

Maximin Degree Domination Number in Graphs

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Abstract: A set $D \subseteq V(G)$ of a graph G is called a *dominating set* G if every vertex $u \in V - D$ is adjacent to at least one vertex $v \in D$. In this paper we define a maximin degree dominating set D of a graph G as a dominating set in which $\delta(\langle D \rangle)$ is maximum. The minimum cardinality of a maximin degree dominating set is the maximin degree domination number of the graph and is denoted by $\gamma_{\delta_{max}}(G)$. We discuss the basic properties of maximin degree dominating sets, maximin degree domination number of some well-known graphs, bounds on $\gamma_{\delta_{max}}(G)$ and the critical aspects of maximin degree domination number with respect to edges and vertices.

Keywords: Maximin degree domination number, Domination, Total domination, Connected domination, critical edges and vertices.

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1. Introduction

We consider here only the finite simple undirected graphs with no loops or multiple edges. For the notations and terminologies used here, the reader is referred to F. Harary¹, unless specified otherwise. We begin with a brief introduction to domination in graphs.

Definition 1.1. In a graph $G=(V, E)$ two vertices u and v are said to dominate each other if they are adjacent to each other in G . A set $D \subseteq V(G)$ is called a dominating set if for every vertex $u \in V - D$, there exists a vertex $v \in D$ such that u and v are adjacent. The minimum

cardinality of a dominating set is called the domination number of the graph G and is denoted by $\gamma(G)$.

In this paper we introduce a new domination parameter called the maximin degree domination number.

Definition 1.2. A dominating set D of a graph G is said to be a maximin degree dominating set if the minimum degree of the induced subgraph $\langle D \rangle$ of D is maximum, where the maximum is taken over all the dominating sets D of the graph. The minimum cardinality of a maximin degree dominating set is called as the maximin degree domination number of the graph and is denoted by $\gamma_{\delta_{max}}(G)$.

Example:

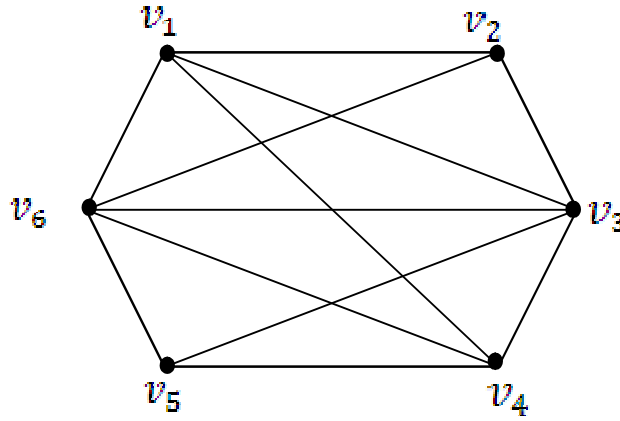


Fig.1.

In Fig.1. there are five maximin degree dominating sets, viz., $D_1 = \{v_1, v_3, v_4, v_5, v_6\}$, $D_2 = \{v_3, v_4, v_1, v_6\}$, $D_3 = \{v_3, v_4, v_5, v_6\}$, $D_4 = \{v_3, v_4, v_1, v_6, v_2\}$ and the vertex set V itself. The minimum degree of the induced subgraphs of all the above three dominating sets is 3. But among the three D_2 has minimum cardinality and hence $\gamma_{\delta_{max}}(G) = 4$ for the graph in Fig.1.

2. Maximin Degree Domination Number of Some Well-Known Graphs

In this section we obtain the value of $\gamma_{\delta_{max}}$ of some class of graphs.

Lemma 2.1. For a regular graph G of order p , $\gamma_{\delta_{max}}(G) = p$.

Proof: This is immediate from the observation that in any r - regular graph G , $\delta(G-v) < r$, for any vertex v of G .

Remark: Converse of the above lemma, however, need not be true. For example, $\gamma_{\delta_{max}}(W_n) = n$, where W_n is the wheel graph.

