

Degree of Approximation of Lipschitz Function by (C, 1) F(a, q) Means of its Fourier Series

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Abstract: In the present paper we obtained the degree of approximation of Lip α class by (C, 1) F(a, q) Mean of its Fourier Series.

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1. Definition and Notations

Let $f \in [-\pi, \pi]$ be periodic with period -2π outside this range. Let the Fourier Series associated with $f(x)$ be given by

$$(1.1) \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

We write

$$(1.2) \quad \phi(x) = \frac{1}{2} \{ f(x+t) + f(x-t) - 2f(x) \}.$$

The family $F(a, q)$ of summability method introduced by Mair¹. The summability matrix $\{C_{nk}\}$ belong to $F(a, q)$ if it satisfies the following conditions:

n is a discrete or continuous parameter : $q = q(n)$ is a positive increasing function which tend to infinity as $n \rightarrow \infty$, 'a' is a positive constant : for

every fixed $\delta : \frac{1}{2} < \delta < \frac{2}{3}$

$$(1.3) \quad C_{nk} = \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \left\{ 1 + O\left(\frac{|k-q|+1}{q}\right) + O\left(\frac{|k-q|^3}{q^2}\right) \right\}$$

as $n \rightarrow \infty$ uniformly in k for $|k-q| \leq q^\delta$, and

$$(1.4) \quad \sum_{|k-q| > q^\delta} (k+1)C_{nk} = O\left\{\exp(-q^\mu)\right\},$$

where μ is some positive number independent of n .

Let

$$(1.5) \quad F_n^{(a,q)}(f; x) = \sum_{k=0}^{\infty} C_{nk} S_k(f; x)$$

denote the $F(a,q)$ mean of the Fourier Series (1.1) of f , where $S_k(f;x)$ is the k^{th} partial sum of (1.1). The family $F(a,q)$ contains the summability methods of generalized Borel, Euler, Taylor, S_a and valiron. It is known (see Kuttner, Rajagopal and Rangachari²)

That

$$(1.6) \quad \sum_{k=0}^{\infty} C_{nk} = 1 + O\left\{q^{-1/2}\right\}.$$

The summability method of Euler, Taylor, S_a and Borel Satisfy (1.6) in the stronger form

$$(1.7) \quad \sum_{k=0}^{\infty} C_{nk} = 1.$$

A function $f \in \text{Lip } \alpha$ if

$$|f(x+t) - f(x)| = O\left(|t|^\alpha\right), \quad \text{for } 0 < \alpha \leq 1.$$

We have

$$(1.8) \quad F_n^{(a,q)} - f(x) = \frac{1}{\pi} \int_0^\pi \frac{\phi(t)}{\sin t/2} \sum_{k=0}^{\infty} C_{nk} \sin\left(k + \frac{1}{2}\right) t dt + O\left(q^{-1/2}\right)$$

and the series $\sum_{k=0}^{\infty} u_k$ is said to be (C, 1) summable to S.

If

$$(1.9) \quad (C, 1) = \frac{1}{n+1} \sum_{k=0}^n S_k \rightarrow S \quad \text{as } n \rightarrow \infty.$$

The (C, 1) transform of the F(a, q) transform is defined as the (C, 1) F(a, q) transform of the partial sum S_n of the $\sum_{k=0}^{\infty} u_k$.

$$(C, F)_n^{(a, q)} = \frac{1}{n+1} \sum_{k=0}^n F_k^{(a, q)} \rightarrow S \quad \text{as } n \rightarrow \infty,$$

where $F_n^{(a, q)}$ denotes the F(a, q) transform of S_n then the series $\sum_{k=0}^{\infty} u_k$ is said

to be summable (C, 1) F(a, q) means or simply summable (C, 1) F(a, q) to S.

