On Totally Geodesic Affine Immersion in Locally Product Riemannian Manifolds

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(Received December 14, 1997)

Abstract: In this paper the totally geodesic affine immersions \( f: (M, \nabla) \to (\overline{M}, \overline{\nabla}) \) are studied in the case when \((\overline{M}, \overline{\nabla})\) is an affine locally product manifold of recurrent curvature. It is proved that \((M, \nabla)\) is flat or of recurrent curvature.

1. Preliminaries

Let \((M, \nabla)\) and \((\overline{M}, \overline{\nabla})\) be connected differentiable manifolds with torsion free affine connection \(\overline{\nabla}\) and \(\nabla\) with a Riemannian metric \(g\) and \(\overline{g}\) respectively. Then Gauss and Wiengarten formulae given by

\[
(1.1) \quad \overline{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad (b) \overline{\nabla}_X \overline{V} = -A_Y X + D_X \overline{V}
\]

for all \(x, y \in TM\) and \(V \in T^1M\), where \(\overline{\nabla}, \nabla\) and \(D\) are respectively the Riemannian, induced Riemannian and induced connections in \(\overline{M}, M\) and the normal bundle of \(T^1M\) of \(M\) respectively. \(B\) is the second fundamental form related to \(A\) by \(g(B(X, Y), U) = g(A_X U, Y)\).

The submanifold \(M\) of \(\overline{M}\) is known to be

(i) totally geodesic in \(\overline{M}\) if \(B = 0\).

(ii) minimal if \(\mu = \text{Trace}(B) / \text{Dim}(M) = 0\), and

(iii) totally umbilical if \(B(X, Y) = g(X, Y)\mu, X, Y \in TM\).

Fundamental Gauss and Codazzi equations for the affine immersion can be written as follows:

\[
(1.2) \quad \overline{R}(X, Y) Z = R(X, Y) Z + A_B(X, Z) Y - A_B(Y, Z) X
\]

\[+ \left( \nabla_X B \right)(Y, Z) - \left( \nabla_Y B \right)(X, Z), \]
\[(1.3) \quad \overline{R}(X, Y) V = \left( \nabla_Y A \right)_Y X - \left( \nabla_X A \right)_Y Y + B\left( A_Y X, Y \right) - B\left( X, A_Y Y \right) + \overline{R}(X, Y) V \]

for vector fields $X$, $Y$ and $Z$ tangent to $M$. Taking the normal component of (1.1a) we obtain the equation of Codazzi as

\[(1.4) \quad \left( \overline{R}(X, Y) Z \right)^1 = \left( \nabla_X B \right)(Y, Z) - \left( \nabla_Y B \right)(X, Z). \]

For a submanifold $M$ of a locally product Riemannian manifold $\overline{M}$ we put

\[FX = tX + fX \quad \text{and} \quad FV = hV + sV \]

where $tX$ is the tangential part of $FX$ and $fX$ the normal part of $FX$. Then $t$ is an endomorphism of the tangent bundle $TM$ and $f$ is a normal bundle value 1-form on the tangent bundle. In this case

\[(1.5) \quad t^2 X = X - hfX, \quad ftX + sfX = 0, \]

\[(1.6) \quad s^2 V = V - fhV, \quad thV + hsV = 0. \]

The covariant derivatives \( \nabla_X B \) and \( \nabla_X A \) are defined by

\[(1.7) \quad \nabla_X B(Y, Z) = D_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z) \]

\[(1.8) \quad \left( \nabla_X A \right)_Y Y = \nabla_X A_Y Y - A_Y \nabla_X Y - A_{D_X Y} Y. \]

2. Riemannian Product Immersion

Let $M^m$ and $M^n$ be Riemannian manifolds of dimension $m$ and $n$ respectively. We consider the product manifold $\overline{M} = M^m \times M^n$ of dimension $m + n$, then $\overline{M}$ admits the product structure tensor field $F$ such that $F^2 = I$, where $I$ the identity tensor and $g(FX, Y) = g(X, FY)$ for any vector field $X$ and $Y$ on $\overline{M}$.

