On Projective Transformations Between Special Finsler Spaces

P.N. Pandey and Reema Verma
Department of Mathematics, University of Allahabad, Allahabad-211002

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Abstract. Aum and Desai, studied the projective transformation between Riemannian spaces. Adati and Miyazawa discussed such transformations between recurrent, symmetric, projective recurrent and projective symmetric spaces in detail. Few results of these authors were extended to recurrent Finsler spaces by Sinha and Faruqui. The aim of the present paper is to extend the results of Adati and Miyazawa to Finsler spaces and to generalize the results of Sinha and Faruqui. The notation for Berwald's covariant differentiation differs from that of Rand and Matsumoto.

1. Preliminaries

Let $\mathcal{F}_n$ be an $n$-dimensional Finsler space equipped with a metric function $\tilde{F}$ satisfying the required conditions, the corresponding symmetric metric tensor $\bar{g}$ and the Berwald's connection $\bar{G}$. Let $\mathcal{F}_n'$ be another Finsler space with metric function $\tilde{F}'$, metric tensor $\bar{g}'$ and the Berwald's connection $\bar{G}'$ such that $\mathcal{F}_n$ is obtained by a projective transformation of $\mathcal{F}_n'$, i.e., the $\mathcal{F}_n$ and the $\mathcal{F}_n'$ are in geodesic correspondence. The projective transformation is characterized by the relation of Berwald's connection coefficients of $\mathcal{F}_n$ and $\mathcal{F}_n'$ such that

$$\bar{G}_{jk}^i = G_{jk}^i - p_j \delta_k^i - p_k \delta_j^i - p_{jk} \bar{x}^i,$$

where $p_j = \frac{\partial}{\partial x^j}$, $p_{jk} = \frac{\partial}{\partial x^k}$ and $\bar{G}_{jk}^i = \frac{\partial}{\partial \bar{x}^j}$. The function $p(x, \bar{x})$ is arbitrary and positively homogeneous of degree one in $\bar{x}^i$. Because of its homogeneity in $\bar{x}^i$, it satisfies

$$\bar{x}^k p_{jk} = \bar{x}^k \frac{\partial}{\partial \bar{x}^k} p_j = 0, \ p_j, \ p_{jk} = 0.$$

The covariant derivative of an arbitrary tensor $T_j^i$ for the connection $\bar{G}$ is given by

$$\bar{\nabla} m T_j^i = \partial m T_j^i - \left(\frac{\partial}{\partial x^i} T_j^m \right) G_{mj}^i \bar{x}^m + T_j^i G_{im}^j - T_j^i G_{jm}^i, \ \bar{\nabla} m = \frac{\partial}{\partial \bar{x}^m} \bar{x}^n,$$

Berwald constructed the tensor $H_{jk}^i$ and the curvature tensor $\bar{H}_{jk}^i$ from deviation tensor $H_{jk}$.

* Unless Otherwise stated, all the geometric objects are supposed to be functions of the line-elements $(\bar{x}, \bar{x}')$. The indices $i, j, k$ take positive integer values from 1 to $n$. 
as follows:

\[(1.4)\]
\[
a) \ H^i_{jk} = \frac{1}{3} \left( \partial_j H^i_k - \partial_k H^i_j \right) \quad b) \ H^i_{jkh} = \partial_j H^i_{kh}
\]

The curvature tensor $H^i_{jkh}$ defined above, is skew-symmetric in last two lower indices $k$ and $h$ and is positively homogeneous of degree zero in $\dot{x}^i$'s. The tensors $H^i_{jk}$ and $H^i_j$ satisfy

\[(1.5)\]
\[
a) \ H^i_{jkh} \dot{x}^h = H^i_{kh} \quad b) \ H^i_{kjh} \dot{x}^k = H^i_{kh} \quad c) \ H^i_{hj} \dot{x}^h = 0.
\]

