Fixed Point Result for a Non-commuting Family of Continuous Maps

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Abstract. In this paper we prove a fixed point theorem for non-commuting family of continuous maps. Our result improve and extend the result of De Marr\textsuperscript{1}, Kirk\textsuperscript{2} and Guo Jing Jiang\textsuperscript{3}. Suitable example is given in support.

1. Introduction

De Marr\textsuperscript{1} established a common fixed point theorem for a family of non-expansive maps in Banach spaces. Later, Kirk\textsuperscript{2} proved a fixed point theorem for non-expansive map in reflexive Banach spaces. Recently Jiang\textsuperscript{3} extended the result of De Marr\textsuperscript{1} for commuting family of continuous maps.

The purpose of this paper is to ensure fixed point in Banach space for a non-commuting (weakly commuting, $R$-weakly commuting) family of continuous maps instead of commuting family of continuous maps.

2. Preliminaries

Let us recall the following definitions

**Definition 2.1.** Let $B$ be a Banach space and $X \subset B$. Let $E$ and $F$ be two self maps of $X$. They are called weakly commuting if

$$\| EFx - FEx \| \leq \| Ex - Fx \|$$

for all $x$ in $X$.

**Definition 2.2.** Let $B$ be a Banach space and $X \subset B$. Let $E$ and $F$ be two self maps of $X$. They are said to be $R$-weakly commuting if there exists some positive real number $R$ such that

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\[ \| EFx - FEx \| \leq R \| Ex - Fx \|. \]

This definition was introduced by Pant in 1994.

**Remark 2.1** Weak commutativity implies R–weak commutativity. But, R–weak commutativity implies weak commutativity only when \( R < 1 \).

**Definition 2.3** Let \( B \) be a Banach space and \( X \subset B \). Then family \( \mathcal{F} \) of self maps is called non–commuting family of self maps if its members satisfy non–commuting conditions (weak commuting, R–weakly commuting) to each other.

Throughout this paper we shall denote the diameter and closed convex hull of a subset \( A \) of \( B \) by \( \delta (A) \) and \( \text{clco} \, A \) respectively.

Following lemmas are needed to prove the main theorem.

**Lemma 2.1** [3] Let \( B \) be a Banach space, \( M \) a compact subset of \( B \) with \( \delta (M) > 0 \). Then there exists an element \( v \in \text{clco} \, M \) such that

\[ \sup \{ \| x - v \| : x \in M \} < \delta (M). \]

**Lemma 2.2** Let \( X_0 \) be a non–empty convex subset of a Banach space \( B \). Let \( E \) and \( F \) be two self maps of \( X_0 \) in \( \mathcal{F} \) satisfying

\[ \| Ex - Fy \| \leq \max \{ \| x - y \|, \min (\| x - Fy \|, \| y - Ex \|) \} \]

for \( x, y \in X_0 \). Assume that there is a compact set \( M \subset X_0 \) such that \( EM = M, FM = M \) and \( \delta (M) > 0 \). Then there exists a non–empty closed convex set \( K \) such that \( K \cap X_0 \) is invariant under \( E \) and \( F \) and \( M \cap (B - K) \neq \emptyset \).

**Proof.** By Lemma 1 there exists an element \( v \in \text{clco} \, M \) such that \( r = \sup \{ \| x - v \| : x \in M \} < \delta (M) \). Let \( \cup (x) = \{ y : \| y - x \| \leq r \) and \( y \in B \)\} for \( x \in M \). \( \cup (x) \) is closed and convex for each \( x \in M \). Let us set \( K = \bigcap_{x \in M} \cup (x) \), then \( K \) is also closed and convex and at least \( v \in K \) i.e. \( K \neq \emptyset \). For any \( x \in K \cap X_0 \) and any \( w \in M \) we have \( x \in \cup (w) \). Also we have \( FM = M \). Then there must exists \( y \in M \) such that \( w = Fy \). By (1), we have

\[ \| Ex - w \| = \| Ex - Fy \| \]

\[ \leq \max \{ \| x - y \|, \min (\| x - Fy \|, \| y - Ex \|) \} \]

\[ \leq \max \{ r, \min (r, \| y - Ex \|) \} \]

\[ = r \]
which implies that $Ex \in \cup (w); Ex \in K \cap X_0$.

