

## The Structure on a Subspace of a Space with an

## $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$ Structure

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(Received January 6, 2017)

**Abstract:** Let  $M^n$  be an almost contact manifold with an  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure of rank  $r$  and let  $N^{n-1}$  be a hypersurface in  $M^n$ . The following theorem is proved: if the dimension of  $T(N^{n-1} \cap f(TN^{n-1}))$  is constant say  $s$ , for all  $p \in N^{n-1}$  possesses a natural  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure of rank  $s$ . It is also proved that the naturally induced  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure is integrable if the structure on  $M^n$  is integrable and if the transversal to  $N^{n-1}$  can be found to lie in the distribution  $M^n$ .

**AMS Subject Classification:** 53C10, 53C15, 53C40, 51H20.

**Keywords:** Hypersurface, Almost contact manifold, Almost paracontact structure, Integrability conditions.

### 1. Introduction

Yano<sup>1</sup> has given the basis of a structure of a tensor field with an equation. Let  $M^n$  be an  $n$  dimensional  $C^\infty$  manifold and  $f$  a tensor of type (1,1) such that

$$(1.1) \quad a_n f^n + a_{n-1} f^{n-1} + a_{n-2} f^{n-2} + \dots + a_2 f^2 + a_1 f^1 = 0,$$

and let the rank of  $f$  be constant say  $r < n$ , on  $M^n$ . We then say that  $M^n$  has  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure of rank  $r$  where  $a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1$  are real or complex constants.

Yano and Ishihara<sup>3</sup> have shown that if  $M^n$  is an almost complex manifold, then a submanifold of  $M^n$  satisfying a certain property, possesses a natural  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure. In particular, Tashiro<sup>[2]</sup> has shown that if the submanifold is a hypersurface, then the induced  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure has a maximal rank (i.e. is an almost contact structure).

Ludden and Blair<sup>4</sup> proved that hypersurface in an almost contact manifold possesses a natural  $f$ -structure which is an almost complex structure if the normal to the hypersurface is in the same direction of the almost contact structure. Finally, it was proved that if the almost contact manifold has additional structure then under some conditions a hypersurface inherits the corresponding analogous structure. Ludden<sup>[5]</sup> and Sato<sup>[6]</sup> have given the explanation about the submanifolds with an  $f$  structure and almost contact structure respectively.

It is known that a hypersurface of an almost contact manifold possesses a natural  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure, which does not have a maximal rank. Mizawa<sup>7</sup> proved that if the submanifold in the manifold with an almost product structure is a hypersurface, then the induced  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure has a maximal rank (i.e. is an almost paracontact structure).

The purpose of this paper is to show that if  $M^n$  has  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure, then the hypersurface of this manifold possesses a natural  $F(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure. Though the study made<sup>8</sup> proves the theorem for  $f(3, -1)$  structure whereas, in this paper the attempt has been made to generalize the theorem to polynomial structure.

In section 3, we shall study the integrability of the induced  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure.

The  $(\psi, \xi, \eta)$ -structure is an almost paracontact structure if we have on  $M^n$ , a tensor field  $\psi$  of type (1,1), a vector field  $\xi$ , and a 1-form  $\eta$  satisfying

$$(1.2) \quad \psi^2 = 1 - \eta \otimes \xi, \quad \eta\psi = 0, \quad \psi\xi = 0, \quad \eta(\xi) = 1.$$

In [7] the following theorem is proved.

**Theorem 1.1.** *Let  $T(M^n)$  denote the tangent space to  $M^n$  at  $p$ . The almost product structure  $\phi$  induces the almost paracontact structure  $\psi$  on a hypersurface  $K^n$  in the following way:  $\phi B = B\psi \oplus (\eta \oplus N)$ ,  $\phi N = B\xi$ , where  $B$  is the differential of the immersion  $IK$  into  $M^n$ , and  $N \in (M^n)_p$ ,  $N \notin T(K)_p$ , for all  $p \in K$ .*

**Theorem 1.2.** *If there is a  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure on  $M^n$ , the operators,  $l = -(a_n f^{n-1} + a_{n-1} f^{n-2} + a_{n-2} f^{n-3} + \dots + a_2 f^1)/a_1$*

$$(1.3) \quad m = I + (a_n f^{n-1} + a_{n-1} f^{n-2} + a_{n-2} f^{n-3} + \dots + a_2 f^1)/a_1$$

*such that  $l = f^2$  and  $m = I - f^2$ , where  $I$  denote the identify operator on  $M^n$ , applied to the tangent space at a point of the manifold, are complimentary projection operators.*

There exists complimentary distributions  $L$  and  $M$  corresponding to the projection operators  $l$  and  $m$  respectively.