The projective curvature tensor $W^i_{jkh}$ and the tensors $W^i_j$ and $W^i_j$ satisfy the following:

\[(1.6)\]
\[
a) \ W^i_{jk} = \frac{1}{3} \left( \partial_j W^i_k - \partial_k W^i_j \right) \quad b) \ W^i_{jkh} = \partial_j W^i_{kh} \quad c) \ W^i_{jkh} \dot{x}^h = W^i_{khj}.
\]
\[
d) \ W^i_{jk} \dot{x}^j = W^i_{kj} \quad e) \ W^i_j \dot{x}^j = 0 \quad f) \ W^i_i = 0 \quad g) \ W^i_{ij} = -W^i_{ji} = 0.
\]
\[
h) \ W^i_{jkh} = W^i_{jikh} = W^i_{jkh} = 0.
\]

The projective curvature tensor $W^i_{jkh}$ is skew-symmetric in last two lower indices and is positively homogeneous of degree zero in $\dot{x}^i$'s. A Finsler space is said to be of scalar curvature if it is isotropic at each point.

2. Two Lemmas

Let us assume that there exists a projective transformation from a non-flat Finsler space $F^*_n$ to a non-flat Finsler space $\overline{F}^*_n$. This means that the connection coefficients of $\overline{F}^*_n$ and $\overline{F}^*_n$ are related by (1.1).

The covariant derivative of the projective deviation tensor $\overline{W}^i_k$ of $\overline{F}^*_n$ is given by

\[
\overline{\nabla}_m \overline{W}^i_k = \partial_m \overline{W}^i_k - \left( \partial_r \overline{W}^i_k \right) G^r_{sm} \dot{x}^s + \overline{W}^r_k G^i_{rm} - \overline{W}^i_r G^r_{km},
\]

which, in view of (1.1) and the invariance of the projective deviation tensor $W^i_k$ under a projective transformation, gives

\[(2.1)\]
\[
\overline{\nabla}_m \overline{W}^i_k = \left( \partial_m W^i_k - \left( \partial_r W^i_k \right) G^r_{sm} \dot{x}^s + W^r_k G^i_{rm} - W^i_r G^r_{km} \right) + p \left( \partial_m W^i_k + 2p_m W^i_k - \delta^i_m p_r W^r_k - p_r W^r_k \dot{x}^i + p_k W^i_m \right).
\]
Using the formula (1.3) in (2.1), we have

\[(2.2) \quad \bar{\mathcal{B}}_m \bar{W}_k^i = \mathcal{B}_m W_k^i + p \partial_m W_k^i + 2\mu_m W_k^i - \delta_m^i \mu_r W_k^r - \mu_m \mu_{kr} W_k^r \dot{x}^i + \mu_k W_m^i.\]

Now we propose the following:

**Lemma 2.1.** If there exists a projective transformation from a non-flat Finsler space \( F_n \) to another non-flat Finsler space \( \bar{F}_n \), then the invariance of the tensor \( \mathcal{B}_m W_k^i \) implies at least one of the following

(i) the transformation is affine,
(ii) both the spaces are of scalar curvature.

**Proof.** If the tensor \( \mathcal{B}_m W_k^i \) is invariant under a projective transformation, i.e., \( \bar{\mathcal{B}}_m \bar{W}_k^i = \mathcal{B}_m W_k^i \), the equation (2.2) reduces to

\[(2.3) \quad p \partial_m W_k^i + 2\mu_m W_k^i - \delta_m^i \mu_r W_k^r - \mu_m \mu_{kr} W_k^r \dot{x}^i + \mu_k W_m^i = 0.\]

Transvecting (2.3) by \( \dot{x}^m \), and using the Euler's theorem for homogeneous functions and the equation (1.2), we have

\[(2.4) \quad 4p W_k^i - \dot{x}^i \mu_r W_k^r = 0.\]

