

Finding the Missing Link between Two Extrema of Wind Speed Patterns: Use of Quintessence Contaminated Pseudo Newtonian Potential

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(Received January 12, 2020)

Abstract: In the literature, two distinct patterns of wind are followed. For adiabatic accretion onto Kerr metric, the wind curve turns almost constant valued, i.e., parallel to the radial distance axis as we go far from the central gravitating object. On the other hand, the accretion model of dark energy candidate shows a wind curve which is parallel to the velocity axis, i.e., the wind speed reaches the value of light's speed at a finite radial distance. So far we fail to find any intermediate wind speed profile style. These works are done with a methodology where the highly nonlinearity grown due to general relativity is avoided by introducing a pseudo Newtonian potential which is a function of the radial distance from the black hole and the spin parameter of the black hole only. In this present work, we are motivated to find some system where the wind speed may stay in the intermediate region of the said extremities. A black hole sitting in a quintessence tub shows repulsive nature as an accretor. Viscosity, on the other hand, does act like a heating up/ frictional element which opposes a smooth inward flow. We check the joint effects on a quintessence black hole as well as viscosity on accretion disc. The variations of density profiles are studied along with the corresponding shear viscosity to entropy density ratio. Results are tallied with the conventional predictions about the values of such quantities.

Keywords: Supermassive Black Holes, Accretion Disc, Adiabatic Gas, Dark Energy, Quintessence, Modified Chaplygin Gas.

PACS : 97.10. Gz; 51.20. +d; 51.30. +i; 95.36. +x; 97.60. Lf.

1. Symbols and Notations Used

a : black hole spin parameter

α : modified Chaplygin gas parameter related to the barotropic part

α_{SS} : Shakura Sunyaev viscosity parameter

β : modified Chaplygin gas parameter related to the negative pressure part
 c : light speed, c_s : sound speed η : shear viscosity coefficient
 F_g : pseudo Newtonian force G : Newton's gravitational constant
 $h(r)$: accretion disc height at radial distance, k_B : Boltzmann's constant
 Λ : cosmological constant M : mass of central black hole
 $M_\odot$: Solar mass m_p : mass of proton
 n : modified Chaplygin gas parameter related to the power of negative pressure part
 Ω : angular speed p : pressure of dark energy
 ϕ : azimuthal coordinate in cylindrical coordinate system
 r : radial coordinate in cylindrical coordinate system
 ρ : density of modified Chaplygin gas s : Entropy density
 σ_T : Thompson scattering cross section t : time coordinate
 $\tau_{r\phi}$: hydrodynamic stress tensor v_ϕ : azimuthal or angular speed of fluid
 v_r : radial inward speed of fluid x : $\frac{r}{r^2}$ $\dot{M}$: Mass accretion rate
 z : vertical coordinate in cylindrical coordinate system
Please Note: Astrophysical quantities are conventionally given in CGS units rather than SI units. We have followed this convention.

2. Introduction

Due Since 1998, although different propositions of conceptual models were done, we are not really able to construct a hundred percent correct theoretical support for the observations of different supernovae of type Ia which reveals late time cosmic acceleration. Stress energy tensor is redesigned to interpret accelerated expansion. The nomenclature of candidates of such a repulsive energy/ hypothetical matter are familiar as dark energy (DE hereafter)¹. Molecular structures of DE were not revealed till date.

Besides DE, another exotic matter/energy is found to exist in literature which is popularly known as dark matter (DM hereafter). This is the "missing link" type of portion of the matter-energy budget of the universe which is responsible for the high galactic rotational curves, gravitational lensing of different regions etc. There exist some references^{2,3} where DM-DE interactive models are proposed to support cosmic acceleration and DE's viscosity. These references have considered "delayed decay of cold DM" (from initial matter dominated flat cosmology with $\Lambda = 0$) into light

undetectable relativistic species. This generates bulk viscosity and a $\Lambda = -1$ epoch near the present time neighborhood. Deceleration for such a model is found to be faster due to the presence of the matter content. Also the model is found to be consistent with the surveys of supernova magnitude-redshift relation^{4,5} and ages from the superannuated stars and globular clusters.

