

Related Fixed Point Theorem for Fuzzy Metric Spaces

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Abstract: In this paper we prove a fixed point theorem for three self mappings in fuzzy metric spaces using some conditions.

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1. Definitions

Definition¹ 1.1. A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is *continuous t -norm* if $*$ is satisfying the following conditions:

- (i) $*$ is commutative and associative,
- (ii) $*$ is continuous,
- (iii) $a * 1 = a$, for all $a \in [0, 1]$,
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition² 1.2. A 3- tuple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t -norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: For all $x, y, z \in X$ and $s, t > 0$

- (FM-1) $M(x, y, t) > 0$,
- (FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
- (FM-3) $M(x, y, t) = M(y, x, t)$,
- (FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
- (FM-5) $M(x, y, .) : (0, \infty) \rightarrow (0, 1]$ is continuous,

$$(FM-6) \quad \lim_{t \rightarrow \infty} M(x, y, t) = I,$$

then M is called a fuzzy metric on X . The function $M(x, y, t)$ denote the degree of nearness between x and y with respect to t .

Example 1.3 Define $a * b = \min\{a, b\}$ and $M(x, y, t) = t/t - |x-y|$, for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space.

Definition² 1.4. Let $(X, M, *)$ be a fuzzy metric space. Then

(a) A sequence $\{x_n\}$ in X converges to x in X if and only if

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = I \text{ for each } t > 0.$$

(b) A sequence $\{x_n\}$ in X is called *Cauchy sequence* if and only if

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = I \text{ for each } p > 0 \text{ and } t > 0.$$

(c) A fuzzy metric space $(X, M, *)$ is said to be *complete* if and only if every Cauchy sequence in X is convergent in X .

Lemma³ 1.5. Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$. If $\exists$ a number $q \in (0, 1)$ such that $M(x_{n+2}, x_{n+1}, qt) \geq M(x_{n+1}, x_n, t)$, for all $t > 0$ and $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X .

Proof. For $t > 0$ and $q \in (0, 1)$ we have

$$M(x_2, x_3, qt) \geq M(x_1, x_2, t) \geq M(x_0, x_1, t/q)$$

or

$$M(x_2, x_3, t) \geq M(x_0, x_1, t/q^2).$$

By simple induction, we have for all $t > 0$ and $n \in \mathbb{N}$

$$M(x_{n+1}, x_{n+2}, t) \geq M(x_1, x_2, t/q^n).$$

Thus for any positive number p and real number $t > 0$, we have

$$\begin{aligned} M(x_n, x_{n+p}, t) &\geq M(x_n, x_{n+1}, t/p) * \dots * M(x_{n+p-1}, x_{n+p}, t/p) \quad [\text{By FM} - 4] \\ &\geq M(x_1, x_2, t/pq^{n-1}) * \dots * M(x_1, x_2, t/pq^{n+p-2}). \end{aligned}$$

Therefore by FM – 6, we have

$$M(x_n, x_{n+p}, t) \geq I * \dots * I \geq I,$$

$\Rightarrow \{x_n\}$ is a Cauchy sequence in X . This completes the proof.

Lemma⁴ 1.6. *Let a fuzzy metric space $(X, M, *)$. If $\forall x, y \in X$ and $t > 0$ with positive number $q \in (0, 1)$ and $M(x, y, qt) \geq M(x, y, t)$, then $x = y$.*

Proof: If for all $t > 0$ and some constant $q \in (0, 1)$, then we have
 $M(x, y, s) \geq M(x, y, s/q) \geq M(x, y, s/q^2) \geq \dots \geq M(x, y, s/q^n)$

for all $s > 0$ and $x, y \in X$. Let $n \rightarrow \infty$, we have $M(x, y, s) = 1$ and thus $x = y$.

Definition 1.7. A function M is *continuous* in fuzzy metric space if and only if whenever $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$ then

$$\lim_{n \rightarrow \infty} M(x_n, y_n, t) = M(x, y, t) \text{ for each } t > 0.$$

Fisher⁵ proved the following theorem for three mappings in complete metric space.

Theorem A. *Let S and T be continuous mappings of a complete metric space (X, d) into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into $S(X) \cap T(X)$ which commute with S and T and satisfy:*

$$d(Ax, Ay) \leq \alpha d(Sx, Ty),$$

for all $x, y \in X$ and $\alpha \in (0, 1)$. Indeed S, T and A have a unique common fixed point.

2. Main Results

In this paper we extend Theorem A to fuzzy metric space. We prove the following:

Theorem 2.1. *Let $(X, M, *)$ be fuzzy metric space and three continuous self mappings $S: X \rightarrow X, T: X \rightarrow X$ and $A: X \rightarrow X$ such that(1)*

$$(i) \quad S(X) \cup T(X) \subseteq A(X),$$

(1)

$$(ii) \quad M(Ax, Ay, qt) \geq \min\{M(Ay, Ty, t), M(Ax, Sx, t), M(Sx, Ty, t)\},$$

$\forall x, y \in X$ and $t > 0$ with positive number $q \in (0, 1)$. Then S, T and A have a unique common fixed point.