Transvecting (2.4) by \( \mu_i \) and using \( \mu_i \dot{x}^i = \dot{x}^i \partial_i \mu = p \), we find \( p \mu_r W_k^r = 0 \); which implies at least one of the following

\[(2.5) \quad \begin{align*}
a) & \quad p = 0, \\
b) & \quad \mu_r W_k^r = 0.
\end{align*}\]

If (2.5a) holds, the transformation is affine. If (2.5a) does not hold, we must have (2.5b). Using (2.5b) in (2.4), we get \( p W_k^i = 0 \); which implies \( W_k^i = 0 \). Since the space \( F_n \) is non-flat and the condition \( W_k^i = 0 \) implies that the space is of scalar curvature (Szabo, Matsumoto, Pandey), the space \( F_n \) is of scalar curvature. In view of Matsumoto's theorem, the space \( \bar{F}_n \) is also of scalar curvature. This completes the proof.

**Lemma 2.2.** If there exists a projective transformation from a non-flat Finsler space \( F_n \) to another non-flat Finsler space \( \bar{F}_n \), then the condition

\[(2.6) \quad \bar{\mathcal{B}}_m \bar{W}_k^i - \mathcal{B}_m W_k^i = L_m W_k^i,\]

where \( L_m \) is a covariant vector field, implies at least one of the following:

(i) both the spaces are of scalar curvature.
(ii) the condition $L_m \dot{x}^m = 4p$ holds.

Proof. Let us assume that there exists a projective transformation from a non-flat Finsler space $F^r_n$ to another non-flat Finsler space $F^r_n$ and the condition (2.6) holds good. In view of (2.6), the equation (2.2) may be written as

$$L_m W^r_k = p \dot{\delta}_m^r W^r_k + 2p_m W^r_k - \dot{\delta}_m^r p_r W^r_k - p_{rm} W^r_k \dot{x}^i + p_k W^i_k.$$  

(2.8)

Contracting the indices $i$ and $m$ in (2.8) and using (1.2), we have

$$L_r W^r_k = p \dot{\delta}_r^r W^r_k - (n - 2) p_r W^r_k.$$  

(2.9)

Contracting the indices $i$ and $j$ in (1.6a) and using (1.6f) and (1.6g), we get $\dot{\delta}_r^r W^r_k = 0$.

Hence we may write (2.9) as

$$L_r W^r_k + (n - 2) p_r W^r_k = 0.$$  

(2.10)

Transvecting (2.8) by $x^m$ and using $x^m \dot{\delta}_m^r = 2W^r_k$ and (1.2), we get

$$\left( L_m \dot{x}^m - 4p \right) W^r_k = -\dot{x}^i p_i W^r_k.$$  

(2.11)

Transvecting (2.11) by $p_j$ and using $p_j \dot{x}^j = p$, we have

$$\left( L_m \dot{x}^m - 3p \right) p_r W^r_k = 0.$$  

(2.12)

The condition (2.12) implies at least one of the following

$$\left( \begin{array}{c} a) \ p_r W^r_k = 0, \\
 b) \ L_r \dot{x}^r - 3p = 0. \end{array} \right.$$  

(2.13)

We claim that (2.13a) holds good. If not, suppose $p_r W^r_k \neq 0$. Then we must have (2.13b).

From (2.11) and (2.13b) we have

$$p W^i_k = \dot{x}^i p_r W^r_k.$$  

(2.14)

Transvecting (2.14) by $L_p$, we have

$$p L_r W^r_k = L_r \dot{x}^r p_r W^r_k.$$  

(2.15)

Multiplying (2.10) by $p$ and using (2.15), we find

$$L_r \dot{x}^r + (n - 2)p = 0.$$  

(2.16)
since \( p_r W^j_k \neq 0 \). Using (2.13b) in (2.16) we get \((n+1)p = 0\). Being the dimension of the space, \( n \) can not be \(-1\). Therefore we have \( p = 0 \); which implies \( p_r = 0, p = 0 \). This gives \( p_r W^j_k = 0 \), a contradiction. Therefore, our supposition is wrong. Thus, we must have (2.13a). In view of (2.13a), (2.11) reduces to \((I_m \tilde{x}^m - 4 p) W^j_k = 0\). This implies atleast one of the following
\[
(2.17) \quad \text{a)} \quad I_m \tilde{x}^m = 4 p, \quad \text{b)} \quad W^j_k = 0.
\]
The condition (2.17b) implies that the space \( F_n \) and \( \tilde{F}_n \) are of scalar curvature.