Among many proposed models/ candidates of DE, modified Chaplygin gas (MCG) is an important one which on regulating its different parameters takes the form of radiation or dust like or quintessence and even phantom era. The equation of state is given as

$$(2.1) \quad p = \alpha\rho - \frac{\beta}{\rho^n}.$$

Viscous effects of a fluid are more realizable when it flows; particularly layer by layer. Most prominent examples are the narrow X-ray binaries where an accretion disc is likely to get formed. The most simple diagram of such a Roche lobe overflow incident can be imagined with some assumptions: Consider that the flow is axis-symmetric. i.e., causing a cylindrical structure around a compact star-preferably a black hole (BH hereafter) where r , ϕ and z are the coordinates. We will consider $\frac{\partial}{\partial\phi} \equiv 0$ and also a stationary disc, i.e., $\frac{\partial}{\partial t} \equiv 0$. We will further consider a thin disc, i.e., $\frac{h(r)}{r} \ll 1$, where $h(r)$ is the disc height. The viscous effect of the disc is also taken to be small, i.e., the radial inward velocity $v_r \ll$ the local Keplerian rotational speed/azimuthal velocity $v_\phi = \Omega r = \frac{GM}{r}$, if the radial momentum is conserved. On the other hand, conservation of angular momentum requires effects of viscous forces to take into account. Keplerian balance always implies differential rotation. A transportation of angular momentum in the direction, perpendicular to the velocity, should be observed due to the differences in velocities at different locations. If this is absent, translational (shear) viscosity will turn down to zero. To simplify the nonzero viscosity, a simplistic description of the physics of accretion disc can be obtained by α_{SS} ⁶ prescription. Though the accretion driven/driving viscosity is of magnetic origin, it is popular to use an effective hydrodynamic description of the related disc presented by the hydrodynamic stress tensor as⁷:

$$(2.2) \quad \tau_{r\phi} = \alpha_{SS}P \Rightarrow \gamma = \alpha_{SS}c_s^2 \left[\frac{dr}{dh} \right]^{-1}.$$

This will be a way to replace the kinematic viscosity by α_{SS} parameter.

The value of α_{SS} is typically assumed to lie between 0.01 to 0.1⁸. Fromang et al.⁹, in 2011, have found a radially varying α_{SS} , the overall size of which was over an order of magnitude lower, peaking at 0.013 and declining to below 0.002. It is clear that, to measure viscosity, we will require to know the variation of density as well. Again accretion density and many other properties are dependent on the spin parameter of the central gravitating BH.

Turner et.al.², in 1984, have shown the range to be $0.996 \leq a \leq 0.998$. Again, for the same Seyfert I galaxy Fink measured the number density of the accretion disc to be 10^{15}cm^{-3} . Turner et.al. have also measured it as $10^{15.95} \text{cm}^{-3}$. Mean molecular weight of Sun is chosen as 0.62 (ionized gas) and hence the density of the taken accretion disc is found to be $\propto 6.2 \times 10^{15} \text{gmcm}^{-3}$.

Bondi or Shakura-Sunyaev considered accretion dynamics by considering Newtonian potential. The essential general relativistic effects on the curvature, i.e. the gravity around a BH was not taken into account. The later work has considered the only general relativistic effect by truncating the innermost edge of the disc at the last stable orbit of the Schwarzschild geometry. Novikov and Thorne¹⁰, in 1973, have developed a complete general relativistic description of a thin Keplerian disc. Ghosh et al¹¹, in 2007, led this trend towards completeness.

