensor satisfy the following

$$\begin{aligned}
 (2.2) \quad (a) \quad & H_{jk}^i = H_{hjk}^i y^h, & (b) \quad & \dot{\partial}_j H_{kh}^i = H_{jkh}^i, \\
 (c) \quad & H_j^i = H_{hj}^i y^h, & (d) \quad & H_{jk}^i = -H_{kj}^i, \\
 (e) \quad & H_i^i = (n-1)H, & (f) \quad & H_k y^k = (n-1)H, \\
 (g) \quad & 3H_{kh}^i = \dot{\partial}_k H_h^i - \dot{\partial}_h H_k^i, & (h) \quad & H_{kh} y^k = H_h, \\
 (i) \quad & H_{rkh}^r = H_{hk} - H_{kh}.
 \end{aligned}$$

Bianchi identities for the components of Berwald Curvature tensor are given by⁹⁻¹⁰

$$\begin{aligned}
 (2.3) \quad (a) \quad & H_{jkh}^i + H_{khj}^i + H_{hjk}^i = 0, \\
 (b) \quad & H_{jkh(m)}^i + H_{jhm(k)}^i + H_{jmk(h)}^i + H_{kh}^r G_{jmr}^i + H_{hm}^r G_{jkr}^i + H_{mk}^r G_{jhr}^i = 0.
 \end{aligned}$$

Relation between the components of Berwald curvature tensor and normal projective curvature tensor is given by¹

$$(2.4) \quad N_{h kj}^i = H_{h kj}^i - \frac{y^j}{(n+1)} \dot{\partial}_h H_{rkj}^r.$$

The Berwald covariant differential operator is defined as

$$(2.5) \quad T_{j(k)}^i = \partial_k T_j^i - (\dot{\partial}_r T_j^i) G_k^r + T_j^r G_{rk}^i - T_r^i G_{jk}^r,$$

where T_j^i are the components of an arbitrary tensor of type (1, 1); (k) denotes the Berwald covariant differential operator.

Relation between the Berwald covariant differential operator and the directional differential operator is given by

$$(2.6) \quad \dot{\partial}_k T_{j(h)}^i - (\dot{\partial}_k T_j^i)_{(h)} = T_j^r G_{rkh}^i - T_r^i G_{jkh}^r.$$

where $\dot{\partial}_j G_{kh}^i = G_{jkh}^i$ are the components of symmetric tensor and satisfy

$$(2.7) \quad G_{jkh}^i y^j = G_{jkh}^i y^k = G_{jkh}^i y^h = 0.$$

Definition 2.1: A Finsler Space F^n with conservative Berwald curvature tensor is characterized by

$$(2.8) \quad H_{jkh(r)}^r = 0, \text{ with } H_{jkh}^i \neq 0.$$

Definition 2.2: A Finsler Space F^n with conservative normal projective curvature tensor is characterized by

$$(2.9) \quad N_{jkh(r)}^r = 0, \text{ with } N_{jkh}^i \neq 0.$$

Contracting the indices i & h in (2.4), we have

$$(2.10) \quad N_{rkj}^r = H_{rkj}^r.$$

Transvecting (2.4) with respect to y^h , we have

$$(2.11) \quad N_{h kj}^i y^h = H_{kj}^i.$$

Contracting the indices i & j in (2.4), we have

$$(2.12) \quad N_{hk} = H_{hk} - \frac{y^j}{(n+1)} \dot{\partial}_h H_{rki}^r.$$

After carrying out some simple calculations, (2.12) reduces to

$$(2.13) \quad N_{jk} = \frac{n}{(n+1)} H_{jk} - \frac{1}{(n+1)} H_{kj} + \frac{(n-1)}{(n+1)} \dot{\partial}_j \dot{\partial}_k H.$$

First Bianchi identity for the components of normal projective Curvature tensor is given by

$$(2.14) \quad N_{jkh}^i + N_{khj}^i + N_{hjk}^i = 0.$$

Normal projective curvature tensor is skew symmetric in its last two lower indices, i.e.