This proves that $K \cap X_0$ is invariant under $E$. Similarly for any $y \in K \cap X_0$ and $z \in M$ we have $y \in \cup (z)$ and also $EM = M$. Then there exists $x \in M$ such that $z = Ex$. Again by (1)

$$\|z - Fy\| = \|Ex - Fy\|$$

$$\leq \max \{\|x - y\|, \min (\|x - Fy\|, \|y - Ex\|)\}$$

$$\leq \max \{r, \min (\|x - Fy\|, r)\}$$

$$= r$$

which implies that $Fy \in \cup (z); Fy \in K \cap X_0$. This prove that $K \cap X_0$ is invariant under $F$.

Now, since $M$ is compact, there exists $l, m \in M$ such that $\|l - m\| = \delta (M) > r$, which shows that

$$m \in \cup (l) \supseteq K; \text{ i.e. } m \in M \cap (B - K) \Rightarrow M \cap (B - K) \neq \emptyset.$$

3. Main Result

Now, we prove main theorem.

**Theorem 3.1** Let $B$ be a Banach space and $X$ be non-empty compact convex subset of $B$. $\mathcal{F}$ is non-empty $R$-weakly commutative family of continuous maps of $X$ into itself. If $\mathcal{F}$ satisfy (1) for $x, y \in X$ and $E, F \in \mathcal{F}$ then the family $\mathcal{F}$ has a common fixed point in $X$.

**Proof.** By the Zorn’s lemma we can obtain a subset $X_0^1$ of $X$ which is minimal with respect to being non-empty closed convex and by each self map of $\mathcal{F}$. Similarly we can obtain a subset $M$ of $X$ which is minimum with respect to being non-empty compact and by each self map of $\mathcal{F}$, and also $M \subset X_0$. Now we will show $EM = M$ for each $E \in \mathcal{F}$ otherwise $F \in \mathcal{F}$ such that $FM$ is non-empty subset of $M$. Now since $EF$ are $R$-weakly commuting, for some positive $R$ we have $EFM = FEM \subset FM$ for each $E \in \mathcal{F}$, i.e. $FM$ is invariant under each $E \in \mathcal{F}$. By the continuity of $F$ and compactness of $M$, we get $FM$ is compact and the minimality of $M$ yields that $FM = M$ which is contradiction.
Now we prove that $M$ is singleton. Let us suppose contrary then $\delta(M) > r$ and by Lemma-2, $K \cap X_0$ is invariant for each map of $\mathcal{F}$. Since $M \cap (B-K) \neq \phi$, $X_0 \cap (B-K) \neq \phi$. Thus $K \cap X_0 \subset X_0$. Since $K$ is closed, $K \cap X_0$ is non-empty compact, convex and proper subset of $X_0$, which is contradiction to the minimality of $X_0$. Thus $M$ is singleton which is fixed point of $\mathcal{F}$.

As a consequence of Theorem 3.1, we can obtain the following corollary with weakly commutative family of continuous maps.

**Corollary 3.1** Let $B$ be a Banach space and $X$ be non-empty compact convex subset of $B$. $\mathcal{F}$ is non-empty weakly commutative family of continuous maps $X$ into itself. If $\mathcal{F}$ satisfy (1) for $x, y \in X$ and $E, F \in \mathcal{F}$ then the family $\mathcal{F}$ has a common fixed point in $X$.

**Proof.** Similar to the prove of theorem 3.1 with $R < 1$.

**Remark 3.1** Theorem 3.1 extend and improve the result of the authors$^{1,2,3}$ and $[3]$ with non-commuting family of continuous maps.

In support of the Theorem 3.1 we give the following example:

**Example.** Let $X = [0, 1]$ with $\|x - y\| = |x - y|$ for $x, y \in X$. Let us define $f_i$ from $X$ to $X$ for each $i \in \mathbb{N}$ as

$$f_i(x) = \frac{x^i}{i}, \quad x \in X, \quad i \in \mathbb{N}.$$ 

Hear for each $i, f_i$ is continuous and $\mathcal{F} = \{f_1, f_2, f_3, \ldots\}$ is a non-empty, non-commuting family of continuous maps family $\mathcal{F}$ satisfy (1) for $x, y \in X$. Thus $\{0\}$ is a common fixed point of $\mathcal{F}$.

**References**