To reduce general relativistic non-linearity, it is helpful to consider stationary flow and to replace the general relativistic effect by the introduction of pseudo Newtonian potentials. Paczynsky & Witta¹², in 1980, proposed such a force which exactly reproduces marginally stable orbit and marginally bound orbit of that infall GR. But this potential does consider only BH's mass. As almost all the celestial objects are rotating, the BHs are also rotating and Mukhopadhyay¹³, in 2002, developed a PNP for a rotating BH for the first time. Sarkar and Biswas¹⁴, in 2019, have constructed a PNP for a rotating BH embedded in quintessence ($p = \omega_q \rho$). This model proposes at most 4.95% error as compared to GR results. Roy and Biswas have modeled an accretion structure with Sarkar & Biswas's potential. For this model, the first requisite cases should be known why accretion onto such a quintessence contaminated BH.

This is obvious to consider supermassive black holes (SMBHs) in the center of galaxies through the cases of such a presence is not justified. We are able to observe SMBHs at redshift $z = 7.54$ which must have formed within less than one billion years¹⁵. Alternative models to BHs, inclusion

of extended objects in classical general relativity¹⁶, consideration of existence of more exotic models, viz “naked singularity”¹⁷ have been considered. So far the motivations of these works were to consider only the gravitational effects of alternatives to BHs and to find out their observational properties in order to distinguish a BH from a so-called BH mimicker. Recently, in the references like¹⁸, a possibility of DM, in the form of bosons, to form self gravitating bound structures in different galaxies are studied. Boshkayev et al ¹⁹, in 2019, have compared the motion of test particles in the gravitational field of both SMBH and DM core. A significant discrepancy in the motion is noticeable 100AU and this increases as we are approaching the center.

To study the corresponding density, we will take the help of the idea of Eddington limit. Let Ξ is the Eddington mass accretion rate of the considered system and ϵ fraction of the mass is supposed to generate energy, then

$$(2.3) \quad \Xi = \frac{4\pi G M m_p}{\epsilon c \sigma_T} \text{ and we choose } M=10^{6+\sigma} M_{\odot} \text{ and } \epsilon = 3 \times 10^{-1-\psi} .$$

Which on simplification gives

$$(2.4) \quad \rho = \frac{4\pi m_p 10^{1+\psi}}{3c\sigma_T} \times \frac{\sqrt{\frac{F_g}{x^3}}}{u c_s} .$$

Once the density is constructed, it is even easy to calculate and analyse the value of $\frac{\eta}{s}$.

Dual holographic nature of states is predicted by strongly interacting quantum field theories. For example, we can choose the systems where BHs are embedded in AdS space. For such a system, a universal lower bound of the shear viscosity coefficient (η) to entropy density (s) ratio is prescribed as^{20- 24}

$$(2.5) \quad \frac{\eta}{s} \gg \frac{1}{4\pi} \frac{\hbar c}{k_B}$$

This lower bound is popularly known as the Kovtun-Starinets-Son (KSS) bound. However, Jakovac²⁵, in 2010, has calculated the η/s ratio mathematically. To do this, he has assumed some physical conditions for the spectral functions and kept the entropy density constant. He observed that the lower bound may not be universal for some systems which carry

quasi-particle constituents with small wave function renormalization constant, high temperature strongly interacting systems or systems with low temperatures and zero mass excitation. As we have discussed in the introductory section of this article, DE may possess a very small amount of shear viscosity. Besides, the entropy density for DE should be high due to its repulsive nature. Entropy for different components of the universe shows²⁶. So for a phenomenon which involves both SMBHs and DE, the η/s ratio may fall and can become lower than the theoretical prediction as well.

The rest of the paper is organized as follows. In section 2, we reorganize the structure of the PNF for a rotating BH embedded in the quintessence universe. Then we construct the mathematical problem for our model. In section 3 we find solutions for radially inward speed, speed of sound and specific angular momentum as a function of radial distance from the BH. We thoroughly analyze these curves as well. In subsection 3, we find the variation of densities of accretion and wind for different parameters and have explained them. Subsection 3 deals with the study of the ratio of shear viscosity to entropy density for our model. In section 4, we concluded in brief.