Theorem 2.2. For any complete bipartite graph $K_{m,n}$,

$$\gamma_{\delta_{max}}(K_{m,n}) = 2 \min\{m, n\}.$$

Proof: Let V_1 and V_2 be the partite sets of $K_{m,n}$ with cardinalities m and n respectively. Without loss of generality, let $m \leq n$. Then the induced subgraph of the set V_1 together with any m number of vertices from the set V_2 forms a complete bipartite graph $K_{m,m}$ which is a regular graph. Note that addition of any vertex of $V(K_{m,n}) - V(K_{m,m})$ to the induced subgraph $K_{m,m}$ will not increase the minimum degree. But the removal of any vertex from $K_{m,m}$ decreases the minimum degree. Hence

$$\gamma_{\delta_{max}}(K_{m,n}) = 2 \min\{m, n\}.$$

Cokayne, Dawes and Hedetniemi² introduced the concept of total domination in graphs as follows.

Definition 2.1. A dominating set D of a graph $G=(V, E)$ is said to be a total dominating set if every vertex of G is adjacent to at least one vertex of D . The minimum cardinality of a total dominating set is the total domination number of the graph and is denoted by $\gamma_t(G)$.

Theorem 2.3. For any tree T , $\gamma_{\delta_{max}}(T) = \gamma_t(T)$, where $\gamma_t(G)$ is the total domination number of the graph G .

Proof: Since any subgraph H of a tree T is a forest and the minimum degree of the induced subgraph $\langle H \rangle$ of H is 0 or 1. Hence it is not possible to obtain a dominating set D of tree T such that the minimum degree of the induced subgraph $\langle D \rangle$ of D is more than 1. But by definition of the total dominating set of a graph, it is the dominating set of the graph whose

induced subgraph has no isolated vertices. Hence any γ_t - set of a tree is also a $\gamma_{\delta_{max}}$ set of the tree. Thus $\gamma_{\delta_{max}}(T) = \gamma_t(T)$, for any tree T .

Corollary 2.3.1. *For any graph G without isolates, $\gamma_t(T) \leq \gamma_{\delta_{max}}(T)$.*

Proof: Without loss of generality, let us assume that the given graph G is connected. Then G contains a spanning tree, say, T . By the Theorem 2.3., we have, $\gamma_{\delta_{max}}(T) = \gamma_t(T)$. Hence the result follows.

Proposition 2.4. *If G is a graph with isolated vertices then $\gamma_{\delta_{max}}(G) = \gamma(G)$.*

Proof: As the isolated vertices of a graph G must be the elements of the γ -sets of the graph G , the minimum degree of induced subgraph of any dominating set of such graphs is always 0. Thus $\gamma_{\delta_{max}}(G) = \gamma(G)$.

3. Bounds on $\gamma_{\delta_{max}}$

Since $\delta(G) \leq \delta(\langle D \rangle)$, for any maximin degree dominating set D of G , every vertex v in a $\gamma_{\delta_{max}}$ -set D of a graph G is adjacent to at least $\delta(G) + 1$ number of vertices of D . Hence $\delta(G)$ is an obvious lower bound for $\gamma_{\delta_{max}}(G)$. The following theorem gives a better lower bound for $\gamma_{\delta_{max}}(G)$.

Theorem 3.1. *For a graph G without isolated vertices, $\gamma_{\delta_{max}}(G) \geq \left\lceil \frac{n\delta(G)}{\Delta(G)+1} \right\rceil$, where $\delta(G)$ and $\Delta(G)$ represents the minimum and maximum degrees of the given graph G respectively.*

Proof: Note that in a graph G each vertex can dominate at most $\Delta(G)$ number of vertices. In a $\gamma_{\delta_{max}}$ -set, every vertex is adjacent to at least $\delta(G)$ number of vertices. Thus $\gamma_{\delta_{max}}(G) \geq \left\lceil \frac{n\delta(G)}{\Delta(G)+1} \right\rceil$.

