Three Dimensional Landsberg Space with Constant Unified Main Scalar

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(Received April 11, 2004)

Abstract: Certain properties of a three dimensional Landsberg space with constant unified main scalar have been discussed in the present paper. A new scalar $K$ has been introduced and its importance has been shown.

1. Introduction

In 1984 F. Ikeda\textsuperscript{1} studied the properties of Finsler Spaces satisfying the condition $L^2C^2 = f(x)$, where $L$ is the fundamental function and $C$ is the length of the torsion vector $C_i$. In his paper\textsuperscript{2} he considered the condition $L^2C^2 = \text{non-zero constant}$, which is stronger than the corresponding condition considered in the-paper\textsuperscript{1}. A two dimensional Berwald space is an example of such a Finsler space with constant function $LC$. However in three dimensional Landsberg space the function $LC$ is not always constant. Some properties of three dimensional Finsler space with non-zero constant function $LC$ have been studied by Ikeda\textsuperscript{3}. The function $LC$ has been called the unified main scalar in the case of three dimensional Finsler space. By introducing a scalar $T$, F. Ikeda has shown that every three dimensional Landsberg space is a Berwald space provided unified main scalar and $T$-scalar are non-zero constants.

The purpose of the present paper is to discuss some more properties of three dimensional Landsberg space with constant unified main scalar. We have introduced a scalar $K$ which has important role in this paper.

In the three dimensional Finsler space we have three essential scalar fields called main scalars $H$, $I$, $J$ and $h$- and $v$-connection vectors $h_i$ and $v_i$. The orthonormal frame field called Moör's frame $(l^i, m^i, n^i)$ plays important role in three dimensional Finsler space\textsuperscript{4}.

2. Scalar Components in Moör's Frame

We consider a three dimensional Finsler space $F^3$ with the fundamental function $L(x, y)$ and the frame $(l, m, n)$ called Moör's frame of $F^3$ where $l$ is the normalized supporting element.
i.e. \( l^i = \frac{\nabla_i}{L} \). \( m^i \) is the normalized torsion vector i.e. \( m^i = c/l^c \). \( n^i \) is constructed by \( g_{ij}l^in^j = 0 \), \( g_{ij}m^in^j = 0 \), \( g_{ij}n^in^j = 1 \) and \( g_{ij} \) is the fundamental tensor of \( F^3 \) defined as
\[ g_{ij} = \frac{1}{2} \tilde{\partial}_i \tilde{\partial}_j \frac{1}{2}, \tilde{\partial}_j \frac{1}{2} = \frac{\partial}{\partial y^j} \).

In the Moór's frame an arbitrary tensor be expressed in terms of scalar components, for instance a tensor \( T(=T) \) of \((1,2)\) type can be written as
\[ T^i_{jk} = T_{a\beta} e_a^i e_{\beta}^j e_{\gamma}^k \]
where \( e_{\varepsilon}^i = l^i \), \( e_{\varepsilon}^i = m^i \), \( e_{\varepsilon}^i = n^i \) and the summation convention is applied to greek indices also. The scalar components \( T_{a\beta} \) are given by
\[ T_{a\beta} = T_{a\beta}^i e_{\alpha}^i e_{\beta}^j e_{\gamma}^k \]
From the equation \( g_{ij} e_{\alpha}^i e_{\beta}^j = \delta_{\alpha \beta} \) we have
\[ g_{ij} = l^i l^j + m^i m^j + n^i n^j \quad \text{(2.1)} \]

Next the C-tensor \( C_{ijk} = \frac{1}{2} \tilde{\partial}_k g_{ij} \) satisfies \( C_{ijk}l^i = 0 \) and symmetric in \( i,j,k \), therefore if \( C_{a\beta} \) are scalar components of \( L \) \( C_{ijk} \) i.e.
\[ LC_{ijk} = C_{a\beta} e_{\alpha}^i e_{\beta}^j e_{\gamma}^k, \]
then we have
\[ LC_{ijk} = H m^i m^j m^k - J s_{i\{jk\}} \{ m^i m^j m^k \} + I s_{i\{jk\}} \{ n^i n^j n^k \} + J n^i n^j n^k, \quad \text{(2.2)} \]
where, \( s_{i\{jk\}} \) denote the cyclic interchange of \( i,j,k \) and summation and
\[ C_{ijk} = 0, C_{222} = H, C_{333} = -C_{223} = J, C_{233} = I. \quad \text{(2.3)} \]

We shall be using Cartan's connection \( CT = (F^i_j, G^i_j, C^i_{jk}) \) in the following section of this paper. The h- and v- covariant derivatives of a tensor field with respect to \( CT \) are indicated by the short and long lines respectively.