$$(2.15) \quad N_{jkh}^i = -N_{jhk}^i.$$

Relation between the components of projective curvature tensor and normal projective curvature tensor is given by

$$(2.16) \quad W_{jkh}^i = N_{jkh}^i + (\delta_k^i M_{hj} - \delta_j^i M_{kh}) - (\delta_h^i M_{kj} - \delta_j^i M_{hk}),$$

where

$$(2.17) \quad M_{kh} = \frac{-1}{n^2 - 1} (nN_{kh} + N_{hk}).$$

Definition 2.3: A Finsler Space F^n with conservative projective curvature tensor is characterized by

$$(2.18) \quad W_{jkh(r)}^r = 0, \text{ with } W_{jkh}^i \neq 0.$$

The components of projective curvature tensor, deviation tensor and torsion tensor satisfy the following

$$(2.19) \quad W_{jkh}^i \dot{x}^j = W_{kh}^i,$$

$$(2.20) \quad W_{kh}^i \dot{x}^k = W_h^i.$$

3. Conservativeness of Berwald Curvature tensor in a Finsler Space with Conservative Normal Projective Curvature Tensor

Let us consider a Finsler Space F^n with conservative normal projective curvature tensor characterized by (2.9).

Differentiating covariantly (2.4) with respect to x^m , we get

$$(3.1) \quad N_{jkh(m)}^i = H_{jkh(m)}^i - \frac{y^j}{(n+1)} (\dot{\partial}_j H_{rkh}^r)_{(m)}.$$

On contracting the indices i & m and using (2.9), we get

$$(3.2) \quad H_{jkh(r)}^r - \frac{y^r}{(n+1)} (\dot{\partial}_j H_{skh}^s)_{(r)} = 0.$$

Transvecting (3.2) by y^j , we get

$$(3.3) \quad H_{kh(r)}^r = 0.$$

Again transvecting (3.3) by y^k , we have

$$(3.4) \quad H_{h(r)}^r = 0.$$

Thus, we have the theorem

Theorem 3.1: *In a Finsler Space with conservative normal projective curvature tensor, Berwald torsion tensor and Berwald deviation tensor are conservative.*

On transvecting Bianchi's second identity (2.3)(b) by y^j and using (2.2)(a) and (2.7), we have

$$(3.5) \quad H_{kh(m)}^i + H_{hm(k)}^i + H_{mk(h)}^i = 0.$$

Contracting the indices i & m and using (3.3), we get

$$(3.6) \quad H_{h(k)} = H_{k(h)}.$$

Differentiate partially (3.5) with respect to y^j , we get

$$(3.7) \quad \partial_j H_{h(k)} = \partial_j H_{k(h)}.$$

Using the commutation formulae (2.6), we get

$$(3.8) \quad H_{jh(k)} = H_{jk(h)}.$$

Hence, we have the theorem

Theorem 3.2: *The Ricci tensor H_{jk} and the vector H_k of a Finsler space with conservative normal projective curvature tensor satisfy $H_{jh(k)} = H_{jk(h)}$ and $H_{h(k)} = H_{k(h)}$ equivalently.*

For Berwald torsion tensor, the commutation formulae (2.6) is given as

$$(3.9) \quad \partial_j H_{kh(m)}^i - H_{jkh(m)}^i = H_{kh}^r G_{jmr}^i - H_{rh}^i G_{jmk}^r - H_{kr}^i G_{jmh}^r.$$

Contracting the indices i & m , we have

$$(3.10) \quad H_{jkh(r)}^r + H_{kh}^r G_{jsr}^s - H_{rh}^s G_{jsk}^r - H_{kr}^s G_{jsh}^r = 0.$$

Rotating cyclically j, k & h in (3.10), we have

$$(3.11) \quad H_{khj(r)}^r + H_{hj}^r G_{ksr}^s - H_{rj}^s G_{ksh}^r - H_{hr}^s G_{ksj}^r = 0.$$

Rotating cyclically j, k & h in (3.10), we have

$$(3.12) \quad H_{hjk(r)}^r + H_{jk}^r G_{hsr}^s - H_{rk}^s G_{hsj}^r - H_{jr}^s G_{hsk}^r = 0.$$