Theorem 3.2. *Let v be a vertex of a graph G without isolated vertices such that v satisfies the following conditions.*

- a. $\deg(v) = \delta(G)$ and
- b. $N(u)$ contains at least $\delta(G)$ number of vertices of degree greater than $\delta(G)$ for all $u \in N(v)$.

Then $v \in V - D$, where D is a $\gamma_{\delta_{max}}$ -set of G .

Proof: Let $u \in N(v)$, where $\deg(v) = \delta(G)$. Suppose u satisfies the second condition in the statement of the theorem. Then $\delta(G - v) \geq \delta(G)$. Hence $V(G) - \{v\}$ forms a maximin degree dominating set of G . This proves the theorem.

Remark: *The vertices satisfying the conditions stated in the above Theorem 3.2., belongs to $V - D$ for a $\gamma_{\delta_{max}}$ -set D of G . Hence we define the following set and obtain an upper bound for $\gamma_{\delta_{max}}(G)$.*

Definition 3.1. Let G be a graph without isolated vertices. The set $M_{\delta}(G)$ is the set of all minimum degree vertices v of G such that $N(u)$ contains at least $\delta(G)$ number of vertices of degree greater than $\delta(G)$ for all $u \in N(v)$.

Corollary 3.2.1. *Let G be any graph without isolates. Then $\gamma_{\delta_{max}}(G) \leq |V - M_{\delta}(G)|$.*

Theorem 3.3. *Let G be a graph with minimum degree $\delta(G) \geq 1$. Let D be a $\gamma_{\delta_{max}}$ -set of G . Then $V - D$ does not contain a maximin degree dominating set of G .*

Proof: Let us consider two cases.

Case 1: $V - D$ is not a dominating set of G .

Then any subset of $V - D$ can not be a dominating set of G and hence the result is true.

Case 2: $V - D$ is a dominating set of G .

Since D is a maximin degree dominating set of G , $\delta(\langle V-D \rangle) \leq \delta(\langle D \rangle)$. Suppose there exists a subset T of $V-D$ such that $|T|=|D|$ and T is a dominating set of G . Then obviously $\delta(\langle T \rangle) \leq \delta(\langle D \rangle)$. If $\delta(\langle T \rangle) = \delta(\langle D \rangle)$ then since T is a dominating set of G , the vertices of minimum degree in $\langle D \rangle$ are adjacent to some vertices of T . Those vertices of T together with the vertices of D forms a new dominating set D' of G such that $\delta(\langle D' \rangle) \geq \delta(\langle D \rangle)$. This is a contradiction to the fact that D is $\gamma_{\delta_{max}}$ -set of G .

Thus T can not be a dominating set of G . Thus $V-D$ does not contain a maximin degree dominating set of G .

Theorem 3.4. Let $\gamma_{\delta_{max}}(G) = k$ and D be a $\gamma_{\delta_{max}}(G)$ -set of a graph G , with $\delta(G) \geq 1$. Then D is a γ -set and γ_t -set of G if and only if every vertex $u_i \in D$ dominates a set $V_i \subseteq V-D$ for $i=1,2,3,\dots,k$ such that, V_i does not contain a vertex v_i that dominates both V_i and V_j , where $i \neq j$

Proof: Let D be a $\gamma_{\delta_{max}}$ -set of a graph G . Suppose that D is a γ -set and γ_t -set of G and that there exists a vertex $u \in D$ which does not dominate any vertex of G uniquely. Then $D - \{u\}$ will still be a dominating set of G , which is a contradiction to our assumption. Therefore each vertex $u_i \in D$ dominates a set $V_i \subseteq V-D$ uniquely. Let $u_i \in D$ dominate a set $V_i \subseteq V-D$ for $i=1,2,3,\dots,k$. Suppose $v_i \in V_i$ dominates each vertex of V_i and each vertex of V_j where $i \neq j$. Since $\delta(\langle D \rangle) \geq 1$, u_i and u_j are adjacent to some vertices of D , and hence u_i and u_j of D can be replaced by v_i . Thus $D' = (D - \{u_i, u_j\}) \cup \{v_i\}$ forms a dominating set of G with $|D'| < |D|$ and $\delta(\langle D' \rangle) = 0$ with $\deg_{(D)}(v_i) = 0$ which is a contradiction. Thus no V_i contains a vertex v_i which dominates V_i and V_j , where $i \neq j$