Let $M$ be a $k$-dimensional submanifold of $\overline{M}$. If $F T_x (M) \subset T_x (M)$ for each point $x$ of $M$, then $M$ is said to be an $F$-invariant in $\overline{M}$. Let $\overline{M}$ be a locally decomposable Riemannian manifold, i.e. $\nabla_X F = 0$. If $M$ is an $F$-invariant submanifold of a locally decomposable Riemannian manifold $\overline{M}$, then $(\nabla_X F) V = 0$ and $sB(X, Y) = B(X, FY)$ Then we have
Theorem 2.1: Let $M$ be an $F$-invariant submanifold of a Riemannian product manifold $\overline{M} = \overline{M}^m \times \overline{M}^n$. Then $M$ is a Riemannian product manifold $M^p \times M^q$ where $M^p$ is a submanifold of $\overline{M}^m$ and $M^q$ is a submanifold of $\overline{M}^n$. $M^p$ and $M^q$ being both totally geodesic in $\overline{M}$.

We denote by the same $F$ the almost product structure on $M$, we now define the curvature tensor $R^1$ of the normal bundle of $M$ by

$$R^1(X, Y) = D_X D_Y - D_Y D_X - D_{[X, Y]},$$

If $R^1 = 0$, the normal connexion of $M$ is said to be flat. It is well known that $R^1 = 0$ if and only if we can choose an orthonormal frame $\{V_a\}$ of the normal bundle $TM^1$ such that $D_{V_a} = 0$ for all $a$.

Lemma 2.2: Let $M$ be an $F$-invariant submanifold of a locally Riemannian product manifold $\overline{M} = \overline{M}^m \times \overline{M}^n$. If the normal connexion of $M$ is flat, then the normal connection of $M^p$ in $\overline{M}^m$ and that of $M^q$ in $\overline{M}^n$ are both flat, where $M = M^p \times M^q$.

Proof: Let $V$ be a vector field in $TM^1$ in $\overline{M}^m$. We can suppose that

$$T_X(\overline{M}^m) = \left\{ X \in T_X(\overline{M}) : FX = X \right\}.$$

For any vector field $X$ tangent to $M$, we have

$$FD_X V = F\nabla_X V + FA_Y X = \nabla_X FV + FA_Y X = -A_{FY} X + D_X FY + FA_Y X = D_X V$$

because $FY = Y$. Therefore, if $V \in TM^1$, then $D_X V = TM^1$ which means that $TM^1$ is parallel. From this we see that the normal connexion of $M^p$ in $\overline{M}^m$ is flat. Similarly, we can see that the normal connexion of $M^q$ in $\overline{M}^n$ is also flat. We assume that $\overline{M}^m$ and $\overline{M}^n$ are complex space forms with constant sectional curvature $c_1$ and $c_2$ and denote them by $\overline{M}^m(c_1)$ and $\overline{M}^n(c_2)$ respectively. Let $M$ be an $F$-invariant submanifold of $M = M^m(c_1) \times M^n(c_2)$. We denote by $R$ the Riemannian curvature tensor of $M$. Then the Gauss equation of $M$ is given by

$$R(X, Y)Z = \frac{1}{16} \left( c_1 + c_2 \right) \left[ g(Y, Z)X - g(X, Z)Y + g(Y, Z) FX - g(FY, Z) FX - g(FX, Z) FY + g(FY, Z) FY \right].$$
\( + 2g(FX, tY) \, tFZ \) + \( \frac{1}{16} \left( c_1 - c_2 \right) [g(FY, Z) \, FX] \\
- g(FX, Z) \, Y + g(Y, Z) \, FX - g(X, Z) \, FY + g(FtX, Z) \, tX \\
- g(FtY, Z) \, tY + g(tY, Z) \, FtX - g(tX, Z) \, FtY + 2g(FX, tY) \, tZ \\
+ 2g(X, tY) \, tfZ \) + \( A_B (Y, Z)^{X-A} B (X, Z) \, Y \).