First we have
\[ l_{ij} = 0, m_{ij} = n_i n_j, n_{ij} = -m_i n_j, \quad \text{(2.4)} \]
\[ Ll_{ij} = h_{ij}, Lm_{ij} = -l_j m_j + n_i n_j, Ln_{ij} = -l_j n_j - m_i n_j, \quad \text{(2.5)} \]
where $h_i$ are components of angular metric tensor, $h_i$ and $v_i$ are component of $h$ and $v$-connection vectors respectively.

The equation (2.4) and (2.5) may be written as

\begin{align}
e_{\alpha\beta\gamma} &= H_{\alpha\beta\gamma} e_{\gamma\mu} e_{\gamma\nu}, \\
Le_{\alpha\beta\gamma} &= V_{\alpha\beta\gamma} e_{\gamma\mu} e_{\gamma\nu}, \tag{2.6}
\end{align}

where $H_{\alpha\beta\gamma}$ and $V_{\alpha\beta\gamma}$ being fixed are given by.

\begin{align}
(H_{\alpha\beta\gamma}) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & h_\gamma & 0 \\ 0 & -h_\gamma & 0 \end{pmatrix}, \\
(V_{\alpha\beta\gamma}) &= \begin{pmatrix} 0 & \delta_\gamma^2 & \delta_\gamma^1 \\ -\delta_\gamma^2 & 0 & v_\gamma \\ -\delta_\gamma^1 & -v_\gamma & 0 \end{pmatrix}, \tag{2.8}
\end{align}

where $h_i$ and $v_i$ are scalar components of $h$- and $v$-connection vectors $h_i$ and $v_i$, respectively, i.e.

\begin{align}
h_i &= h_\gamma e_{\gamma\mu}, \\
v_i &= v_\gamma e_{\gamma\mu}. \tag{2.9}
\end{align}

The first scalar component $v_1$ of $v$-connection vector $v_i$ vanishes identically.

If $T_{\alpha\beta\gamma\delta}$ be scalar components of $T_{ijkh}$ then we have

\begin{align}
T_{\alpha\beta\gamma\delta} &= (\delta_\gamma^k T_{\alpha\beta\gamma}) e_{\delta\mu}^k + T_{\mu\beta\gamma} H_{\mu\gamma\delta} + T_{\alpha\mu\gamma} H_{\mu\gamma\delta} + T_{\alpha\beta\mu} H_{\mu\gamma\delta}, \tag{2.10}
\end{align}

where, $\delta_\gamma^k = \delta_\gamma^k - G^i_k \delta_i$

Similarly if $T_{\alpha\beta\gamma\delta}$ be scalar component of $L T_{ijkh}$ then

\begin{align}
T_{\alpha\beta\gamma\delta} &= L(\delta_\gamma^k T_{\alpha\beta\gamma}) e_{\delta\mu}^k + T_{\mu\beta\gamma} V_{\mu\gamma\delta} + T_{\alpha\mu\gamma} V_{\mu\gamma\delta} + T_{\alpha\beta\mu} V_{\mu\gamma\delta}. \tag{2.11}
\end{align}

The scalar components $C_{\alpha\beta\gamma\delta}$ of $L C_{ijhk}$ are given by

\begin{align}
C_{1\beta\gamma\delta} &= 0, \\
C_{2226} &= H_{\delta} - J_{\delta} + 3J_{\delta}, \\
C_{2236} &= -J_{\delta} + (H - 2I) h_6, \\
C_{2336} &= I_{\delta} - 3J_{\delta}, \\
C_{3336} &= J_{\delta} + 3J_{\delta}, \tag{2.12}
\end{align}

where $H_{\delta}$ for instance is the $h$-scalar derivative of $H$ i.e. $H_{\delta} = (\delta_{\beta}^i H) e_{\delta}^i$. 

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On the other hand, the scalar components $C_{\alpha \beta \gamma \delta}$ of $LC_{\alpha \beta \gamma \delta}$ is completely symmetric therefore it gives

\begin{align}
(H-2l)v_2 - 3Jv_3 &= J_2 + H_3, \\
3Jv_2 + (H-2l)v_3 &= I_2 = J_3, \\
3Jv_2 + 3Jv_3 &= -J_2 + I_3.
\end{align}

(2.13)

We quote the following theorem which has been proved by M. Matsumoto in page 197.

**Theorem (2.1).** The $h$-curvature tensor $R_{\alpha \beta \gamma \delta}$ of $CT$ in any three dimensional Finsler space is written in the form

\begin{equation}
R_{\alpha \beta \gamma \delta} = Q_{(\alpha \beta \gamma \delta)} + g_{\alpha \beta} L_{\gamma \delta} + g_{\gamma \delta} L_{\alpha \beta},
\end{equation}

(2.14)

where $L_{\alpha \beta}$ is defined as

\begin{equation}
L_{\alpha \beta} = R_{\alpha \beta} - \frac{R}{4} R_{\gamma \delta}.
\end{equation}

(2.15)

$R_{\alpha \beta}$ is the $h$-Ricci tensor defined by $R_{\alpha \beta} = R^h_{\alpha \beta}$ and $R = R_{\alpha \beta} g^{\alpha \beta}$.

