A Note on Horseshoe Lemma for M-Projective and M-Injective Modules

A. S. Ranadive

Department of Mathematics, Guru Ghasidas University, Bilaspur (M.P.), India

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Abstract: In this paper we have dualized a result of Singh et. al.\(^1\) We also generalize one result of same author to get Horseshoe lemma for M-projective and M-injective modules.

Preliminaries

Throughout this paper \(R\) denotes a ring with unity and all the modules considered are left unitary modules over \(R\).

The following definitions are due to Azumayya et al.\(^2\)

**Definition 1:** An \(R\)-module \(U\) is called M-projective if given a diagram

\[
\begin{array}{c}
U \\
\downarrow f \\
M \twoheadrightarrow N \\
\alpha
\end{array}
\]

of \(R\)-modules and \(R\)-homomorphisms with exact row there is a \(R\)-homomorphism \(g : U \rightarrow M\) such that the resulting diagram is commutative.

**Definition 2:** An \(R\)-module \(U\) is called M-injective if given a diagram

\[
\begin{array}{c}
O \rightarrow N \rightarrow M \\
\alpha \\
\downarrow f \\
U
\end{array}
\]

of \(R\)-modules and \(R\)-homomorphisms with exact row there is a \(R\)-homomorphism \(g : M \rightarrow U\) such that the resulting diagram is commutative.

**Proposition 1:** Any \(R\)-module \(U\) is M-projective if and only if given a diagram

\[
\begin{array}{c}
M \\
\downarrow f \\
Y \rightarrow Z \\
\alpha \\
\beta
\end{array}
\]

of \(R\)-modules and \(R\)-homomorphisms with exact row and \(\beta \circ f = 0\) there exists a \(R\)-homomorphism \(g : U \rightarrow M\) such that the resulting diagram is commutative.\(^1\)
**Proposition 2**: An $R$-module $U$ is $M$-injective if and only if given a diagram of $R$-modules and $R$-homomorphisms of the form

\[
\begin{array}{c}
\alpha \\
Z \rightarrow \ Y \rightarrow M \\
f \downarrow \\
U
\end{array}
\]

in which row is exact and $f|_{\text{Im} \alpha}$ is monic there exists a $R$-homomorphism $g : M \rightarrow U$ such that the resulting diagram is commutative.

**Proof**: Let $U$ be $M$-injective. Since row is exact $f|_{\text{ker} \beta}$ is also monic so we have a $R$-homomorphism $f' : \text{Im} \beta \rightarrow U$ given by $f'(x) = f(y)$ where $y \in Y$ is such that $\beta(y) = x$. Thus if $i : \text{Im} \beta \rightarrow M$ is the natural injection we have the diagram

\[
\begin{array}{c}
0 \rightarrow \text{Im} \beta \rightarrow M \\
f \downarrow \\
U \leftarrow M
\end{array}
\]

in which the row is exact. Since $U$ is $M$-projective there exists a $R$-homomorphism $g' : M \rightarrow U$ such that $g' \circ i = f'$. If $f : Y \rightarrow \text{Im} \beta$ is defined by $f(y) = \beta(y)$ we have the commutative diagram

\[
\begin{array}{c}
\alpha \\
Z \rightarrow \ Y \rightarrow M \\
j \downarrow \\
0 \rightarrow \text{Im} \beta \rightarrow M \\
f \downarrow \\
U \leftarrow M
\end{array}
\]

where $f' \circ j = f$ and $I : M \rightarrow M$ is the identity homomorphism. Converse is easily seen to be true by letting $Z = 0$.

**Remark**: Here we observe that the proposition 2 above dualizes the proposition 1 of Singh et. al. However we can’t dualize the proposition 2 of Singh et. al.

**Definition 3**: For any module $M$ let $C_p(M)$ (respectively $C_l(M)$) denotes the class of $M$-projective (respectively $M$-injective) modules.

**Proposition 3**: $C_p(M)$ (respectively $C_l(M)$) is closed under direct sums (respectively direct products) and direct summands (respectively direct factors).

2. Main Result

**Proposition 4**: Consider the diagram of $R$-modules and $R$-homomorphisms of the form
in which the row is exact and columns are complexes with each \( U_i \) and \( U_i'' \) as $M$-projective, $i = 0, 1, 2, \ldots$. Then, there exists a $M$-projective resolution of $M$ and chain maps so that the columns form an exact sequence of complexes.

**Proof:** Since for each $i$, $U_i$ and $U_i''$ are $M$-projective it follows that $U_i' \oplus U_i''$ is also $M$-projective for each $i$. Also for each $i$ the sequence

\[
0 \to U_i' \to U_i' \oplus U_i'' \to U_i'' \to 0
\]

is split exact where $f_i$ is the $i$-th canonical injection and $g_i$ is the $i$-th canonical projection. It now follows from proposition 3 that there exists a map $\gamma_0 : U_i' \oplus U_i \to M$ such that the resulting squares are commutative. The remaining proof now follows by induction using Lemma 6.20 and $3 \times 3$ lemma\(^4\).

**Proposition 5:** Consider the diagram of $R$-modules and $R$-homomorphisms of the form

\[
\begin{array}{c}
0 \\
\downarrow f \\
M' \end{array} \to \begin{array}{c}
M \\
\downarrow \alpha_0 \\
U_0 \end{array} \to \begin{array}{c}
M'' \\
\downarrow \beta_0 \\
U_1 \end{array} \to \begin{array}{c}
0 \\
\downarrow f \\
M' \end{array}
\]

in which the row is exact and columns are complexes with each $U_i'$ and $U_i''$ as $M$-injective, $i = 0, 1, 2, \ldots$. Then there exists a $M$-injective resolution of $M$ and chain maps so that the columns form an exact sequence of complexes.
**Proof**: Since
\[ \bigoplus_{i \in I} A_i \subseteq \bigcap_{i \in I} A_i \quad \text{and} \quad \bigoplus_{i \in I} A_i = \bigcap_{i \in I} A_i \]
if \( I \) finite it follows that the proof of proposition 4 can be dualized to prove the result using proposition 3.

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**References**