Adding (3.10), (3.11) and (3.13) and using (2.2)(d), we get

$$(3.13) \quad H_{kh}^r G_{jsr}^s + H_{hj}^r G_{ksr}^s + H_{jk}^r G_{hsr}^s = 0.$$

Transvecting (3.13) by y^j , we get

$$(3.14) \quad H_k^r G_{hsr}^s - H_h^r G_{ksr}^s = 0.$$

$$(3.15) \quad H_k^r G_{hsr}^s = H_h^r G_{ksr}^s.$$

Thus, we have the following theorem

Theorem 3.3: *The identities (3.13) and (3.15) hold in a Finsler space with conservative normal projective curvature tensor.*

Consider

$$\begin{aligned} (3.16) \quad (\partial_j H_{skr(h)}^s - \partial_k H_{sjr(h)}^s) y^r &= [\partial_j (H_{rk(h)} - H_{kr(h)}) - \partial_k (H_{rj(h)} - H_{jr(h)})] y^r \\ &= \partial_j H_{rk(h)} y^r - \partial_j \partial_k H_{r(h)} y^r - \partial_k H_{rj(h)} y^r + \partial_k \partial_j H_{r(h)} y^r \\ &= \partial_j (H_{rk(h)} y^r) - H_{jk(h)} - \partial_k (H_{rj(h)} y^r) + H_{kj(h)} \\ &= \partial_j (H_{k(h)}) - H_{jk(h)} - \partial_k (H_{j(h)}) + H_{kj(h)} \\ &= H_{jk(h)} - H_{jk(h)} - H_{kj(h)} + H_{kj(h)} \\ &= 0. \end{aligned}$$

Hence we have

$$(3.17) \quad \partial_j H_{skr(h)}^s y^r = \partial_k H_{sjr(h)}^s y^r.$$

For Berwald curvature tensor, the commutation formulae (2.6) is given as

$$(3.18) \quad \partial_l H_{jkh(m)}^i - (\partial_l H_{jkh}^i)_{(m)} = H_{jkh}^r G_{lmr}^i - H_{rkh}^i G_{lmj}^r - H_{jrh}^i G_{lmk}^r - H_{jkr}^i G_{lmh}^r.$$

Transvecting (3.18) by y^m , we get

$$(3.19) \quad y^m \partial_l H_{jkh(m)}^i - y^m (\partial_l H_{jkh}^i)_{(m)} = 0.$$

$$(3.20) \quad y^m \partial_l H_{jkh(m)}^i = y^m (\partial_l H_{jkh}^i)_{(m)}.$$

Contracting the indices i & j in (3.20), we get

$$(3.21) \quad y^m \dot{\partial}_l H_{rkh(m)}^r = y^m \left(\dot{\partial}_l H_{rkh}^r \right)_{(m)}.$$

Thus, we have the following theorem

Theorem 3.4: *In a Finsler space with conservative normal projective curvature tensor the identities (3.17), (3.20) and (3.21) hold.*

From (3.21), we have

$$\begin{aligned}
 (3.22) \quad y^r \dot{\partial}_j H_{skh(r)}^s &= y^r \left(\dot{\partial}_j H_{skh}^s \right)_{(r)} \\
 &= y^r \left(\dot{\partial}_j \left(H_{hk} - H_{kh} \right) \right)_{(r)} \\
 &= y^r \dot{\partial}_j \left(H_{hk} \right)_{(r)} - y^r \dot{\partial}_j \left(H_{kh} \right)_{(r)} \\
 &= y^r \dot{\partial}_j H_{hk(r)} - y^r \dot{\partial}_j H_{kh(r)} \\
 &= y^r \dot{\partial}_j \dot{\partial}_h H_{k(r)} - y^r \dot{\partial}_j \dot{\partial}_k H_{h(r)} \\
 &= (n-1) \left\{ \dot{\partial}_j \dot{\partial}_h H_{(k)} - \dot{\partial}_k \dot{\partial}_h H_{(j)} \right\} - \left\{ H_{jk(h)} - H_{kj(h)} \right\} \\
 &= \dot{\partial}_h \left\{ \dot{\partial}_j H_{k(r)} y^r - \dot{\partial}_k H_{j(r)} y^r \right\} + \left\{ H_{jk(h)} - H_{kj(h)} \right\} - \left\{ H_{jk(h)} - H_{kj(h)} \right\} \\
 &= \dot{\partial}_h \left\{ H_{jk(r)} y^r - H_{kj(r)} y^r \right\} \\
 &= \dot{\partial}_h \left\{ H_{skj(r)}^s y^r \right\} \\
 &= \dot{\partial}_h H_{skj(r)}^s y^r + H_{skj(h)}^s.
 \end{aligned}$$