Shyam Lal and Prem Narain Singh³ obtained approximation of Lip $(\xi(t), p)$ function by (C, 1) (E, 1) means of its Fourier Series. They proved the following theorem:

Theorem A: $f: R \rightarrow R$ is 2π - periodic and Lip $(\xi(t), p)$ function belonging to Lip $(\xi(t), p)$ then the degree of approximation of f by (C, 1) (E, 1) means of Fourier Series Satisfies.

$$(2.1) \quad \|(CF)_n^{(a, q)} - f(x)\|_p = O\left(\xi\left(\frac{1}{n+1}\right)(n+1)^{1/p}\right).$$

Provided $\xi(t)$ satisfy the following conditions:

$$(2.2) \quad \left\{ \int_0^{1/(n+1)} \left(\frac{t\phi(t)}{\xi(t)} \right)^p dt \right\}^{1/p} = O\left(\frac{1}{n+1}\right),$$

$$(2.3) \quad \left\{ \int_{1/(n+1)}^{\pi} \left(\frac{t^{-\delta}\phi(t)}{\xi(t)} \right)^p dt \right\}^{1/p} = O((n+1)^{\delta}),$$

where δ is an arbitrary number such that $q(1-\delta)-1 > 0$ condition (2.2) and (2.3) holds uniformly in x and generalizing the above result $(CF)_n^{(a,q)}$ are $(C,1)$ $F(a,q)$ means of Fourier Series (1.1).

Shyam Lal and J.K. Kushuwaha⁴ obtained the degree of approximation of function of Lip α class by product summability mean of the form $(C,1)$ (E,q) . Their theorem is as follows.

Theorem B: If $f : R \rightarrow R$ is 2π -periodic Lebesgue integrable on $[-\pi, \pi]$ and belonging to the Lipschitz class then the degree of approximation of f by the $(C,1)$ (E,q) product means of its Fourier series satisfies for $n = 0, 1, 2, \dots$

$$(2.4) \quad \|(CE)_n^q(x) - f(x)\|_\infty = O((n+1)^{-\alpha}) \quad \text{for } 0 < \alpha < 1.$$

Generalising the above results and applying the product summability mean of the form $(C,1)$ $F(a,q)$ we have obtained the degree of approximation for Lipschitz class in the present paper.

3. Main Theorem

Let n be the integral part of $q = q(n)$.

Theorem: If $f : R \rightarrow R$ is 2π -periodic Lebesgue integrable on $[-\pi, \pi]$ and belonging to the Lipschitz class then the degree of approximation of f by the $(C,1)$ $F(a,q)$ product means of Fourier Series satisfies for $n = 0, 1, 2, \dots$

$$(3.1) \quad \|(C, F)_n^{(a,q)} - f(x)\|_\infty = O((n+1)^{-\alpha}), \quad \text{for } 0 < \alpha < 1.$$

For the proof of our theorem we have required the following lemmas:

Lemma 1: If $q = q(n)$ is an integer valued function of n then for $\frac{1}{2} < \delta < \frac{2}{3}$ we have

$$(3.2) \quad \int_0^\pi \frac{\phi(t)}{\sin \frac{t}{2}} \sum_{|k-q| \leq q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \sin\left(k + \frac{1}{2}\right) t dt$$

$$= \int_0^\pi \frac{\phi(t)}{\sin \frac{t}{2}} \exp\left(\frac{-qt^2}{4a}\right) \sin\left(q + \frac{1}{2}\right) t dt + O\left(q \exp(-aq^{2\delta-1})\right).$$

This is due to Shrivastava and Verma⁵.