### 3. Projective Recurrent Spaces

In this section we discuss the projective transformation (1.1) from a projective recurrent Finsler space \( F_n \) to another projective recurrent Finsler space \( \tilde{F}_n \) characterized by the conditions
\[
(3.1) \quad \mathcal{B}_m W^j_{jk}, = \lambda^j_m W^j_{jk} \quad W^j_{jk} \neq 0
\]
and
\[
(3.2) \quad \mathcal{B}_m \tilde{W}^j_{jk} = \lambda^j_m \tilde{W}^j_{jk} \quad \tilde{W}^j_{jk} \neq 0
\]
respectively. The covariant vectors \( \lambda^j_m \) and \( \lambda^j_m \) are said to be recurrence vectors. Projective recurrent Finsler spaces are actually non-flat spaces because the vanishing of curvature tensor implies the vanishing of \( W^j_{jk} \) and for a projective recurrent space \( W^j_{jk} \neq 0 \). Transvecting (3.1) and (3.2) by \( X^i \) and using (1.6c) and (1.6d), we have
\[
(3.3) \quad \mathcal{B}_m \tilde{W}^j_{h} = \lambda^j_m \tilde{W}^j_{h} \quad \tilde{W}^j_{h} \neq 0
\]
and
\[
(3.4) \quad \mathcal{B}_m W^j_{h} = \lambda^j_m W^j_{h} \quad W^j_{h} \neq 0.
\]
Since the projective deviation tensor \( W^j_k \) is invariant with respect to a projective transformation, i.e., \( \tilde{W}^j_k = W^j_k \), we have
\[
(3.5) \quad \mathcal{B}_m \tilde{W}^j_k - \mathcal{B}_m W^j_k = (\lambda^j_m - \lambda^j_m) W^j_k.
\]
If the recurrence vectors are same, we have
\[ \mathcal{B}_m W^i_k - \tilde{\mathcal{B}}_m W^i_k = 0, \]

i.e., the tensor \( \mathcal{B}_m W^i_k \) is invariant. Hence, in view of Lemma 2.1, we have at least one of the following:

(i) \( F_n \) and \( \tilde{F}_n \) are of scalar curvature,

(ii) the transformation is affine.

Since \( \tilde{F}_n \) and \( F_n \) are projective recurrent spaces, the projective deviation tensors \( W^i_k \) and \( W^i_k \) cannot vanish identically. Therefore \( F_n \) and \( \tilde{F}_n \) cannot be of scalar curvature. Hence the (ii) holds. This leads to:

**Theorem 3.1.** The projective transformation from a projective recurrent space \( F_n \) to another projective recurrent space \( \tilde{F}_n \) with same recurrence vectors is necessarily affine.

If the recurrence vectors \( \tilde{\lambda}_m \) and \( \lambda_m \) are not equal, i.e., \( \tilde{\lambda}_m \neq \lambda_m \); suppose \( \tilde{\lambda}_m - \lambda_m = L_m \). Then we have \( \mathcal{B}_m \tilde{W}^i_k - \mathcal{B}_m W^i_k = L_m W^i_k \). Since \( \tilde{W}^i_k = W^i_k \neq 0 \), \( \tilde{F}_n \) and \( F_n \) are not of scalar curvature. Therefore, in view of Lemma 2.2, we have \( L_m \tilde{x}^m = 4p \) i.e. \( (\tilde{\lambda}_m - \lambda_m) \tilde{x}^m = 4p \). Thus, we have:

**Theorem 3.2.** If a projective recurrent Finsler space \( F_n \) with recurrence vector \( \lambda_m \) is transformed to a projective recurrent Finsler space \( \tilde{F}_n \) with recurrence vector \( \tilde{\lambda}_m \) by the projective transformation (1.1), then we have \( (\tilde{\lambda}_m - \lambda_m) \tilde{x}^m = 4p \).