Conversely, since $\delta(G) \geq 1$, it is natural that, $\delta(\langle D \rangle) \geq 1$. Hence, $\delta(\langle D \rangle) \geq 1$ and D is a total dominating set of G . Also from the statement

of the theorem, every vertex of D uniquely dominates some vertex of $V - D$. Therefore D is the smallest dominating set with $\delta(\langle D \rangle) \geq 1$. Hence D is a γ -set and a γ_t -set of the graph G .

4. Critical Aspects of Maximin Degree Domination Number

E. Sampathkumar and Prabha S. Neeralagi³ introduced the critical aspects as follows:

Definition 4.1. Let t be any parameter defined on the graph G and e be any element of graph G . Then the element e is said to be

1. t -critical if $t(G - e) \neq t(G)$
2. t^+ -critical if $t(G - e) > t(G)$
3. t^- -critical if $t(G - e) < t(G)$.
4. t -fixed if e belongs to every t -set.
5. t -free if e belongs to some t -set but not to all t -sets.
6. t -totally free if e belongs to no t -set.

Motivated by the above definition, we define the following

Definition 4.2. Let e be any element of the graph G . Then the element e is said to be

1. $\gamma_{\delta_{max}}$ -critical if $\gamma_{\delta_{max}}(G - e) \neq \gamma_{\delta_{max}}(G)$.
2. $\gamma_{\delta_{max}}^+$ if $\gamma_{\delta_{max}}(G - e) > \gamma_{\delta_{max}}(G)$.
3. $\gamma_{\delta_{max}}^-$ -critical if $\gamma_{\delta_{max}}(G - e) < \gamma_{\delta_{max}}(G)$
4. $\gamma_{\delta_{max}}$ -redundant if $\gamma_{\delta_{max}}(G - e) = \gamma_{\delta_{max}}(G)$
5. $\gamma_{\delta_{max}}$ -free if e belongs to some $\gamma_{\delta_{max}}$ -set but not to all $\gamma_{\delta_{max}}$ -sets.
6. $\gamma_{\delta_{max}}$ -fixed if e belongs to every $\gamma_{\delta_{max}}$ -sets
7. $\gamma_{\delta_{max}}$ -totally free if e belongs to no $\gamma_{\delta_{max}}$ -set.

Theorem 4.1. Every element of a regular graph G is $\gamma_{\delta_{max}}^-$ -critical.

Proof. Note that removal of any edge or any vertex from a regular graph G results in graph G' with $\delta(G') = \delta(G) - 1$. Hence G' contains at least two vertices, say u and v , of degree $\delta(G) - 1$. Then the subgraphs

$\langle V - \{u\} \rangle$ or $\langle V - \{v\} \rangle$ will form a maximin degree dominating set of G' . Hence $\gamma_{\delta_{max}}(G') \leq |V| - 1 < \gamma_{\delta_{max}}(G)$. This proves the result.

Theorem 4.2. *Every edge of a star graph is $\gamma_{\delta_{max}}$ -redundant.*

Proof. Let e be any edge of the star graph $K_{1,n}$. Then the removal of e from $K_{1,n}$ results in a graph, $K_{1,n} - e = K_{1,n-1} \cup K_1$. Then $\gamma_{\delta_{max}}(K_{1,n} - e) = 2 = \gamma_{\delta_{max}}(K_{1,n})$.

Theorem 4.3. *Let G be a graph of order n such that $\gamma_{\delta_{max}}(G) < n$. If an edge $e = uv$ of G is $\gamma_{\delta_{max}}^-$ -critical, then for every $\gamma_{\delta_{max}}$ -set D , any one of the following two conditions hold.*

1. $u \in D, v \in V - D$ implies $N(v) \cap D = \{u\}$
2. $u, v \in D$.