## 2. The Structure on the Hypersurface

**Theorem 2.1.** *Let  $M^n$  be a manifold with a  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure of rank  $r$  and let  $N^n$  be a hypersurface in  $M^n$ ,  $m = n-1$ . If the dimension of  $T(M^n)_p \cap f(T(M^n))_p$  is constant, say  $s$ , for all  $p \in M^n$ , then  $N^m$  possesses a natural  $F(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure of rank  $s$ .*

**Proof :** Let  $C$  be a transversal defined on  $N^m$ , i.e.  $C \in T(M^n)_p$ , but  $C \notin T(M^n)_p$  for all  $p \in M^n$ . Let  $B$  be a differential of the imbedding of  $N^m$  in  $M^n$ . Then  $B$  is a map of  $T(N^m)$  into  $T_r(M^n)$ , where  $T_r(M^n)$  denotes the restriction of  $T(M^n)$ , the tangent bundle of  $M^n$  to  $N^m$ . Then we can find a locally 1-form  $C^*$  defined on  $N^m$  such that:

$$B^{-1}B = I, BB^{-1} = I - C^* \otimes C,$$

$$C^*B = B^{-1}C = 0, C^*(C) = I.$$

Let  $F$  be defined locally on  $T(N^m)$  by  $F = B^{-1}fB$ . Then,

$$\begin{aligned}
 (2.1) \quad FX &= B^{-1}fBX \\
 F^2X &= B^{-1}fBB^{-1}fBX = B^{-1}f(BB^{-1})f(BX) \\
 &= B^{-1}f(I - C^* \otimes C)f(BX) \\
 &= B^{-1}f^2(BX) - B^{-1}fC^* \otimes Cf(BX) \\
 &= B^{-1}f^2(BX) - C^*f(BX)B^{-1}f \otimes C \\
 &= B^{-1}f^2(BX) - C^*f(BX)B^{-1}fC.
 \end{aligned}$$

If  $C$  is in distribution  $M$ , then  $fC = 0$ . We have that

$$(2.2) \quad F^2X = B^{-1}f^2BX,$$

Similarly

$$\begin{aligned}
 F^3X &= B^{-1}fBB^{-1}f^2BX \\
 &= B^{-1}f(BB^{-1})f^2(BX) \\
 &= B^{-1}f(I - C^* \otimes C)f^2(BX) \\
 &= B^{-1}f^2(BX) - B^{-1}fC^* \otimes Cf^2(BX) \\
 &= B^{-1}f^2(BX) - C^*f(BX)B^{-1}f \otimes C \\
 &= B^{-1}f^2(BX) - C^*f^2(BX)B^{-1}fC.
 \end{aligned}$$

If  $C$  is in distribution  $M$ , then  $fC = 0$ . We have that

$$(2.3) \quad F^3 X = B^{-1} f^3 B X.$$

Similarly

$$(a_n F^n + a_{n-1} F^{n-1} + a_{n-2} F^{n-2} + \dots + a_2 F^2 + a_1 F^1) X$$

$$B^{-1} (a_n f^n + a_{n-1} f^{n-1} + a_{n-2} f^{n-2} + \dots + a_2 f^2 + a_1 f^1) B X = 0.$$

{Using the relation (1.1), (2.1), (2.2), and (2.3) }

Yano<sup>1</sup> has shown that an  $f$  structure of maximal rank is an almost complex structure if this rank is even or an almost contact structure if this rank is odd.