### 4. Recurrent Spaces

This section is devoted to the study of the projective transformation (1.1) from a recurrent Finsler space \( F_n \) to another recurrent Finsler space \( \tilde{F}_n \) characterized by

\[ \mathcal{B}_m H^i_{jkh} = \lambda_m H^i_{jkh}, \quad H^i_{jkh} \neq 0 \]  

(4.1)

and

\[ \mathcal{B}_m \tilde{H}^i_{jkh} = \tilde{\lambda}_m \tilde{H}^i_{jkh}, \quad \tilde{H}^i_{jkh} \neq 0 \]  

(4.2)

respectively. The covariant vectors \( \lambda_m \) and \( \tilde{\lambda}_m \) are recurrence vectors of \( F_n \) and \( \tilde{F}_n \) respectively. It has been observed that the projective deviation tensor \( W^i_k \) of a recurrent space is recurrent with same recurrence vector. In view of this result, we find

\[ a) \mathcal{B}_m W^i_h = \lambda_m W^i_h, \quad b) \mathcal{B}_m \tilde{W}^i_h = \tilde{\lambda}_m \tilde{W}^i_h. \]  

(4.3)
It has been proved by Pandey\textsuperscript{10} that a recurrent Finsler space of scalar curvature does not exist. We also know that a Finsler space is of zero projective curvature if and only if it is of scalar curvature. From these two theorems we may conclude that a recurrent Finsler space can not have zero projective curvature, and hence zero projective deviation tensor. Thus, $W_k^j \neq 0$ and $\overline{W}_k^j \neq 0$. Proceeding in the similar way of proof of Theorems 3.1 and 3.2, we may prove:

**Theorem 4.1.** The projective transformation from a recurrent Finsler space $F_n$ to another recurrent Finsler space $\overline{F}_n$ with same recurrence vectors is necessarily affine.

**Theorem 4.2.** If a recurrent Finsler space $F_n$ with recurrence vector $\lambda_m^*$ is transformed to a recurrent space $\overline{F}_n$ with recurrence vector $\overline{\lambda}_m^*$ by the projective transformation (1.1), we have $\overline{\lambda}_m^* - \lambda_m^* = 4\rho_m$.

5. Projective Symmetric Spaces

In this section we will study the projective transformation from a non-flat projective symmetric space $F_n$ to another non-flat projective symmetric space $\overline{F}_n$ characterized by

\begin{equation}
(5.1) \quad \begin{align*}
a) & \quad \mathcal{B}_m W_{jkh} = 0, \\
b) & \quad \overline{\mathcal{B}}_m \overline{W}_{jkh} = 0,
\end{align*}
\end{equation}

respectively\textsuperscript{11}. Transvection of (5.1a) and (5.1b) by $x^j x^k$ gives

\begin{equation}
(5.2) \quad \begin{align*}
a) & \quad \mathcal{B}_m W_h^j = 0, \\
b) & \quad \overline{B}_m \overline{W}_h^j = 0,
\end{align*}
\end{equation}

which shows that $\overline{\mathcal{B}}_m \overline{W}_h^j = \mathcal{B}_m W_h^j$. Therefore, in view of Lemma 2.1, we may conclude:

**Theorem 5.1.** If there exists a projective transformation from a non-flat projective symmetric space $F_n$ to another non-flat projective symmetric space $\overline{F}_n$, then we have at least one of the following conditions:

(i) the transformation is affine,

(ii) both spaces $F_n$ and $\overline{F}_n$ are of scalar curvature.