and the Codazzi equation by

\[
(2.3) \quad \left( \nabla_X B \right)(Y, Z) - \left( \nabla_Y B \right)(X, Z) \\
= \frac{1}{16} \left( c_1 + c_2 \right) [g(tY, Z) \, fX - g(tX, Z) \, fY + 2g(X, tY) \, fZ \\
+ g(FtY, Z) \, fFX - g(FtX, Z) \, fFY + 2g(FX, tY) \, fFZ] \\
+ \frac{1}{16} \left( c_1 - c_2 \right) [g(FtY, Z) \, fXF - g(FtX, Z) \, fFY + g(tY, Z) \, fFX \\
- g(tX, Z) \, fFY + 2g(FX, tY) \, fFZ + 2g(X, tY) \, fFZ].
\]

3. Totally Geodesic Immersion

Since for a totally geodesic immersion \( \nabla_X Y = \nabla_X Y \), the Gauss equation becomes

\[
(3.1) \quad R(X, Y) \, Z = \frac{1}{16} \left( c_1 + c_2 \right) [g(Y, Z) \, X - g(X, Z) \, Y \\
+ g(tY, Z) \, tX - g(tX, Z) \, tY + 2g(X, tY) \, tZ + g(FY, Z) \, FX \\
- g(FX, Z) \, FY + g(FtY, Z) \, FtX - g(FtX, Z) \, FtY \\
+ 2g(FX, tY) \, FtZ] + \frac{1}{16} \left( c_1 - c_2 \right) [g(FY, Z) \, X - g(FX, Z) \, Y \\
+ g(Y, Z) \, FX - g(X, Z) \, FY + g(FtX, Z) \, tX - g(FtY, Z) \, tY \\
+ g(tY, Z) \, FtX - g(tX, Z) \, FtY + 2g(FX, tY) \, tZ \\
+ 2g(X, tY) \, tfZ].
\]
In this case the Ricci Tensor $S$ of $M$ is given by

$$
S(X, Y) = \frac{1}{16} \left( c_1 + c_2 \right) [(k - 2) g(X, Y) + g(\mathcal{F}X, Y) \text{Tr}F + 6g(\mathcal{F}X, Y)] + \frac{1}{16} \left( c_1 - c_2 \right) [(k - 2) g(\mathcal{F}X, Y) + g(X, Y) \text{Tr}F + 6g(\mathcal{F}X, Y)].
$$

Assume that $f$ is an affine immersion. We define covariant derivative $\nabla^2 A$ by

$$
(\nabla^2_{XY} A)_V W = (\nabla_X (\nabla_Y A))_V W - (\nabla_{\nabla_X Y} A)_V W
$$

for arbitrary vector fields $X, Y, Z, W$ tangent to $M$ and $V$ a normal vector field. After a simple calculation we have

$$
(\nabla^2_{XY} A)_V W = \nabla_X \nabla_Y A_V W - \nabla_W A_V \nabla_Y W
$$

$$
- \nabla_X A_{D_Y V} W - \nabla_Y A_V \nabla_X W + A_V \nabla_Y \nabla_X W
$$

$$
+ A_{D_Y V} \nabla_X W - \nabla_Y A_{D_X V} W + A_{D_X V} \nabla_Y W + A_V \nabla_{\nabla_X Y} W
$$

$$
+ A_{D_Y D_X V} W - \nabla_{\nabla_X Y} A_V W + A_{D_{\nabla_X Y} V} W.
$$

In consequence of (3.5) we have

$$
(\nabla^2_{XY} A)_V W = R(X, Y) A_V W - A_V R(X, Y) W = A_{R^1(X, Y) V} W
$$

Theorem 3.1: We have

$$
\overline{R}(X, Y) Z = R(X, Y) Z.
$$

$$
\overline{R}(X, Y) V = - \left( \nabla_X A \right)_V Y + \left( \nabla_Y A \right)_V X + R^1(X, Y) V.
$$