Thus we have

$$(3.23) \quad y^r \dot{\partial}_j H_{skh(r)}^s - y^r \dot{\partial}_h H_{skj(r)}^s = H_{skj(h)}^s.$$

Or

$$(3.24) \quad y^r \left(\dot{\partial}_h H_{sjk(r)}^s - \dot{\partial}_j H_{shk(r)}^s \right) = H_{skj(h)}^s.$$

Transvecting (3.24) by y^k , we get

$$(3.25) \quad y^r \left(y^k \dot{\partial}_h H_{sjk(r)}^s - y^k \dot{\partial}_j H_{shk(r)}^s \right) = y^k H_{skj(h)}^s.$$

Using (3.17) in (3.25), we have

$$(3.26) \quad y^k H_{skj(h)}^s = 0.$$

$$(3.27) \quad y^k (H_{jk(h)} - H_{kj(h)}) = 0.$$

$$(3.28) \quad y^r H_{jr(h)} = H_{j(h)}.$$

Differentiating partially (3.28) with respect to y^k , we get

$$(3.29) \quad y^r \dot{\partial}_k H_{jr(h)} + H_{jk(h)} = \dot{\partial}_k H_{j(h)}.$$

Interchanging the indices j & k in (3.29), we have

$$(3.30) \quad y^r \dot{\partial}_j H_{kr(h)} + H_{kj(h)} = \dot{\partial}_j H_{k(h)}.$$

From (3.29) and (3.30), we have

$$(3.31) \quad y^r \dot{\partial}_k H_{jr(h)} + H_{jk(h)} - y^r \dot{\partial}_j H_{kr(h)} - H_{kj(h)} = \dot{\partial}_k H_{j(h)} - \dot{\partial}_j H_{k(h)}.$$

Using the commutation formulae (2.6) in (3.31), we get

$$(3.32) \quad y^r \left(\dot{\partial}_k H_{jr} \right)_{(h)} - H_{sr} G_{kjh}^s y^r + H_{jk(h)} - y^r \dot{\partial}_j H_{kr(h)} - H_{kj(h)} = H_{kh(j)} - H_s G_{kjh}^s - H_{jh(k)} + H_s G_{kjh}^s.$$

$$(3.33) \quad y^r \left(\dot{\partial}_j H_{kr} \right)_{(h)} - H_{sr} G_{kjh}^s y^r + H_{jk(h)} - y^r \dot{\partial}_j H_{kr(h)} - H_{kj(h)} = H_{kh(j)} - H_{jh(k)}.$$

$$(3.34) \quad y^r \dot{\partial}_j H_{kr(h)} + H_{sr} G_{jkh}^s y^r - H_{sr} G_{kjh}^s y^r + H_{jk(h)} - y^r \dot{\partial}_j H_{kr(h)} - H_{kj(h)} = H_{kj(h)} - H_{jk(h)}.$$

$$(3.35) \quad H_{jk(h)} = H_{kj(h)}.$$

Thus, we conclude

Theorem 3.5: *In a Finsler space with conservative normal projective curvature tensor, identity (3.24) holds and Ricci tensor H_{jk} of a Finsler space with conservative normal projective curvature tensor satisfy $H_{jk(h)} = H_{kj(h)}$.*

From (3.2), we have

$$(3.36) \quad H_{jkh(r)}^r = \frac{y^r}{(n+1)} \left(\dot{\partial}_j H_{skh}^s \right)_{(r)}.$$

Using (3.21) in (3.36), we have

$$(3.37) \quad H_{jkh(r)}^r = \frac{y^r}{(n+1)} \dot{\partial}_j H_{skh(r)}^s.$$

$$(3.38) \quad H_{jkh(r)}^r = \frac{y^r}{(n+1)} \dot{\partial}_j \left(H_{hk(r)} - H_{kh(r)} \right).$$