6. Symmetric Spaces

Let us discuss the projective transformation from a non-flat symmetric space $F_n$ to another non-flat symmetric space $\overline{F}_n$ characterized by

\begin{equation}
(6.1) \quad \begin{align*}
a) & \quad \mathcal{B}_m H_{jkh}^j = 0, \\
b) & \quad \overline{\mathcal{B}}_m \overline{H}_{jkh}^j = 0,
\end{align*}
\end{equation}

respectively. The projective deviation tensor $W_k^j$ of a symmetric Finsler space is a covariant constant\textsuperscript{12}. Hence we have $\overline{B}_m \overline{W}_h^j = \mathcal{B}_m W_h^j = 0$; which, in view of Lemma 2.1
implies at least one of the following:
(i) the transformation is affine,
(ii) both spaces are of scalar curvature.

But according to Pandey\textsuperscript{10}, a symmetric Finsler space $F_n (n > 2)$ of scalar curvature is a Riemannian space of constant Riemannian curvature provided it is non-flat. Hence we conclude:

**Theorem 6.1.** Let two non-flat symmetric spaces $F_n (n > 2)$ and $F'_n (n > 2)$ be related by a projective transformation. Then we have at least one of the following:
(i) the transformation is affine,
(ii) Finsler spaces $F_n$ and $F'_n$ are Riemannian spaces of constant Riemannian curvature.

7. Different Type of Spaces

In this section we discuss the projective transformation from a non-flat Finsler space $F_n$ to another non-flat Finsler space $F'_n$ such that
(i) $F_n$ is recurrent and $F'_n$ is symmetric.
(ii) $F_n$ is projective recurrent and $F'_n$ is projective symmetric.
(iii) $F_n$ is projective recurrent and $F'_n$ is recurrent.
(iv) $F_n$ is recurrent and $F'_n$ is projective symmetric.
(v) $F_n$ is symmetric and $F'_n$ is projective symmetric.

**Case (i).** Suppose the Finsler space $F_n$ is recurrent and $F'_n$ is symmetric. Since the projective deviation tensor $W^i_k$ is recurrent in a recurrent space and covariant constant in a symmetric space, we have

\begin{equation}
(7.1) \quad \text{a)} \, \mathcal{B}_m W^i_k = \lambda_m W^i_k, \quad \text{b)} \, \mathcal{B}_m W^i_k = 0,
\end{equation}

$\lambda_m$ being recurrence vector. Thus

\begin{equation}
(7.2) \quad \mathcal{B}_m W^i_k - \mathcal{B}_m W^i_k = -\lambda_m W^i_k.
\end{equation}

In view of Lemma 2.2, we have at least one of the following
(i) $F_n$ and $F'_n$ both are of scalar curvature.
(ii) $-\lambda_m = 4p_m$.

Pandey\textsuperscript{10} proved that a recurrent Finsler space of scalar curvature does not exist. In view of this result (i) is not possible. Hence, we may conclude:

**Theorem 7.1.** Let a recurrent Finsler space $F_n$ be related with a symmetric Finsler
space $\overline{F}_n$ by a projective transformation. Then the recurrence vector $\lambda_m$ satisfies $\lambda_m = -4p_m$.

**Case (ii).** Suppose $F_n$ is projective recurrent and $\overline{F}_n$ projective symmetric. Then we have $\overline{\beta}_m W_k^j = \lambda_m W_k^j$ and $\overline{\beta}_m W_k^j = 0$. This means $\overline{\beta}_m W_k^j - \overline{\beta}_m W_k^j = -\lambda_m W_k^j$. Therefore, in view of Lemma 2.2, we have at least one of the following:

(i) both spaces are of scalar curvature,
(ii) $-\lambda_m = 4p_m$.

But for a projective recurrent space $W_k^j \neq 0$. Thus condition (i) does not hold. Hence, we may conclude:

**Theorem 7.2.** Let a projective recurrent space $F_n$ be related with a projective symmetric Finsler space $\overline{F}_n$ by a projective transformation. Then the recurrence vector $\lambda_m$ satisfies $\lambda_m = -4p_m$.