3. Mathematical Modeling of Viscous Accretion onto Rotating SMBH

We write our pseudo-Newtonian potential as

$$(3.1) \quad F_x = \frac{N(x, a, \omega_q)}{D(x, a, \omega_q)},$$

where $N(x, a, \omega_q) = \{\alpha(x, a, \omega_q) + \beta(x, a, \omega_q) - \sqrt{\gamma(x, a, \omega_q)}\}^2$, with

$$\alpha(x, a, \omega_q) = \zeta(x, a, \omega_q)\{3\omega_q(a^2 + x^2) + 3x^2 - 8x + a^2\},$$

$$\beta(x, a, \omega_q) = (2a^2 + 6x - 8)\zeta^2(x, a, \omega_q)\left(\frac{A_q^{-2}}{a}\right) - 2aA_q^2x$$

$$\gamma(x, a, \omega_q) = 2\eta^4(x, \omega_q)\{(x^2 - 2x + a^2)x^{3\omega_q} - A_qx\}^2\{A_q(3\omega_q + 1) + 2\zeta(x, a, \omega_q)/(aA_q)\},$$

Along with $\zeta(x, a, \omega_q) = aA_qx^{3\omega_q}$ and $\eta(x, \omega_q) = x^{3(\omega_q-1)}$

The denominator is written as

$$D(x, a, \omega_q) = x^3 \{ \phi(x, a, \omega_q) + 2x\psi^2(x, a, \omega_q) \}$$

where

$$\phi(x, a, \omega_q) = a\zeta(x, a, \omega_q)(3\omega_q + 1) + 2a^2x^{6\omega_q}$$

and

$$\psi(x, a, \omega_q) = \zeta(x, a, \omega_q)x^{-3\omega_q}a^{-1} + (2 - x)x^{3\omega_q}.$$

The continuity equation for our system reduces to

$$(3.2) \quad \frac{d}{dx} \{ x u \rho h(x) \} = 0.$$

The components of Navier-Stokes equations can be combined with equation (2.5) and modified Chaplygin gas equation of state (2.1) to be seen as

$$(3.3) \quad \frac{du}{dx} = \frac{u}{x^3} \frac{\{\lambda^2 - x^3 F_g(x) \Psi(x, a, \omega_q) + x^3 \mu(x, a, \omega_q) c_s^4\}}{\Psi(x, a, \omega_q) u^2 - 2c_s^4},$$

where $\Psi(x, a, \omega_q) = c_s^2 + (\alpha - c_s^2)n + \alpha$ and $\mu(x, a, \omega_q) = (3 - \frac{x}{F_g} \frac{dF_g}{dx})$.

$$(3.4) \quad \frac{dc_s}{dx} = \left[\frac{1}{2} \mu(x, a, \omega_q) + \frac{1}{u} \frac{du}{dx} \right] \frac{(n+1)c_s(c_s^2 - \alpha)}{\Psi(x, a, \omega_q)}$$

and

$$(3.5) \quad \begin{aligned} \frac{d\lambda}{dx} = & \frac{x\alpha_{ss}}{u} \left[\frac{1}{2x} \left(5 - \frac{x}{F_g} \frac{dF_g}{dx} \right) \left\{ u^2 + \frac{(1+n)\alpha - c_s^2}{n} \right\} + 2u \frac{du}{dx} \right. \\ & \left. + \frac{2c_s}{c_s^2 - \alpha} \right] \frac{dc_s}{dx} \left\{ \left(u^2 + \frac{(1+n)\alpha - c_s^2}{n} \right) \frac{1}{c_s} - (c_s^2 + u^2) \left(\frac{1}{1+n} \frac{2c_s}{c_s^2 - \alpha} \right) \right\} \frac{dc_s}{dx}. \end{aligned}$$

Equation (3.3) simply says that the denominator may turn out to be zero in the region (0,1). As the fluid speed is almost zero when we are far enough from the BH and is almost equal to the speed of light when we are at the event horizon, we may speculate that there exist a critical radial

distance x_c where the denominator vanishes. Equivalently, the numerator should also vanish to keep the flow physical and continuous. At $x = x_c$, using L'Hospital's rule we may obtain a second degree first order differential equation.