Here $Q_{(\alpha \beta \gamma \delta)}$ denote the interchange of indices $\alpha, \beta$ and subtraction.

The scalar component $L_{\alpha \beta}$ of $L_{\alpha \beta}$ are given by

\begin{equation}
L_{\alpha \beta} = \frac{1}{2} \delta_{\alpha \beta} \ast R_{\gamma \delta} = \ast R_{\beta \alpha}.
\end{equation}

(2.16)

where $\ast R_{\alpha \beta}$ are scalar components of $\ast R_{\alpha \beta}$ and $\ast R_{\alpha \beta}$ is given by $R_{\alpha \beta \gamma \delta} = \epsilon_{\alpha \beta \gamma \delta} \ast R_{\gamma \delta}$.

3. **Constant Unified Main Scalar**

Contracting (2.2) with $g^{\alpha \beta}$, we get

\begin{equation}
LC_i = (H+1)m_i,
\end{equation}

(3.1)

which shows that the unified main scalar $LC_i$ is equal to $H+1$.

We now assume that the unified main scalar $LC_i$ is non-zero constant then by addition of first and third equations of (2.13) we get, $(H+1)v_2 = 0$, which gives $v_2 = 0$. Hence we have the following:
Proposition (3.1). In a three-dimensional Finsler space with non-zero constant unified main scalar the $v$-connection vector $v_i$ may be written as $v_i = v_j n_i$.

A Finsler space is called a Landsberg space, if the Berwald connection $\Gamma$ is $h$-metrical which is equivalent to $P_{ijk} = 0$. Since

$$P_{ijk} = Q_{(ij)} \{ C_{ijk} + C_{ij} P_{nk} \},$$

$$P_{ijk} = Q_{(ij)} \{ \dot{C}_{jk} P_{ik} + P_{khr} C_{ij} \},$$

we have the following.

Proposition (3.2). A Finsler space is a Landsberg space if and only if $C_{ijk}$ is completely symmetric tensor.

If $F^3$ is a three-dimensional Landsberg space then $C_{aibj} = 0$ and in view of proposition (3.2) it follows that $C_{aibj} = C_{abji}$ which gives

$$\begin{aligned}
(a) & \quad H_j + 3Jh_j = 0 \\
(b) & \quad -J_j +(H-2i)h_j = 0 \\
(c) & \quad I_j - 3Jh_j = 0 \\
(d) & \quad J_j + 3Jh_j = 0
\end{aligned}$$

(3.4)

$$\begin{aligned}
(a) & \quad (H-2i)h_j - 3Jh_j = J_2 + H_3 \\
(b) & \quad 3Jh_j + (H-2i)h_j = I_2 + J_3 \\
(c) & \quad 3Jh_j + 3Jh_j = -J_2 + I_3.
\end{aligned}$$

(3.5)

Now, if unified main scalar LC($-H+1$) is non-zero constant then addition of (3.4) b and (3.4) d gives $h_j = 0$ and hence $H_j = I_j = J_j = 0$ further addition of (3.5) a and (3.5) c gives $h_j = 0$. Hence from (3.5) it follows that

$$-3Jh_j = J_2 + H_3, \quad (H-2i)h_j = I_2 + J_3.$$

(3.6)

If all the main scalar $H, I, J$ are constant then from (3.6) it follows that either $h_j$ is equal to zero or $J = 0$ and $H = 2i$. Summarizing these results we get,

Theorem (3.1). In a three-dimensional Landsberg space with non-zero constant unified main scalars we have $h_j = I_j = J_j = H_j = 0$.

Theorem (3.2). In a three-dimensional Landsberg space with non-zero constant main scalars either $h$-connection vector vanishes or $J = 0$ and $H = 2i$. 
The h-covariant derivative of (3.1) gives $L C_{ij} = (H+I)n_i h_j$, where we have used the equation (2.4) and the fact that $L_k = 0$ and $H+I$ is constant. Since in a Landsberg space $C_{ij}$ is symmetric tensor, the above equation gives that the h-connection vector $h_i$ is parallel to $n_i$ and we have the following:

**Theorem (3.3).** If the unified main scalar of three dimensional Landsberg space $F^3$ is non-zero constant then $C_{ij}$ and h-connection vector $h_i$ are represented by

$$C_{ij} = L^1 K n_i n_j, \quad h_i = (H+I)^{-1} K n_i,$$

respectively.

The function $K$ which occurs in theorem (3.3) will be called K-scalar and is given by $K = (H+I)h_3$.