Using (3.35) in (3.38), we have

$$(3.39) \quad H_{jkh(r)}^r = 0.$$

Thus, we have the theorem

Theorem 3.6: *In a Finsler space with conservative normal projective curvature tensor, the Berwald curvature tensor is also conservative.*

4. Conservativeness of Projective Curvature tensor in a Finsler Space with Conservative Normal Projective Curvature Tensor

Let us consider a Finsler Space F^n with conservative normal projective curvature tensor characterized by (2.9).

Differentiating covariantly (2.13) with respect to x^h , we get

$$(4.1) \quad N_{jk(h)} = \frac{n}{(n+1)} H_{jk(h)} - \frac{1}{(n+1)} H_{kj(h)} + \frac{(n-1)}{(n+1)} \left(\dot{\partial}_j \dot{\partial}_k H \right)_{(h)}.$$

Interchanging the indices j & k in (4.1), we have

$$(4.2) \quad N_{kj(h)} = \frac{n}{(n+1)} H_{kj(h)} - \frac{1}{(n+1)} H_{jk(h)} + \frac{(n-1)}{(n+1)} \left(\dot{\partial}_k \dot{\partial}_j H \right)_{(h)}.$$

Thus we have

$$(4.3) \quad N_{jk(h)} - N_{kj(h)} = H_{jk(h)} - H_{kj(h)}.$$

Using (3.35) in (4.3), we get

$$(4.4) \quad N_{jk(h)} = N_{kj(h)}.$$

Interchanging the indices k & h in (4.1) then subtracting it from (4.1), we have

$$\begin{aligned}
 (4.5) \quad N_{jk(h)} - N_{jh(k)} &= \frac{1}{(n+1)} (H_{hk(j)} - H_{kh(j)}) + \frac{(n-1)}{(n+1)} \left\{ (\dot{\partial}_j \dot{\partial}_k H)_{(h)} - (\dot{\partial}_j \dot{\partial}_h H)_{(k)} \right\} \\
 &= \frac{1}{(n+1)} (H_{hk} - H_{kh})_{(j)} + \frac{(n-1)}{(n+1)} \left\{ \dot{\partial}_j (\dot{\partial}_k H)_{(h)} + (\dot{\partial}_r H) G_{jkh}^r \right. \\
 &\quad \left. - \dot{\partial}_j (\dot{\partial}_h H)_{(k)} - (\dot{\partial}_r H) G_{jkh}^r \right\} \\
 &= \frac{1}{(n+1)} (H_{hk} - H_{kh})_{(j)} + \frac{(n-1)}{(n+1)} \dot{\partial}_j \left\{ (\dot{\partial}_k H)_{(h)} - (\dot{\partial}_h H)_{(k)} \right\} \\
 &= \frac{1}{(n+1)} \left\{ H_{hk(j)} - H_{kh(j)} + \dot{\partial}_j (H_{kh(r)} y^r - H_{hk(r)} y^r) \right\} \\
 &= \frac{1}{(n+1)} \dot{\partial}_j \left\{ (H_{kh(r)} - H_{hk(r)}) y^r \right\}
 \end{aligned}$$

Using (3.35) in (4.5), we have

$$(4.6) \quad N_{jk(h)} - N_{jh(k)} = 0.$$

$$(4.7) \quad N_{jk(h)} = N_{jh(k)}.$$

Interchanging the indices j & h in (4.1), we have

$$(4.8) \quad N_{hk(j)} = \frac{n}{(n+1)} H_{hk(j)} - \frac{1}{(n+1)} H_{kh(j)} + \frac{(n-1)}{(n+1)} (\dot{\partial}_h \dot{\partial}_k H)_{(j)}.$$