$$\textbf{Lemma 2 :} \quad \text{Let } M_n(t) = \frac{1}{(n+1)\pi} \sum_{k=0}^n \frac{\exp\left(\frac{-qt^2}{4a}\right) \sin\left(q + \frac{1}{2}\right)t}{\sin \frac{t}{2}}$$

$$\text{then } M_n(t) = O(n+1) \quad \text{for } 0 < t < \frac{\pi}{n+1}.$$

$$\textbf{Proof :} \quad \text{Using } \sin nt \leq n \sin t, \quad \text{for } 0 < t < \frac{\pi}{n+1}$$

$$\text{then } M_n(t) = \frac{1}{(n+1)\pi} \sum_{k=0}^n \frac{\exp\left(\frac{-qt^2}{4a}\right) \sin\left(q + \frac{1}{2}\right)t}{\sin \frac{t}{2}}$$

$$\leq \frac{1}{(n+1)} \sum_{k=0}^n \frac{(2q+1) \cdot \sin \frac{t}{2}}{\sin \frac{t}{2}}$$

$$(3.3) \quad = O(n+1).$$

$$\textbf{Lemma 3:} \quad \text{Let } M_n(t) = \frac{1}{(n+1)\pi} \sum_{k=0}^n \left[\exp\left(\frac{-qt^2}{4a}\right) \frac{\sin\left(q + \frac{1}{2}\right)t}{\sin \frac{t}{2}} \right]$$

$$\text{then } M_n(t) = O\left(\frac{1}{t}\right) \quad \text{for } \frac{\pi}{n+1} < t < \pi.$$

$$\textbf{Proof:} \quad \sin \frac{t}{2} \geq \left(\frac{t}{\pi}\right) \quad \text{and} \quad \left| \sin\left(q + \frac{1}{2}\right)t \right| \leq 1 \quad \text{for } \frac{\pi}{n+1} \leq t \leq \pi$$

$$M_n(t) = \frac{1}{(n+1)\pi} \sum_{k=0}^n \left[\exp\left(\frac{-qt^2}{4a}\right) \cdot \frac{1}{\frac{t}{\pi}} \right]$$

$$(3.4) \quad = O\left(\frac{1}{t}\right).$$

$$\textbf{Lemma 4:} \quad \text{Let } K_n(t) = \frac{1}{(n+1)\pi} \sum_{k=0}^n \frac{O(q \exp(-aq^{2\delta-1}))}{\sin \frac{t}{2}}$$

then $K_n(t) = O(n+1)$ for $0 \leq t \leq \frac{\pi}{(n+1)}$.

Proof: Using $\left| \sin \frac{t}{2} \right| \leq \left(\frac{t}{2} \right)$ for $0 \leq t < \frac{\pi}{(n+1)}$

$$\begin{aligned} K_n(t) &= \frac{1}{(n+1)\pi} \sum_{k=0}^n \frac{O(q \exp(-aq^{2\delta-1}))}{t/2} \\ (3.5) \quad &= O(n+1). \end{aligned}$$

Lemma 5: Let $K_n(t) = \frac{1}{(n+1)\pi} \sum_{k=0}^n \frac{O(q \exp(-aq^{2\delta-1}))}{\sin t/2}$

then $K_n(t) = O\left(\frac{1}{t}\right)$, for $\frac{\pi}{(n+1)} \leq t \leq \pi$.

Proof: Using $\left| \sin \frac{t}{2} \right| \geq \left(\frac{t}{\pi} \right)$ for $\frac{\pi}{(n+1)} \leq t \leq \pi$

$$\begin{aligned} K_n(t) &= \frac{1}{(n+1)\pi} \sum_{k=0}^n \frac{O(q \exp(-aq^{2\delta-1}))}{t/\pi} \\ (3.6) \quad K_n(t) &= O\left(\frac{1}{t}\right). \end{aligned}$$

4. Proof of Theorem

Following E.C. Titchmarsh (1939, P-403). The K^{th} partial sum $S_k(f;x)$ of the Fourier Series is given by

$$(4.1) \quad S_k(f;x) - f(x) = \frac{1}{\pi} \int_0^\pi \frac{\phi(t)}{\sin t/2} \sin\left(k + \frac{1}{2}\right)t dt.$$