Proof. Suppose $e = uv$ is a $\gamma_{\delta_{max}}^-$ -critical edge of the graph G . Let us assume that none of the two conditions is true for e . Then there exists a particular $\gamma_{\delta_{max}}$ -set D of G such that $u \in D, v \in V - D$ but $N(v) \cap D \neq \{u\}$. Since D is dominating set of G , we must have $|N(v) \cap D| \geq 2$. Thus v is vertex of $V - D$ which has at least two neighbors in D and hence removal of the edge e does not affect the minimum degree of the subgraph $\langle D \rangle$ and domination property of D . This is a contradiction. $\square$

Corollary 4.3.1. *For any graph G , suppose there exists a $\gamma_{\delta_{max}}$ -set D such that $e = uv$ is an edge of G with $u \in D, v \in V - D$ and $|N(v) \cap D| \geq 2$ then the edge e is $\gamma_{\delta_{max}}$ -redundant edge of G .*

Theorem 4.4. *A $\gamma_{\delta_{max}}$ -totally free vertex v of a graph G is $\gamma_{\delta_{max}}$ -critical if and only if $V(G - v) - D$ contains a set A of vertices with $\deg_{G-v}(u) \geq \delta(\langle D \rangle)$ for all $u \in A$, which are adjacent to a set $D' \subseteq D$, where D' is a dominating set of $G - v$ and $|D'| \neq |D| - A$.*

Proof: Suppose v is a $\gamma_{\delta_{max}}$ -totally free vertex of G which is also $\gamma_{\delta_{max}}$ -critical. Suppose $\deg_{G-v}(u) < \delta(\langle D \rangle)$ for all $u \in V(G-v) - D$. Then induced subgraph of any dominating set of $G-v$ will not have minimum degree greater than $\delta(\langle D \rangle)$. Also since v is a $\gamma_{\delta_{max}}$ -totally free vertex of G , removal of v from G will not affect the minimum degree of $\langle D \rangle$. Hence $\gamma_{\delta_{max}}(G-v) = |D| = \gamma_{\delta_{max}}(G)$. This is a contradiction to the fact that v is $\gamma_{\delta_{max}}$ -critical. Hence $V(G-v) - D$ contains few vertices of degree greater than or equal to $\delta(\langle D \rangle)$. Let $B = \{u \in V(G-v) - D \mid \deg_{G-v}(u) \geq \delta(\langle D \rangle)\}$. Since v is $\gamma_{\delta_{max}}$ -totally free vertex of G , $v \notin D$ for any $\gamma_{\delta_{max}}$ -set D of G and removal of v from G does not affect the minimum degree of any $\gamma_{\delta_{max}}$ -set of G . Let $D' \subseteq D$ be a dominating set of $G-v$. Suppose $A \subseteq B$ be the set of vertices which are adjacent to all the vertices of D' . Then $A \cup D'$ is a $\gamma_{\delta_{max}}$ -set of $G-v$. If $|D'| = |D| - |A|$ then $\gamma_{\delta_{max}}(G-v) = |A \cup D'| = |A| + |D'| = |D| = \gamma_{\delta_{max}}(G)$. This is a contradiction to the fact that v is $\gamma_{\delta_{max}}$ -critical. Hence $|D'| \neq |D| - |A|$. Conversely suppose v is a $\gamma_{\delta_{max}}$ -totally free vertex of G . Let $D' \subseteq D$ be a dominating set of $G-v$ and $V(G-v) - D$ contains a set A of vertices with $\deg_{G-v}(u) \geq \delta(\langle D \rangle)$ for all $u \in A$. Suppose every vertex of A is adjacent to each vertex of D' and $|D'| \neq |D| - |A|$. Then $A \cup D'$ forms a maximin degree dominating set of $G-v$ and also $\gamma_{\delta_{max}}(G-v) = |A \cup D'| = |A| + |D'| = |D| = \gamma_{\delta_{max}}(G)$. Thus v is $\gamma_{\delta_{max}}$ -critical.

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