From (4.1) & (4.8), we have

$$\begin{aligned}
 (4.9) \quad N_{jk(h)} - N_{hk(j)} &= \frac{n}{(n+1)} (H_{jh(k)} - H_{hj(k)}) + \frac{(n-1)}{(n+1)} \left\{ (\dot{\partial}_j \dot{\partial}_k H)_{(h)} - (\dot{\partial}_h \dot{\partial}_k H)_{(j)} \right\} \\
 &= \frac{(n-1)}{(n+1)} \left\{ (\dot{\partial}_j \dot{\partial}_k H)_{(h)} - (\dot{\partial}_h \dot{\partial}_k H)_{(j)} \right\} \\
 &= \frac{(n-1)}{(n+1)} \left\{ \dot{\partial}_k (\dot{\partial}_j H)_{(h)} + (\dot{\partial}_r H) G_{jkh}^r - \dot{\partial}_k (\dot{\partial}_h H)_{(j)} - (\dot{\partial}_r H) G_{jkh}^r \right\}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(n-1)}{(n+1)} \dot{\partial}_k \left\{ \left(\dot{\partial}_j H \right)_{(h)} - \left(\dot{\partial}_h H \right)_{(k)} \right\} \\
&= \frac{1}{(n+1)} \left\{ \dot{\partial}_k \left(H_{jh(r)} y^r - H_{hj(r)} y^r \right) \right\} \\
&= \frac{1}{(n+1)} \dot{\partial}_k \left\{ \left(H_{jh(r)} - H_{hj(r)} \right) y^r \right\}.
\end{aligned}$$

Using (3.35) in (4.9), we have

$$(4.10) \quad N_{jk(h)} = N_{hk(j)}.$$

Differentiating covariantly (2.17) with respect to x^j , we get

$$(4.11) \quad M_{kh(j)} = \frac{-1}{n^2 - 1} (nN_{kh(j)} + N_{hk(j)}).$$

Interchanging the indices j & h in (4.11), we have

$$(4.12) \quad M_{kj(h)} = \frac{-1}{n^2 - 1} (nN_{kj(h)} + N_{jk(h)}).$$

From (4.11) & (4.12), we have

$$(4.13) \quad M_{kh(j)} - M_{kj(h)} = \frac{-n}{n^2 - 1} (N_{kh(j)} - N_{kj(h)}) + \frac{-1}{n^2 - 1} (N_{hk(j)} + N_{jk(h)}).$$

Using (4.7) & (4.10) in (4.13), we have

$$(4.14) \quad M_{kh(j)} = M_{kj(h)}.$$

Interchanging the indices k & h in (4.12), we have

$$(4.15) \quad M_{hj(k)} = \frac{-1}{n^2 - 1} (nN_{hj(k)} + N_{jh(k)}).$$

From (4.12) & (4.15), we have

$$\begin{aligned}
(4.16) \quad M_{hj(k)} - M_{kj(h)} &= \frac{-1}{n^2 - 1} (nN_{hj(k)} + N_{jh(k)}) + \frac{1}{n^2 - 1} (nN_{kj(h)} + N_{jk(h)}) \\
&= \frac{-1}{n^2 - 1} \left\{ n(N_{hj(k)} - N_{kj(h)}) + (N_{jh(k)} - N_{jk(h)}) \right\}.
\end{aligned}$$

Using (4.7) & (4.10) in (4.16), we have

$$(4.17) \quad M_{hj(k)} = M_{kj(h)}.$$

Interchanging the indices k & h in (4.11), we have

$$(4.18) \quad M_{hk(j)} = \frac{-1}{n^2 - 1} (nN_{hk(j)} + N_{kh(j)}).$$

From (4.11) & (4.18), we have

$$(4.19) \quad M_{kh(j)} - M_{hk(j)} = \frac{-1}{n^2 - 1} \{ n(N_{kh(j)} - N_{hk(j)}) + (N_{hk(j)} - N_{kh(j)}) \}.$$

Using (4.4) in (4.19), we have

$$(4.20) \quad M_{kh(j)} = M_{hk(j)}.$$

Thus, we have the following theorem

Theorem 4.1: *The identities (4.4), (4.7), (4.10), (4.14), (4.17) and (4.20) hold in a Finsler space with conservative normal projective curvature tensor.*