The $F(a,q)$ transform $F_n^{(a,q)}$ of $S_k(f;x)$ is given by

$$\begin{aligned}
F_n^{(a,q)} - f(x) &= \frac{1}{\pi} \int_0^\pi \frac{\phi(t)}{\sin t/2} \sum_{k=0}^{n=\infty} C_{nk} \sin\left(k + \frac{1}{2}\right) t dt + O\left(q^{-1/2}\right) \\
(4.2) \quad &= \frac{1}{\pi} \int_0^\pi \frac{\phi(t)}{\sin t/2} \left[\sum_{|k-q| \leq q^\delta} + \sum_{|k-q| > q^\delta} \right] C_{nk} \sin\left(k + \frac{1}{2}\right) t dt + O\left(q^{-1/2}\right).
\end{aligned}$$

The (C,1) F(a,q) transform of $S_k(f; x)$ by $(CF)_n^{(a,q)}$, we have

$$\begin{aligned}
(4.3) \quad (CF)_n^{(a,q)} - f(x) &= \frac{1}{(n+1)\pi} \sum_{k=0}^n \int_0^\pi \frac{\phi(t)}{\sin t/2} \left[\sum_{|k-q| \leq q^\delta} + \sum_{|k-q| > q^\delta} \right] C_{nk} \sin\left(k + \frac{1}{2}\right) t dt \\
&\quad + O\left(q^{-1/2}\right) \\
&= I_1 + I_2 + O\left(q^{-1/2}\right),
\end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{1}{(n+1)\pi} \sum_{k=0}^n \left[\int_0^\pi \frac{\phi(t)}{\sin t/2} \sum_{|k-q| > q^\delta} C_{nk} \sin\left(k + \frac{1}{2}\right) t dt \right] \\
&= \frac{1}{(n+1)\pi} \sum_{k=0}^n \left[\int_0^{\pi/n+1} + \int_{\pi/n+1}^\pi \right] \frac{\phi(t)}{\sin t/2} \sum_{|k-q| > q^\delta} C_{nk} \sin\left(k + \frac{1}{2}\right) t dt,
\end{aligned}$$

$$(4.4) \quad I_2 = I_{2.1} + I_{2.2}$$

Now

$$|I_{2.1}| \leq \frac{1}{(n+1)\pi} \sum_{k=0}^n \left[\int_0^{\pi/n+1} \phi(t) \sum_{|k-q| > q^\delta} \frac{C_{nk} \sin\left(k + \frac{1}{2}\right) t}{\sin t/2} dt \right].$$

Using $\sin nt \leq n \sin t$, for $0 < t < \frac{\pi}{(n+1)}$

$$\begin{aligned}
&= \frac{1}{(n+1)\pi} \sum_{k=0}^n \left[\int_0^{\pi/n+1} \phi(t) \sum_{|k-q|>q^\delta} C_{nk} \frac{(2k+1)\sin t/2}{\sin t/2} dt \right] \\
&= \frac{1}{(n+1)} \sum_{k=0}^n (2k+1) \int_0^{\pi/n+1} \phi(t) dt \quad \left[\because \sum_{|k-q|>q^\delta} C_{nk} = 1 \right] \\
&= O(n+1) \int_0^{\pi/n+1} |t|^\alpha dt \\
&= O(n+1) \left[\frac{t^{\alpha+1}}{\alpha+1} \right]_0^{\pi/n+1} \\
&= O(n+1) \frac{1}{(\alpha+1)} \cdot \frac{\pi^{\alpha+1}}{(n+1)^{\alpha+1}} \\
(4.5) \quad &= O(n+1)^{-\alpha}
\end{aligned}$$

Now

$$I_{2.2} = \frac{1}{(n+1)\pi} \sum_{k=0}^n \left[\int_{\pi/n+1}^{\pi} \frac{\phi(t)}{\sin t/2} \sum_{|k-q|>q^\delta} C_{nk} \sin\left(k + \frac{1}{2}\right) t dt \right]$$

