

Eigen function Expansion Method for Solving One-Group Neutron Diffusion Equation

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Abstract: This paper introduces an analytical technique to obtain the analytical approximate solution of the Neutron Diffusion Equation with fixed source. The main properties of this method lie in its flexibility and ability to solve differential equations accurately and conveniently. The explicit solution of the Neutron Diffusion equation has been presented in the closed form by an infinite series and then the numerical solution has been represented graphically. Mathematically, flux distribution is the solution of diffusion equation with specific boundary conditions. Regarding the utilization of Eigen function Expansion Method, we employ Fourier basis for expressing the initial approximation due to the structure of the problem and its boundary conditions.

The simplest group diffusion problems involve only, one group of neutrons, which for simplicity, are assumed to be all thermal neutrons. A more accurate procedure, particularly for thermal reactors, is to split the neutrons into two groups; in which case thermal neutrons are included in one group called the thermal or slow group and all the other are included in fast group. The neutrons within each group are lumped together and their diffusion, scattering, absorption and other interactions are described in terms of suitably average diffusion coefficients and cross-sections, which are collectively known as group constants. The solution obtained in explicit form which is suitable for computer programming and other purposes such as analysis of flux distribution in a square critical reactor.

The Eigen function Expansion Method is relatively new approach to provide an analytical approximate solution to neutron diffusion equation. This method is particularly valuable tools for solving problems useful for Applied Scientist and Engineers because it provide approximate analytical solutions as well as numerical approximate solutions. It was successfully applied to autonomous ordinary and partial differential equations and other fields. This is an efficient analytical method using which we obtain the solution for Neutron Diffusion Model with specified boundary conditions. This method is efficient to calculate and convenient to use for finding flux distribution in one-group neutron diffusion equation. Neutron flux distribution and flux are computed in mathematical software package with fewer rounds off errors and cited graphically. By using this method we expanding the distribution function using Fourier series and then computed the result by *Mathematica* software package.

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1. Introduction

In this present analysis, we have solved the one-group neutron diffusion equation via Eigen function Expansion method¹. This method is applied to obtain analytical solution of the neutron diffusion equation with fixed source. The explicit solution for one-group neutron diffusion equation has been solved and we presented it in the closed form by an infinite series. The numerical solution has been cited graphically. This present method is efficient and convenient to calculate the analytical solution for Neutron Diffusion Equation².

2. The Physical Model- The One Group Neutron Diffusion Equation with Fixed Source

We consider the following time independent fixed source one group neutron diffusion equation³ for a homogeneous region where geometry with the vacuum boundary conditions

$$\nabla^2 \phi(\vec{r}) - \kappa^2 \phi(\vec{r}) = -\frac{S(\vec{r})}{D}, \quad \vec{r} \in V, \quad \phi(\vec{r}) = 0, \quad \vec{r} \in S$$

where $\phi(\vec{r})$ is the neutron flux, $S(\vec{r})$ is the neutron source, Σ_a is the absorption cross section and the diffusion constant D is given by inverse diffusion length $\kappa^2 = \Sigma_a / D$.

We consider the neutron diffusion equation with the fixed source for a two dimensional system with a square geometry⁴, it is symmetric with respect to both x and y -axes. In this scenario, neutron diffusion equation together with the boundary conditions given by eq. (2.1) reduces to

$$(2.1) \quad \frac{\partial^2 \phi(x, y)}{\partial x^2} + \frac{\partial^2 \phi(x, y)}{\partial y^2} - \kappa^2 \phi(x, y) = -\frac{S}{D}$$

$$(2.2) \quad \left. \frac{\partial \phi(x, y)}{\partial x} \right|_{x=0} = 0, \quad \phi(x, y)|_{x=a} = 0$$

$$(2.3) \quad \left. \frac{\partial \phi(x, y)}{\partial y} \right|_{y=0} = 0, \quad \phi(x, y)|_{y=a} = 0$$

3. Application of Eigen Function Expansion Method for Neutron Diffusion Equation

The one group neutron diffusion equation given by eq. (2.2) is

$$(3.1) \quad \frac{\partial^2 \phi(x, y)}{\partial x^2} + \frac{\partial^2 \phi(x, y)}{\partial y^2} - \kappa^2 \phi(x, y) = -\frac{S}{D}.$$