Differentiating covariantly (2.16) with respect to x^m , we get

$$(4.21) \quad W_{jkh(m)}^i = N_{jkh(m)}^i + (\delta_k^i M_{hj(m)} - \delta_j^i M_{kh(m)}) - (\delta_h^i M_{kj(m)} - \delta_j^i M_{hk(m)}).$$

Contracting the indices i & m and using (2.9), we get

$$(4.22) \quad W_{jkh(r)}^r = (M_{hj(k)} - M_{kh(j)}) - (M_{kj(h)} - M_{hk(j)}).$$

$$(4.23) \quad W_{jkh(r)}^r = (M_{hj(k)} - M_{kj(h)}) - (M_{kh(j)} - M_{hk(j)}).$$

Using (4.14) & (4.14) in (4.23), we get

$$(4.24) \quad W_{jkh(r)}^r = 0.$$

Transvecting (4.24) by y^j & using (2.19), we have

$$(4.25) \quad W_{kh(r)}^r = 0.$$

Transvecting (4.25) by y^k & using (2.20), we have

$$(4.26) \quad W_{h(r)}^r = 0.$$

Thus, we have the theorem

Theorem 4.2: *In a Finsler space with conservative normal projective curvature tensor, the projective curvature tensor, torsion tensor & deviation tensor are also conservative.*

5. Finsler Space with Conservative Berwald Curvature Tensor

Consider a Finsler Space F^n with conservative Berwald curvature tensor defined by (2.8). Then the Berwald torsion tensor and Berwald deviation tensor are also conservative⁸.

Also in a Finsler space with conservative Berwald curvature tensor, the identities $H_{jh(k)} = H_{jk(h)}$ and $H_{h(k)} = H_{k(h)}$ are satisfied⁸.

Thus, we have the theorem

Theorem: *In a Finsler space with conservative Berwald curvature tensor the identities (3.17), (3.20), (3.21), (3.25) & (3.35) hold.*

Consider equation (3.1). On contracting the indices i & m and using (2.8), we get

$$(5.1) \quad N_{jkh(r)}^r = -\frac{y^r}{(n+1)} (\dot{\partial}_h H_{skh}^s)_{(r)}.$$

Using (3.21) in (5.1), we get

$$(5.2) \quad N_{jkh(r)}^r = -\frac{y^r}{(n+1)} \dot{\partial}_h H_{skh(r)}^s.$$

$$(5.3) \quad N_{jkh(r)}^r = -\frac{y^r}{(n+1)} \dot{\partial}_h (H_{hk(r)} - H_{kh(r)}).$$

Using (3.35) in (5.3), we have

$$(5.4) \quad N_{jkh(r)}^r = 0.$$

Thus, we conclude

Theorem 5.1: *In a Finsler space with conservative Berwald curvature tensor, the normal projective curvature tensor is also conservative.*

Hence the identities (4.4), (4.7), (4.10), (4.14), (4.17) and (4.20) also hold in a Finsler space with conservative Berwald curvature tensor.

Differentiating covariantly (2.16) with respect to x^m , we get

$$(5.5) \quad W_{jkh(m)}^i = N_{jkh(m)}^i + (\delta_k^i M_{hj(m)} - \delta_j^i M_{kh(m)}) - (\delta_h^i M_{kj(m)} - \delta_j^i M_{hk(m)}).$$

Contracting the indices i & m and using (2.9), we get

$$(5.6) \quad W_{jkh(r)}^r = (M_{hj(k)} - M_{kh(j)}) - (M_{kj(h)} - M_{hk(j)}).$$

$$(5.7) \quad W_{jkh(r)}^r = (M_{hj(k)} - M_{kj(h)}) - (M_{kh(j)} - M_{hk(j)}).$$

Using (4.14) & (4.14) in (5.7), we get

$$(5.8) \quad W_{jkh(r)}^r = 0.$$

Transvecting (5.8) by y^j & using (2.19), we have

$$(5.9) \quad W_{kh(r)}^r = 0.$$

Transvecting (5.9) by y^k & using (2.20), we have

$$(5.10) \quad W_{h(r)}^r = 0.$$

Thus, we have the theorem

Theorem 5.2: *In a Finsler space with conservative Berwald curvature tensor, the projective curvature tensor, torsion tensor & deviation tensor are also conservative.*

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