Using $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$ and $\left| \sin\left(k + \frac{1}{2}\right) t \right| \leq \left(k + \frac{1}{2}\right) t$, for $\frac{\pi}{(n+1)} \leq t \leq \pi$

$$\begin{aligned}
|I_{2.2}| &\leq \frac{1}{(n+1)\pi} \sum_{k=0}^n \left[\int_{\pi/n+1}^{\pi} \frac{\phi(t)}{t/\pi} \sum_{|k-q|>q^\delta} C_{nk} \left(k + \frac{1}{2}\right) t dt \right] \\
&= \frac{1}{(n+1)} O(\exp(-q^\mu)) \cdot \int_{\pi/n+1}^{\pi} |\phi(t)| dt.
\end{aligned}$$

$$\begin{aligned}
& \left[\because \sum_{|k-q| > q^\delta} C_{nk} \left(k + \frac{1}{2} \right) = O(\exp(-q^\mu)) \right] \\
&= \frac{1}{(n+1)} \int_{\pi/n+1}^{\pi} |t|^\alpha dt \\
&= \frac{1}{(n+1)} \cdot \left[\frac{t^{\alpha+1}}{\alpha+1} \right]_{\pi/n+1}^{\pi} \\
&= \frac{1}{(n+1)} \cdot \frac{1}{(\alpha+1)} \cdot \left[\frac{\pi^{\alpha+1}}{(n+1)^{\alpha+1}} - \pi^{\alpha+1} \right]
\end{aligned}$$

$$(4.6) \quad = O(n+1)^{-\alpha}$$

Now

$$\begin{aligned}
I_1 &= \frac{1}{(n+1)\pi} \sum_{k=0}^n \int_0^{\pi} \frac{\phi(t)}{\sin t/2} \sum_{|k-q| \leq q^\delta} C_{nk} \sin \left(k + \frac{1}{2} \right) t dt \\
&= \frac{1}{(n+1)\pi} \sum_{k=0}^n \int_0^{\pi} \frac{\phi(t)}{\sin t/2} \sum_{|k-q| \leq q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \left\{ 1 + \left(\frac{|k-q|+1}{q} \right) + O\left(\frac{|k-q|^3}{q^2} \right) \right\} \sin \left(k + \frac{1}{2} \right) t dt
\end{aligned}$$

$$(4.7) \quad = I_{1.1} + I_{1.2} + I_{1.3}$$

Now

$$\begin{aligned}
|I_{1.2}| &\leq \frac{1}{(n+1)\pi} \sum_{k=0}^n \int_0^{\pi} \frac{\phi(t)}{\sin t/2} \sum_{|k-q| \leq q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \cdot O\left(\frac{|k-q|+1}{q} \right) \sin \left(k + \frac{1}{2} \right) t dt \\
&= \frac{1}{(n+1)\pi} \sum_{k=0}^n \left[\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi} \right] \frac{\phi(t)}{\sin t/2} \sum_{|k-q| \leq q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) O\left(\frac{|k-q|+1}{q} \right) \sin \left(k + \frac{1}{2} \right) t dt \\
&= \frac{1}{(n+1)\pi} \sum_{k=0}^n \int_0^{\pi/n+1} \frac{|\phi(t)|}{t} \sum_{|k-q| \leq q^\delta} \left\{ \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) O\left(\frac{|k-q|+1}{q} \right) \left(|k-q| + q + \frac{1}{2} \right) t dt \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(n+1)\pi} \sum_{k=0}^n \int_{\pi/n+1}^{\pi} \frac{|\phi(t)|}{t} \sum_{|k-q| \leq q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) O\left(\frac{|k-q|+1}{q}\right) dt \\
(4.8) \quad & = I_{1.21} + I_{1.22} .
\end{aligned}$$