Theorem 3.2: For a totally geodesic immersion

$$
(\nabla^2_{XY} \overline{R})(X, Y) Z = (\nabla^2_{XY} R)(X, Y) Z.
$$
(3.10) \[
\left( \overline{\nabla}_W \overline{R} \right) (X, Y) V = (R(X, Y) A)_{\nu} W + A_{\nu} R(X, Y) W
- \left( \overline{\nabla}_W^2 X \right)_{\nu} Y + \left( \overline{\nabla}_W^2 Y \right)_{\nu} X + \left( \overline{\nabla}_W \overline{R}^1 \right) (X, Y) V.
\]

Proof: The relation is a direct consequence of formulae and
\[
\left( \overline{\nabla}_W \overline{R} \right) (X, Y) Z = \overline{\nabla}_W \overline{R} (X, Y) Z - \overline{R} \left( \overline{\nabla}_W X, Y \right) Z
- \overline{R} \left( X, \overline{\nabla}_W Y \right) W - \overline{R} (X, Y) \overline{\nabla}_W Z
= \nabla_W R(X, Y) Z - R \left( \nabla_W X, Y \right) Z - R \left( X, \nabla_W Y \right) W
- R(X, Y) \nabla_W Z = \left( \nabla_W R \right) (X, Y) Z.
\]

Using \( \overline{\nabla} Y = \nabla X \), (3.4) and (1.1b), we get
\[
\overline{\nabla}_W \overline{R}(X, Y) V = \overline{\nabla}_W \left( \left( \nabla_Y A \right)_{\nu} X - \left( \nabla_X A \right)_{\nu} Y + R^1 (X, Y) V \right)
= \nabla_W \left( \nabla_Y A \right)_{\nu} X - \nabla_W \left( \nabla_X A \right)_{\nu} Y + \nabla_W \left( R^1 (X, Y) V \right)
= \nabla_W \left( \nabla_Y A \right)_{\nu} X - \nabla_W \left( \nabla_X A \right)_{\nu} Y - A R^1 (X, Y) V W
+ D_W R^1 (X, Y) V
\]
and
\[
R \left( \overline{\nabla}_W X, Y \right) V = \overline{R} \left( \nabla_W X, Y \right) V
= \left( \nabla_Y A \right)_{\nu} \nabla_W X - \left( \nabla_{\nabla_W X} A \right)_{\nu} Y + R^1 \left( \nabla_W X, Y \right) V
\]
\[
\overline{R} \left( X, \overline{\nabla}_W Y \right) V = \overline{R} \left( X, \nabla_W Y \right) V
= \left( \nabla_{\overline{\nabla}_W Y} A \right)_{\nu} X - \left( \nabla_X A \right)_{\nu} \nabla_W Y + R^1 \left( X, \nabla_W Y \right) V
\]
\[ \tilde{\nabla} V = \tilde{\nabla} V \]

\[ \quad = \tilde{\nabla} (X, Y) V = \tilde{\nabla} (X, Y) D V \]

\[ \quad = - \tilde{\nabla} (X, Y) A V + \tilde{\nabla} (X, Y) D V \]

\[ \quad = - \tilde{\nabla} (X, Y) A V + \left( \nabla V A \right) D V \]

\[ \quad = - \left( \nabla V A \right) D V + R (X, Y) D V \]

Applying the above and (3.4) to the formulae

\[ \left( \tilde{\nabla} \tilde{\nabla} \right) (X, Y) V = \tilde{\nabla} (X, Y) V - \tilde{\nabla} \left( \nabla V X, Y \right) V - \tilde{\nabla} \left( X, \nabla V Y \right) V \]

\[ \quad - \tilde{\nabla} (X, Y) V = \tilde{\nabla} \left( \nabla V X, Y \right) V - \tilde{\nabla} \left( \nabla V X, Y \right) V - \tilde{\nabla} \left( X, \nabla V Y \right) V \]

\[ \quad = \tilde{\nabla} (X, Y) V \]

\[ \quad = \tilde{\nabla} (X, Y) V + \tilde{\nabla} (X, Y) V \]