Next we consider the Ricci identity $^4 C_{i j k} - C_{i k j} = -C_{i} R'_{j k} - C_{i k} R'_{j k}$ substituting equation (2.14) into this identity and paying attention to equation (2.4) and theorem (3.3) we get,

$$L^1 \left[ K_n_j - K_i n_k - K^2 (H+I)^{-1} (m_j n_k - m_k n_j) \right] n_i,$$

$$= -C_{i} (g_{i k} L'_{k} + \delta'_{i k} L_{j} - g_{i k} L'_{j} - \delta'_{i k} L_{j}) - C_{i k} (y_{i k} L'_{j} + \delta'_{i k} L_{j} - y_{i j} L'_{j} - \delta'_{i j} L_{j}).$$

where we have used the fact that $R_{j k} = g^{i h} R_{h j k} y^h = R_{h j k} y^h$ and $L_n = g^{i h} L_{j k}$.

Since $C_{i} = C_{m i}$ in view of (2.5) and proposition (3.1) its v-covariant derivatives gives

$$C_{i} = C L^1 [y_j n_i - m_j] n_i.$$  

Since $L\alpha\beta$ are scalar components of $L_{\alpha\beta}$ and $L_{\alpha\beta}$ are given by equation (2.16) we may write

$$L_{ij} = \frac{1}{2} (\ast R_{22} + \ast R_{33} - \ast R_{11}) n_i n_j + \frac{1}{2} (\ast R_{11} + \ast R_{33} - \ast R_{22}) m_i m_j$$

$$+ \frac{1}{2} (\ast R_{11} + \ast R_{22} - \ast R_{33}) n_i n_j - \ast R_{22} l_i l_j$$

$$- \ast R_{11} l_i l_j - \ast R_{33} m_i m_j - \ast R_{22} n_i n_j.$$  

Now we assume that $K$ is constant then from (3.7), (3.8) and (3.9) we get

$$
\begin{align*}
(a) \quad K^2 &= - (H+I)^2 (\ast R_{11} + \ast R_{22}), \\
(b) \quad \ast R_{12} + \ast R_{22} y_3 &= 0, \\
(c) \quad \ast R_{13} + \ast R_{33} y_3 &= 0.
\end{align*}
$$

(3.10)
From these results, we have the following:

**Theorem (3.4).** Let $F^3$ be a three dimensional Landsberg space. If the unified main scalar of $F^3$ is non-zero constant and the $K$-scalar of $F^3$ is constant, then $(*R_{ij} + v_j R_{ij})$ is constant and the equation (3.10) holds.

Now we consider the Ricci identity:

$$C_{i,j,k} - C_{i,k,j} = - C_{r s} P_{r s}^{i} - C_{i,j} C_{i,j}^{r} - C_{i,j} P_{i,j}^{r}.$$ 

If $F^3$ is a Landsberg space then $P_{i,j}^{i} = 0 = P_{i,j}^{j}$. Therefore, in view of equation (2.2) and theorem (3.3), the above Ricci identity reduces to

$$C_{i,j,k} - C_{i,k,j} = L^{-1} K \{ J n_{j} n_{m} m_{k} - J n_{j} m_{k} m_{j} + n_{j} n_{m} n_{k} - J n_{j} n_{m} n_{k} \}.$$ 

(3.11)

Now assuming the $K$-scalar to be a constant, in view of equations (2.4), (2.5), (3.8) and theorem (3.3), the L.H.S. of equation (3.11) may be written as

$$C_{i,j,k} - C_{i,k,j} = - L^{-2} K [ n_{j} n_{l} n_{k} + n_{j} n_{k} n_{l} + n_{k} n_{l} n_{j} + n_{j} n_{l} n_{k} + n_{j} n_{l} n_{k} - (H + 1) L^{-2} n_{j} n_{k} n_{l}].$$ 

(3.12)

Comparing equations (3.11) and (3.12), we get

$$\begin{align*}
(H+I)v_{2,2} &= -LKJ \\
(H+I)v_{2,3} &= LKJ \\
(H+I)v_{3,2} &= LKI - Ky_j \\
(H+I)v_{3,3} &= LKJ \\
(H+I)v_{3,3} &= -K
\end{align*}$$

(3.13)

Since $v_{2,2} = 0$ and $h_j = (H+I)^{-1} K$ equation (3.13) gives

$$J = 0, \quad v_{3,3} = 0, \quad v_{3,3} v_{2,3} = h_j v_j.$$ 

Hence we have the following:

**Theorem (3.5).** Let $F^3$ be a three dimensional Landsberg space. If the unified main scalar of $F^3$ is non-zero constant and the $K$-scalar of $F^3$ is constant then $J = 0$, $v_{3,3} = 0$, $v_{3,3} v_{2,3} = h_j v_j$. 

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