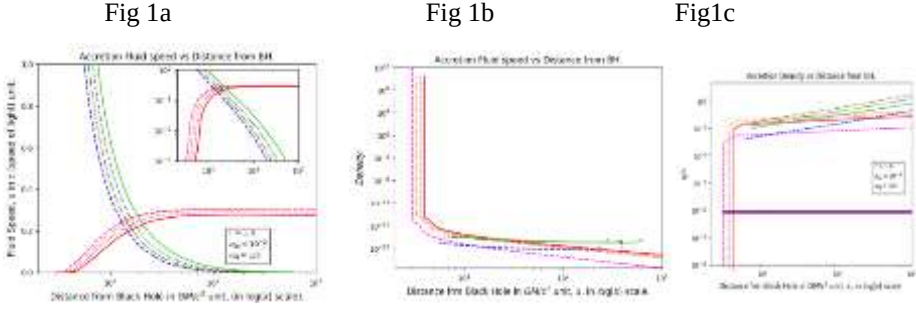
4. Solutions: Velocity Profiles, Density Profiles and the $\frac{\eta}{s}$ Profiles

We have plotted the velocity, density and $\frac{\eta}{s}$ profiles for accretion with $\lambda_c = 2.7, \alpha_{ss} = 10^{-2}$ of adiabatic fluid with $\Gamma = 1.6$, in figures 1a-1c ($\omega_q = \frac{1}{3}, A_q = 0.01$); figures 2a-2c ($\omega_q = -\frac{2}{3}, A_q = 10^{-10}$) and figures 3a-3c ($\omega_q = -1, A_q = 10^{-10}$).

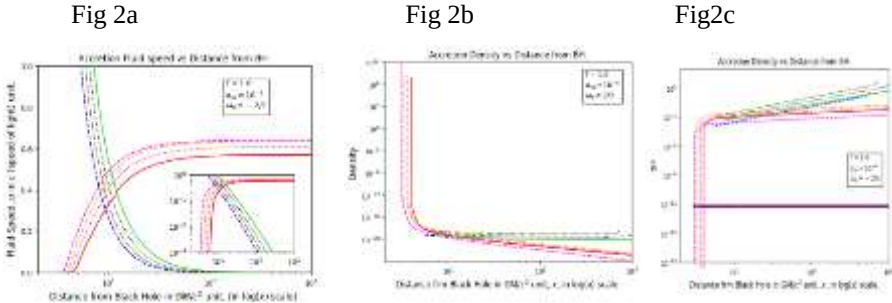
Accretion speed is rising as we move towards the BH. Wind speed is low near the BH. As we go far, it increases and then becomes almost constant. Point to be noted that the accretion becomes fainter as we move far from the BH. This fainting rate increases as the value of the spin parameter increases. The more is the rotation, the nearer the sensible wind flow starts from the BH. So, inclusion of viscosity reduces the angular momentum transport efficiency which ultimately causes the reduction of the physical radius of the disc.

These three sets of graphs conclude that a viscous accretion onto a quintessence contaminated BH is weakened/ shortened by four factors: the rotation of the BH, negativity of ω_q in which the BH is embedded, value of α_{ss} , i.e., the viscosity imposed and the negativity inserted in EoS of accreting fluid. If all these factors increase, the accretion disc faints causing a weaker feeding process.