**Case (iii).** Suppose the Finsler space $F_n$ is projective recurrent and $\overline{F}_n$ is recurrent. Since the projective deviation tensor is recurrent in a projective recurrent space as well as in a recurrent space we have

a) $\overline{\beta}_m W_k^j = \lambda_m W_k^j, W_k^j \neq 0$

b) $\overline{\beta}_m W_k^j = \overline{\lambda}_m W_k^j, W_k^j \neq 0$,

where $\lambda_m$ and $\overline{\lambda}_m$ are recurrence vectors. Thus

$$\overline{\beta}_m W_k^j - \overline{\beta}_m W_k^j = \overline{\lambda}_m W_k^j - \lambda_m W_k^j$$

(7.4) since the projective deviation tensor is an invariant under a projective transformation i.e., $W_k^j = \overline{W}_k^j$.

$$\overline{\beta}_m W_k^j - \overline{\beta}_m W_k^j = (\overline{\lambda}_m - \lambda_m) W_k^j$$

(7.5)

If $\overline{\lambda}_m \neq \lambda_m$, in view of Lemma 2.2, we have at least one of the following

(i) $F_n$ and $\overline{F}_n$ both are of scalar curvature,
(ii) $\overline{\lambda}_m - \lambda_m = 4p_m$.

But recurrent as well as projective recurrent spaces can not be of scalar curvature. Therefore, we conclude:

**Theorem 7.3.** Let a projective recurrent Finsler space $F_n$ with recurrence vector $\lambda_m$ be transformed to a recurrent Finsler space $\overline{F}_n$ with recurrence vector $\overline{\lambda}_m$ by a projective transformation. If $\overline{\lambda}_m = \lambda_m$, we have $\overline{\lambda}_m - \lambda_m = 4p_m$.
If $\bar{\lambda}_m = \lambda_m$. Then $\bar{\mathcal{B}}_m W_k^j - \mathcal{B}_m W_k^j = 0$. Since a recurrent or a projective recurrent Finsler space cannot be of scalar curvature, we conclude that the transformation is necessarily affine. Hence we have:

**Theorem 7.4.** The projective transformation from a projective recurrent Finsler space $F_n$ to a recurrent Finsler space $\bar{F}_n$ with same recurrence vectors is necessarily affine.

**Case (iv).** Suppose $F_n$ is recurrent and $\bar{F}_n$ is projective symmetric. Since the projective deviation tensor $W_k^j$ is recurrent in a recurrent space and it is a covariant constant in a projective symmetric space, we have $\mathcal{B}_m W_k^j - \mathcal{B}_m W_k^j = -\lambda_m W_k^j$. Since a recurrent space cannot be of scalar curvature, the Lemma 2.2 leads to

**Theorem 7.5.** If a recurrent Finsler space $F_n$ with recurrence vector $\lambda_m$ is transformed to a projective symmetric space $\bar{F}_n$ by a projective transformation, the recurrence vector satisfies $\lambda_m = -4\rho_m$.

**Case (v).** In this case $F_n$ is symmetric and $\bar{F}_n$ is projective symmetric. Since the projective deviation tensor is covariant constant in both the spaces, $\mathcal{B}_m W_k^j - \mathcal{B}_m W_k^j = 0$. In view of Lemma 2.1, we have at least one of the following

(i) the transformation is affine,
(ii) both the spaces are of scalar curvature.

Pandey\(^{10}\) proved that a symmetric Finsler space of scalar curvature is a Riemannian space of constant Riemannian curvature. In view of this result, we may conclude:

**Theorem 7.6.** If a symmetric Finsler space $F_n$ is transformed to a projective symmetric space $\bar{F}_n$ by a projective transformation, we have at least one of the following

(i) the transformation is affine,
(ii) $F_n$ is a Riemannian space of constant Riemannian curvature and $\bar{F}_n$ is of scalar curvature.

References