We solve the neutron diffusion equation (3.1) by applying Eigenfunction expansion method, considering first homogeneous system as

$$(3.2) \quad \frac{\partial^2 \phi(x, y)}{\partial x^2} + \frac{\partial^2 \phi(x, y)}{\partial y^2} - \kappa^2 \phi(x, y) = 0.$$

Next, we apply separation of variable method

$$(3.3) \quad \phi(x, y) = F(x)G(y).$$

By replacing eq. (3.3) in eq. (3.2), the variables are separated in the following way

$$(3.4) \quad \kappa^2 - \frac{1}{F(x)} \frac{d^2 F}{dx^2} = \frac{1}{G(y)} \frac{d^2 G}{dy^2} = -\lambda^2.$$

The differential equation for $G(y)$ is given by

$$(3.5) \quad \frac{d^2 G}{dy^2} + \lambda^2 G = 0.$$

Therefore, the solution for eq. (3.5) is

$$(3.6) \quad G(y) = A \cos(\lambda y) + B \sin(\lambda y),$$

where A and B are arbitrary constants.

Now, for $G(y)$, we obtain from the boundary conditions given by eq. (2.3)

$$\left. \frac{dG}{dy} \right|_{y=0} = 0,$$

$$(3.7) \quad G(y)|_{y=a} = 0.$$

Hence, we obtain the Eigen value of this problem is given by

$$(3.8) \quad \lambda_n = \frac{(2n+1)\pi}{2a}, \quad n = 0, 1, 2, \dots$$

For these Eigen values the corresponding Eigen functions are

$$(3.9) \quad G_n(y) = \cos\left(\frac{(2n+1)\pi}{2a}y\right), \quad n = 0, 1, 2, \dots$$

$$= \cos(\beta_n y), \quad \text{where } \beta_n = \frac{(2n+1)\pi}{2a}.$$

Consider, the general form of the solution as

$$(3.10) \quad \phi(x, y) = \sum_{n=0}^{\infty} \phi_n(x, y)$$

$$= \sum_{n=0}^{\infty} F_n(x) G_n(y).$$

Next, we also expand $\frac{S}{D}$ as a generalized Fourier series of Eigen functions

$$\frac{S}{D} = \sum_{n=0}^{\infty} s_n G_n(y),$$

where the Fourier cosine coefficients s_n is given by

$$s_n = \frac{2}{a} \int_0^a \frac{S}{D} \cos(\beta_n y) dy, \quad n = 1, 2, \dots$$

or

$$s_n = \frac{2S(-1)^n}{aD\beta_n}.$$

Now, substituting eq. (3.10) and eq. (3.11) in the neutron diffusion eq. (3.1), we obtain

$$(3.14) \quad \sum_{n=0}^{\infty} \frac{d^2 F_n(x)}{dx^2} G_n(y) + \sum_{n=0}^{\infty} F_n(x) \frac{d^2 G_n(y)}{dy^2} - \kappa^2 \sum_{n=0}^{\infty} F_n(x) G_n(y) = - \sum_{n=0}^{\infty} s_n G_n(y)$$

After substituting eq. (3.9) in eq. (3.14), we get

$$(3.15) \quad \sum_{n=0}^{\infty} G_n(y) \left[\frac{d^2 F_n(x)}{dx^2} - (\beta_n^2 + \kappa^2) F_n(x) + s_n \right] = 0,$$

where $G_n(y) \neq 0$ and $\alpha_n^2 = \beta_n^2 + \kappa^2$, then we obtain

$$(3.16) \quad \frac{d^2 F_n(x)}{dx^2} - \alpha_n^2 F_n(x) + s_n = 0.$$

The solution for eq.(3.16) is

$$(3.17) \quad F_n(x) = c_1 \cosh(\alpha_n x) + c_2 \sinh(\alpha_n x) + \frac{s_n}{\alpha_n^2}.$$

Now, consider the boundary conditions in eq. (2.3), we have

$$(3.18) \quad F_n(x) \Big|_{x=a} = 0,$$

$$\frac{dF_n(x)}{dx} \Big|_{x=0} = 0.$$

Using eq. (3.18), we obtain the arbitrary constants

$$(3.19) \quad c_1 = \frac{-s_n}{\alpha_n^2} \frac{1}{[1 - \cosh(\alpha_n a)]},$$

$$c_2 = 0$$

Substituting the value of c_1 and c_2 in eq. (3.17), we get

$$(3.20) \quad F_n(x) = \frac{s_n}{\alpha_n^2} \left[1 - \frac{\cosh(\alpha_n x)}{\cosh(\alpha_n a)} \right]$$