Now

$$\begin{aligned}
I_{1.21} &= \frac{1}{(n+1)\pi} \sum_{k=0}^n O\left(\sqrt{q} \int_0^{\pi/n+1} |\phi(t)| dt\right) \\
&= \frac{1}{(n+1)\pi} \sum_{k=0}^n O\left(\sqrt{q} \int_0^{\pi/n+1} |t|^\alpha dt\right) \\
&= \frac{1}{(n+1)\pi} \sum_{k=0}^n O\left(\sqrt{q} \left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\pi/n+1}\right) \\
&= \frac{1}{(n+1)\pi} \sum_{k=0}^n O\left(\frac{\sqrt{q}}{(\alpha+1)} \cdot \frac{\pi^{\alpha+1}}{(n+1)^{\alpha+1}}\right) \\
&= \frac{1}{(n+1)\pi} (n+1) O\left(\frac{\sqrt{q}}{(n+1)^{\alpha+1}}\right) \\
&= O\left(\frac{\sqrt{n}}{(n+1)^{\alpha+1}}\right) \\
(4.9) \quad &= O(n+1)^{-\alpha} .
\end{aligned}$$

Now

$$I_{1.22} = \frac{1}{(n+1)\pi} \sum_{k=0}^n O\left(\frac{1}{\sqrt{q}} \int_{\pi/n+1}^{\pi} \frac{|\phi(t)|}{t} dt\right)$$

$$\begin{aligned}
&= \frac{1}{(n+1)\pi} \sum_{k=0}^n \cdot O \left(\frac{1}{\sqrt{q}} \int_{\pi/n+1}^{\pi} \frac{|t|^\alpha}{t} dt \right) \\
&= \frac{1}{(n+1)\pi} \sum_{k=0}^n \cdot O \left(\frac{1}{\sqrt{q}} \int_{\pi/n+1}^{\pi} t^{\alpha-1} dt \right) \\
&= \frac{1}{(n+1)\pi} \sum_{k=0}^n \cdot O \left(\frac{1}{\sqrt{q}} \left[\frac{t^\alpha}{\alpha} \right]_{\pi/n+1}^{\pi} \right) \\
&= \frac{1}{(n+1)\pi} (n+1) \left(\frac{1}{\sqrt{q}} \left[\pi^\alpha - \frac{\pi^\alpha}{(n+1)^\alpha} \right] \right)
\end{aligned}$$

$$(4.10) \quad = O(n+1)^{-\alpha} .$$

Now

$$\begin{aligned}
I_{1.1} &= \frac{1}{(n+1)\pi} \sum_{k=0}^n \int_0^{\pi} \frac{\phi(t)}{\sin t/2} \sum_{|k-q| \leq q^\delta} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \cdot \sin\left(k + \frac{1}{2}\right) t dt \\
&= \frac{1}{(n+1)\pi} \sum_{k=0}^n \int_0^{\pi} \frac{\phi(t)}{\sin t/2} \left[\exp\left(\frac{-qt^2}{4a}\right) \sin\left(q + \frac{1}{2}\right) t dt + O\left(q \exp(-aq^{2\delta-1})\right) \right]
\end{aligned}$$

$$(4.11) \quad = I_{1.11} + I_{1.12} .$$

Now

$$I_{1.11} = \frac{1}{(n+1)\pi} \sum_{k=0}^n \int_0^{\pi} \frac{\phi(t)}{\sin t/2} \left[\exp\left(\frac{-qt^2}{4a}\right) \sin\left(q + \frac{1}{2}\right) t dt \right] .$$

Applying Lemma (2)

$$\begin{aligned}
&= \int_0^{\pi} \phi(t) M_n(t) dt \\
&= \left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi} \right) \phi(t) M_n(t) dt
\end{aligned}$$

$$(4.12) \quad = I_{1.111} + I_{1.112}.$$

Now

$$\begin{aligned}
 I_{1.111} &= \int_0^{\pi/n+1} \phi(t) M_n(t) dt \\
 &= O(n+1) \int_0^{\pi/n+1} \phi(t) dt \\
 &= O(n+1) \int_0^{\pi/n+1} t^\alpha dt \\
 &= O(n+1) \cdot \left[\frac{t^{\alpha+1}}{\alpha+1} \right]_0^{\pi/n+1} \\
 &= O(n+1) \cdot \frac{\pi^{\alpha+1}}{(n+1)^{\alpha+1} \cdot (\alpha+1)} \\
 (4.13) \quad &= O(n+1)^{-\alpha}
 \end{aligned}$$