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Let $f$ be an affine immersion. For a 1-form $\rho$ on the normal bundle $N(M)$ and its first and second covariant derivatives with respect to the connection $D$ are defined by

\[
\left( D_X \rho \right) (V) = X(\rho(V)) - \rho(D_X V),
\]

\[
\left( D^2_{XY} \rho \right) = D_X \left( D_Y \rho \right) - D_{[X,Y]} \rho
\]

respectively. Assuming $R^1(X, Y) = D^2 X Y \rho - D^2 Y X \rho$, we obviously have:

**Theorem 3.4**: If the second derivative of the normal connection is symmetric, then the curvature tensor of the normal connection of $M$ vanish identically.

If $f$ is umbilical i.e., $A(X) = \rho(X)$ for certain 1-form $\rho$, then

\[
\left( \nabla_X A \right)_Y = \left( D_X \rho \right)(V) Y, \quad \left( \nabla^2 \right)_{XY} A = \left( D^2_{XY} \rho \right)(V) Z
\]

and

\[
(R(X, Y) A)_Z = \left( R^1(X, Y) \rho \right)(V) Z.
\]

**Proposition 3.1**: Let $f : (M, \nabla) \to (\bar{M}, \bar{\nabla})$ be a totally geodesic affine immersion, where $(M, \nabla)$ is an affine locally product Riemannian manifold of recurrent curvature, say $\bar{\nabla} R = \bar{\phi} \odot \bar{R}$, then we have

\[
A_Y R(X, Y) W = - (R(X, Y) A)_W W - \left( \nabla^2_{YX} A \right)_Y X
\]

\[
+ \left( \nabla^2_{XY} A \right)_Y + \phi(W) \left( \left( \nabla_X A \right)_Y X - \left( \nabla_X A \right)_Y Y \right)
\]

(3.11)

\[
(\nabla W R^1)(X, Y) = \phi(W) R^1(X, Y) V
\]

In particular when $f$ is additionally umbilical then

\[
(3.12)
\]

\[
\rho(V) R(X, Y) W = - \left( R^1(X, Y) \rho \right)(V) W - \left( D^2_{XY} \rho \right)(V)
\]

\[
- \phi(W) \left( D_Y \rho \right)(V) X + \left( D^2_{WX} \rho \right)(V) Y - \phi(W) \left( D_X \rho \right)(V) Y
\]

(3.13)
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Proof: (3.11) and (3.12) are consequences of the formulae $\bar{\nabla}_X Y = \nabla_X Y$, (3.6) and the assumption $\bar{\nabla} R = \Phi \otimes \bar{R}$. In this case $(R (X, Y) A) Z = (R^1 (X, Y) p) (V) Z$ becomes (3.13).

We shall study the existence of a certain class of $f$ invariant submanifolds in a complex space form of non-null holomorphic sectional curvature.

A proper $F$ invariant submanifold $M$ of a locally product Riemannian manifold $M$ is a $F$ invariant with both distributions $\mathbb{V}$ and $\mathbb{V}^T$ of non-null dimensions. Also $M$ is totally umbilical if there exists a normal vector field $L$ such that the second fundamental form $B$ satisfies $B (X, Y) = g (X, Y) L$, for any vector fields $X, Y$ tangent to $M$.

Now we propose:

Theorem 3.5: There exists no totally umbilical proper $F$ invariant submanifolds of an elliptic or hyperbolic complex space.

Proof: Suppose there exists a totally umbilical proper $F$-invariant submanifold $M$ of a complex space form $M (c_1 \neq 0, c_2 \neq 0)$. Let $X$ and $Y$ be two non-null vector field, from $\mathbb{V}$ and $\mathbb{D}$ respectively then, for the normal part of $R (X, F X) Y$, we get $[R (X, F X) Y]_N \neq 0$. On the other hand, since $M$ is totally umbilical, the Codazzi-equations give $[R (X, F X) Y] = g (F X, Y) D_X L - g (X, Y) D_{F X} L = 0$. Thus, we get a contradiction. This completes the proof.

References