Proof. Since $S(X) \subseteq A(X)$ and $T(X) \subseteq A(X)$. We define a sequence $\{x_n\}$ such that

$$Sx_{2n} = Ax_{2n+1} \text{ and } Tx_{2n+1} = Ax_{2n+2}, \quad \forall n \in \mathbb{N}.$$

We shall prove that $\{Ax_n\}$ is a Cauchy sequence. Suppose $x = x_{2n}$, $y = x_{2n+1}$

$$\begin{aligned} M(Ax_{2n}, Ax_{2n+1}, qt) &\geq \min\{M(Ax_{2n+1}, Tx_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t), \\ &\quad M(Sx_{2n}, Tx_{2n+1}, t)\}, \\ &= \min\{M(Ax_{2n+1}, Ax_{2n+2}, t), M(Ax_{2n}, Ax_{2n+1}, t), \\ &\quad M(Ax_{2n+1}, Ax_{2n+2}, t)\}, \quad [\text{from (1)}] \\ &\geq \min\{M(Ax_{2n}, Ax_{2n+1}, t/q), M(Ax_{2n-1}, Ax_{2n}, t/q), \\ &\quad M(Ax_{2n}, Ax_{2n+1}, t/q)\}, \\ &\geq \min\{M(Ax_{2n-1}, Ax_{2n}, t/q^2), M(Ax_{2n-1}, Ax_{2n}, t/q), \\ &\quad M(Ax_{2n-1}, Ax_{2n}, t/q^2)\}, \\ &= M(Ax_{2n-1}, Ax_{2n}, t/q), \end{aligned}$$

Therefore $M(Ax_{2n}, Ax_{2n+1}, qt) \geq M(Ax_{2n-1}, Ax_{2n}, t/q)$.

By induction $M(Ax_{2k}, Ax_{2m+1}, qt) \geq M(Ax_{2k-1}, Ax_{2m}, t/q), \quad \forall k, m \in \mathbb{N}$

Case.1 if $2m+1 > 2k$, then

$$\begin{aligned} M(Ax_{2k}, Ax_{2m+1}, qt) &\geq M(Ax_{2k-1}, Ax_{2m}, t/q) \\ &\dots\dots\dots \\ &\geq M(Ax_0, Ax_{(2m+1)-2k}, t/q^{2k}). \end{aligned}$$

Case.2 if $2k > 2m+1$, then

$$\begin{aligned} M(Ax_{2k}, Ax_{2m+1}, qt) &\geq M(Ax_{2k-1}, Ax_{2m}, t/q) \\ &\dots\dots\dots \\ &\geq M(Ax_{2k-(2m+1)}, Ax_{0-2k}, t/q^{2m+1}). \end{aligned}$$

From both cases, we conclude that

$$M(Ax_n, Ax_{n+p}, qt) \geq M(Ax_0, Ax_p, t/q^n), \quad \forall n, p \in \mathbb{N}$$

Thus $\{Ax_n\}$ is a Cauchy sequence. Since X is complete, therefore $\exists z \in X$ such that

$$z = \lim_{n \rightarrow \infty} Ax_{2n} = \lim_{n \rightarrow \infty} Sx_{2n-1} = \lim_{n \rightarrow \infty} Tx_{2n-1}.$$

$$\Rightarrow Az = Sz = Tz.$$

Now

$$\begin{aligned} M(Az, A^2z, qt) &\geq \min\{M(AAz, TAz, t), M(Az, Sz, t), \\ &\quad M(Sz, TAz, t)\} \text{ [from (1)]} \\ &= M(Sz, TAz, t) \\ &= M(Sz, ATz, t) \\ &\geq M(Sz, AAz, t) \\ &= M(Sz, A^2z, t) \\ &\geq M(Sz, A^2z, t/q) \\ &\quad \cdot \\ &\geq M(Sz, A^2z, t/q^n) \end{aligned}$$

$$\Rightarrow M(Az, A^2z, qt) \geq M(Sz, A^2z, t/q^n)$$

Since $\lim_{n \rightarrow \infty} M(Sz, A^2z, t/q^n) = 1$, therefore $Az = A^2z$.

Thus z is a common fixed point of A , S and T .

Uniqueness of z : Let(hyp.) $w \neq z$ be another common fixed point of A , S and T . From (1) we have

$$\begin{aligned} M(Az, Aw, qt) &\geq \min\{M(Aw, Tw, t), M(Az, Sz, t), M(Sz, Tw, t)\}, \\ \Rightarrow M(z, w, qt) &\geq \min\{M(Aw, Aw, t), M(Az, Az, t), M(Az, Aw, t)\}, \\ \Rightarrow M(z, w, qt) &\geq M(z, w, t). \end{aligned}$$

Therefore $z = w$. Hence our hypothesis is false and z is unique.

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