**Corollary 2.1** *Let  $M^n$  be a manifold with almost contact structure  $(\psi, \xi, \eta)$  and let  $M^{n-1}$  be a hypersurface of  $M^n$ . Then the natural tensor field  $\psi|_M = B^{-1} \psi B$  defines an almost complex structure on  $M^n$  if and only if  $\xi$  is nowhere tangent to  $M^n$ .*

**Proof:** The if part is proved in theorem (2.1).

If  $\xi$  is everywhere tangent to  $M^n$  ( recall that we are considering hypersurfaces  $M^n$  where  $\xi$  is everywhere or nowhere tangent to  $M^n$ ) then  $B^{-1} \psi B$  is an  $f$ -structure of rank  $n-2$  and hence is not an almost complex structure.

**Theorem 2.2.** *If  $(f, \xi, \eta)$  is an almost paracontact structure on  $M^n$  then  $N^{n-1}$  possesses a natural  $F(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure if  $\xi$  is tangent to  $N^{n-1}$ . The hypersurface  $N^{n-1}$  possesses a natural almost product structure if  $n+1$  is not a tangent to  $N^{n-1}$ .*

**Proof :** When  $\xi$  is not a tangent to  $N^{n-1}$ ,  $\xi$  can be chosen for a pseudonormal. Then we have from theorem (2.1). That  $T(N^{n-1}) \cap f(TN^{n-1}) = T(N^{n-1})$  and  $\text{rank } f = \dim N^{n-1} = n-1$ . The

almost paracontact structure  $f$  has a maximal rank, i.e.  $f$  is an almost product structure.

### 3. Integrability Conditions

The structure  $f$  is integrable if  $[f, f] = 0$ , where  $[f, f]$  denotes the Nijenhuis tensor of  $f$ , i.e.

$$(3.1) \quad [f, f](X, Y) = [fX, fY] - [fX, Y] - [X, fY] + f^2[X, Y],$$

where  $X$  and  $Y$  are vector fields on  $M^n$ .

In this section we shall assume that  $f$  is an  $f(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure on  $M^n$  and  $F = B^{-1}fB$  is the naturally induced  $F(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure.

**Theorem 3.1.** *Let  $N^{n-1}$  be a hypersurface in  $M^n$ , and suppose  $f$  is integrable. If, locally, the transversal to  $N^{n-1}$  can be found to lie in the distribution  $M$ , then the induced  $F(a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1)$  structure on  $N^{n-1}$  is integrable.*

**Proof :** We see that

$$\begin{aligned} [F, F](X, Y) &= [FX, FY] - [FX, Y] - [X, FY] + F^2[X, Y] \\ &= [B^{-1}fBX, B^{-1}fBY] - [B^{-1}fBX, Y] - [X, B^{-1}fBY] + B^{-1}fBF[X, Y] \\ &= B^{-1}[BB^{-1}fBX, BB^{-1}fBY] - B^{-1}[BB^{-1}fBX, BY] \\ &\quad - B^{-1}[BX, BB^{-1}fBY] + B^{-1}fBB^{-1}f[BX, BY]. \\ &= B^{-1}[(I - C^* \otimes C)fBX, (I - C^* \otimes C)fBY] - B^{-1}[(I - C^* \otimes C)fBX, BY] \\ &\quad - B^{-1}[(BX, BB^{-1}fBY] + B^{-1}f[(I - C^* \otimes C)f[BX, BY]. \\ &= B^{-1}\{[f, f](BX, BY) - [C^*(fBX)C, fBY] - [fBX, C^*(fBY)C] \\ &\quad + [C^*(fBX)C, C^*(fBY)C] + f[C^*(fBX)C, BY] \} \end{aligned}$$

$$+ f[BX, C^* (fBY)C] - C^* f[BX, BY]fC \}.$$

where we have used the fact that  $B(X, Y) = [BX, BY]$ , for vector fields  $X, Y$  on  $N^{n-1}$ , and that  $B^{-1}B = I$ ,  $BB^{-1} = I - C^* \otimes C$ . If the transversal  $C$  lies in the distribution  $M$ , then the form  $C^*$  can be chosen so that  $C^* = 0$ . We see that:  $[F, F](X, Y) = B^{-1}([f, f](BX, BY)) = 0$ .

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