Now, Applying Lemma (3)

$$\begin{aligned}
 I_{1.112} &= \int_{\pi/n+1}^{\pi} \phi(t) M_n(t) dt \\
 &= \int_{\pi/n+1}^{\pi} O\left(\frac{1}{t}\right) \cdot |t|^\alpha dt \\
 &= \left[\frac{t^\alpha}{\alpha} \right]_{\pi/n+1}^{\pi} \\
 &= \frac{1}{\alpha} \left[\pi^\alpha - \frac{\pi^\alpha}{(n+1)^\alpha} \right]
 \end{aligned}$$

$$(4.14) \quad = O(n+1)^{-\alpha}.$$

Now

$$I_{1.12} = \frac{1}{(n+1)\pi} \sum_{k=0}^n \int_0^{\pi} \frac{\phi(t)}{\sin \frac{t}{2}} \cdot O(q \exp(-aq^{2\delta-1})) dt$$

$$= \int_0^{\pi} \phi(t) K_n(t) dt$$

$$= \left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi} \right) \phi(t) K_n(t) dt$$

$$(4.15) \quad = I_{1.121} + I_{1.122}.$$

Now, applying Lemma (4)

$$I_{1.121} = \int_0^{\pi/n+1} \phi(t) K_n(t) dt$$

$$= O(n+1) \cdot \int_0^{\pi/n+1} \phi(t) dt$$

$$= O(n+1) \int_0^{\pi/n+1} |t|^{\alpha} dt$$

$$= O(n+1) \cdot \left[\frac{t^{\alpha+1}}{\alpha+1} \right]_0^{\pi/n+1}$$

$$= O(n+1) \cdot \frac{\pi^{\alpha+1}}{(n+1)^{\alpha+1} \cdot (\alpha+1)}$$

$$(4.16) \quad = O(n+1)^{-\alpha}$$

Now

$$I_{1.122} = \int_{\pi/n+1}^{\pi} \phi(t) K_n(t) dt$$

Applying Lemma (5)

$$= \int_{\pi/n+1}^{\pi} O\left(\frac{1}{t}\right) \cdot |t|^{\alpha} dt$$

$$= \left[\frac{t^{\alpha}}{\alpha} \right]_{\pi/n+1}^{\pi}$$

$$= \frac{1}{\alpha} \left[\pi^{\alpha} - \frac{\pi^{\alpha}}{(n+1)^{\alpha}} \right]$$

$$(4.17) \quad = O(n+1)^{-\alpha}.$$

Now

$$I_{1.3} = \frac{1}{(n+1)\pi} \sum_{k=0}^n \int_0^{\pi} \frac{\phi(t)}{\sin t/2} \sum_{|k-q| \leq q^{\delta}} \sqrt{\frac{a}{\pi q}} \exp(-aq^{-1}(k-q)^2) \cdot O\left(\frac{|k-q|^3}{q^2}\right) \sin\left(k + \frac{1}{2}\right) t dt.$$

Similarly

$$(4.18) \quad I_{1.3} = O(n+1)^{-\alpha}.$$

Now combining from (4.3) to (4.18)

$$\|(CF)_n^{(a,q)}(x) - f(x)\| = O((n+1)^{-\alpha}) \quad \text{for } 0 < \alpha < 1.$$

Thus

$$\|(CF)_n^{(a,q)}(x) - f(x)\|_{\infty} = \sup_{-\pi \leq x \leq \pi} |(CF)_n^{(a,q)}(x) - f(x)|$$

$$= O\left((n+1)^{-\alpha}\right) \quad \text{for } 0 < \alpha < 1.$$

This completes the proof of the theorem.

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