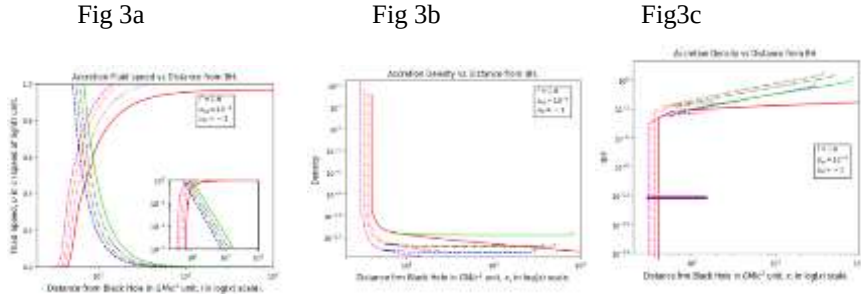
Legends for the figures 1a-3c : Red and green solid lines : Wind and Accretion at $a=0$ respectively; Purple and black dotted lines : Wind and Accretion at $a=0.5$ respectively; Orange and olive dashed dotted lines : Wind and Accretion at $a=0.9$ respectively; Pink and Blue dashed-dashed lines : Wind and Accretion at $a=0.998$ respectively



Figures 1a, 1b and 1c are fluid speed, density and $\frac{\eta}{s}$ profiles respectively for adiabatic accretion for $\Gamma = 1.6, \lambda_c = 2.7, \alpha_{SS} = 10^{-2}, \omega_q = \frac{1}{3}, A_q = 0.01$.



Figures 2a, 2b and 2c are fluid speed, density and $\frac{\eta}{s}$ profiles respectively for adiabatic accretion for $\Gamma = 1.6, \lambda_c = 2.7, \alpha_{SS} = 10^{-2}, \omega_q = -\frac{2}{3}, A_q = 10^{-10}$.



Figures 3a, 3b and 3c are fluid speed, density and $\frac{\eta}{s}$ profiles respectively for adiabatic accretion for $\Gamma = 1.6, \lambda_c = 2.7, \alpha_{SS} = 10^{-2}, \omega_q = -1, A_q = 10^{-10}$.

5. Brief Discussions and Conclusions

This present article can be treated as a detailed study of the viscous accreti-

on onto a rotating black hole embedded in a quintessence universe and the consequent thermodynamic phenomena. To construct the mathematical model we have chosen a particular type of black hole which has mass and rotation as signature properties along with a special type of background. Quintessence is a hypothetical fluid which is theorized to create repulsive force responsible for late time cosmic acceleration. We choose a rotating black hole solution which carries effects of the quintessence universe in it. The gravitational effect of such a black hole is implied through a pseudo Newtonian potential. This is done as direct general relativistic nonlinear differential equations are difficult to solve. Viscous effect is adopted through the Shakura and Sunyaev α_{SS} effects.

We follow that if the viscosity is high the accretion branch's fluid speed steeply falls down as we go far from the central black hole. Wind speed increases as we increase viscosity. But the radial distance wise shift is small. At a finite distance fluid speed becomes equal to that of light. As we increase the quintessential effect, wind speed increases.

Density profiles are found to be very interesting. At the edges of the disc, approximately at the order of thousand Schwarzschild radius distance the density is found to be very low. But of course, this was higher than the density of the universe. At the nearer vicinity of the SMBH, we see the wind density to raise upto the order of $10^{12} \text{ gm cm}^{-3}$. This quite matches with the observational results. Finally, we study the η/s ratio and follow that this ratio turns out to be less than the theoretical predictions. Present day speculation about the shear viscosity of DE supports this result. Interestingly, we achieve the result where η/s is lower than the pre-predicted value for adiabatic accretion. Only the dark energy contamination is considered for the BH metric itself, not onto the accreting fluid's property. Our results strongly state that far late times BHs, accretion of adiabatic fluid can even reduce the η/s ratio.

Acknowledgments: This research is supported by the project grant of the Government of West Bengal, Department of Higher Education, Science and Technology and Biotechnology (File no:- ST /P/S&T /16G – 19/2017). PB thanks Swami Vivekananda Merit Cum Means Fellowship of West Bengal State Government, India.

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