Now, we substitute eqs. (3.9) and (3.20) in eq. (3.10), the general solution for one group neutron diffusion equation with fixed source is

$$(3.21) \quad \begin{aligned} \phi(x, y) &= \sum_{n=0}^{\infty} \phi_n(x, y) \\ &= \frac{2S}{aD} \sum_{n=0}^{\infty} \frac{(-1)^n}{\beta_n \alpha_n^2} \cos(\beta_n y) \left[1 - \frac{\cosh(\alpha_n x)}{\cosh(\alpha_n a)} \right] \end{aligned}$$

4. Numerical Results and Discussions

In the present analysis, we consider a square reactor core with edge length $2a = 50$ cm and apply Eigenfunction expansion method⁵ to obtain the numerical solution for one quadrant of the system which is sufficient owing to the symmetry regarding the vacuum conditions at the left and upper boundaries together with the reflector conditions at the right and lower boundaries as expressed in eqs. (2.2) and (2.3).

In this numerical discussion, we assume the constants of the reactor as presented in Table 1.

Table-1

Constants	a (cm)	D (cm)	Σ_a (cm ⁻¹)	S
Value	25	1.77764	1	1

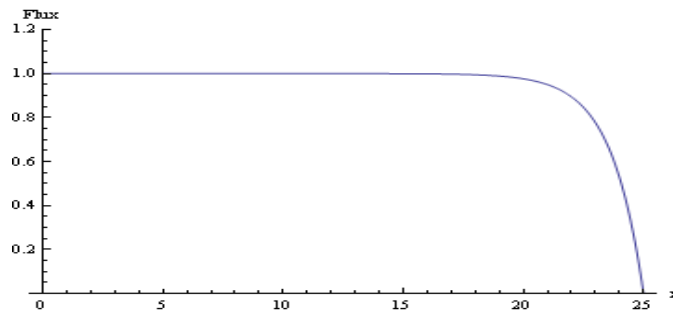


Fig. 1 Neutron Flux for $y = 0$

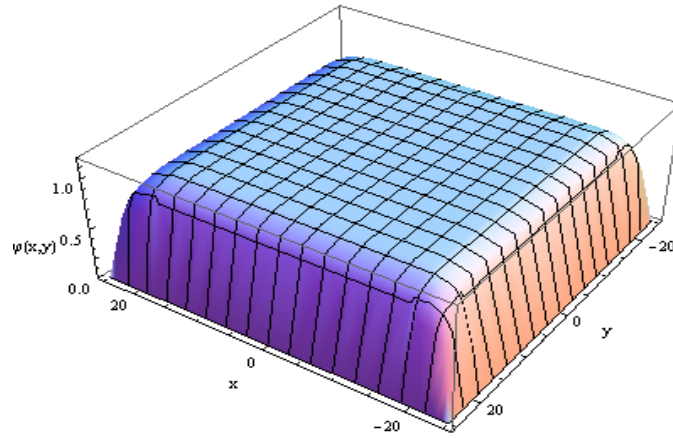


Fig. 2 Neutron Flux Distribution

We present the numerically computed result obtained by Eigenfunction Expansion Method in Fig. 1 for $y=0$ and the Neutron Flux distribution $\phi(x,y)$ in Fig. 2. Figures 1–2 have been drawn using the *Mathematica* Software.

5. Conclusion

In this paper, Eigen function Expansion Method has been successfully applied to find the analytical approximate as well as numerical solutions of the Neutron Diffusion equation with fixed source. The expansion method is straightforward. The components of the series solution can be easily computed using any mathematical symbolic package. The expansion method does not require discretization of the variables, i.e., time and space, it is not affected by computational round off errors and one is not faced with necessity of large computer memory and time. Consequently, the computational size will be reduced. It is also worth noting that the advantage of the expansion methodology is that it displays a fast convergence of the solution.

The Eigen function Expansion Method has been known as a powerful tool for solving many functional equations such as ordinary, partial differential equations.

In this work, we demonstrate that the above method is well suited to solve Neutron Diffusion Model with specified boundary conditions. The results are shown graphically. In our work we use the *Mathematica* Package to calculate the series obtained from the Eigen function Expansion Method.

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