

Some Fixed Point Results for ϕ -Maps in A -Metric Space

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Abstract: In this paper, we prove the existence and uniqueness of fixed points for mappings satisfying ϕ -maps in the setting of A -metric spaces. Our results generalize various existing fixed point results in A -metric spaces.

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1. Introduction

In order to study the existence or non-existence of fixed point, researchers are searching for generalizations of metric space in different directions. A -metricspace is one such generalization introduced by Abbas et.al.¹. Abbas et.al. obtained it by generalizing the concept of S -metric space which was introduced by Sedghi et.al.². There are various results on generalizations of metric space for these one can see research papers³⁻¹¹ and references therein.

In this paper, we prove some fixed point theorems for mappings satisfying ϕ -maps in the setting of A -metric spaces.

2. Preliminaries

Abbas et.al.¹ introduced the concept of A -metric space as generalization of metric space and S -metric space.

Definition¹ 2.1: Let X be a nonempty set. A function $A: X^n \rightarrow [0, \infty)$ is called an A -metric on X if for any $x_i, a \in X$, $i = 1, 2, \dots, n$, the following conditions hold:

$$(2.1) \quad A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \geq 0,$$

$$(2.2) \quad A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0 \text{ if and only if } x_1 = x_2 = x_3 = \dots = x_{n-1} = x_n,$$

$$(2.3) \quad \begin{aligned} A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) &\leq A(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) \\ &\quad + A(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) \\ &\quad + A(x_3, x_3, x_3, \dots, (x_3)_{n-1}, a) \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\quad + A(x_{n-1}, x_{n-1}, x_{n-1}, \dots, (x_{n-1})_{n-1}, a) \\ &\quad + A(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a). \end{aligned}$$

The pair (X, A) is called an A -metric space.

Lemma¹ 2.1: Let (X, A) be A -metric space. Then

$$A(x, x, x, \dots, x, y) = A(y, y, y, \dots, y, x) \text{ for all } x, y \in X.$$

Example 2.1: Let $X = \mathbb{R}$. Define a function $A: X^n \rightarrow [0, \infty)$ by

$$\begin{aligned} A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) &= |x_n + x_{n-1} + \dots + x_2 - (n-1)x_1| \\ &\quad + |x_n + x_{n-1} + \dots + x_3 - (n-2)x_2| \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\quad + |x_n + x_{n-1} + x_{n-2} - 3x_{n-3}| \\ &\quad + |x_n + x_{n-1} - 2x_{n-2}| + |x_n - x_{n-1}|. \end{aligned}$$

Then (X, A) is an A -metric space.

Lemma¹ 2.2: Let (X, A) be A -metric space. Then for all $x, y \in X$ we have

$$A(x, x, x, \dots, x, z) \leq (n-1)A(x, x, x, \dots, x, y) + A(z, z, z, \dots, z, y)$$

and

$$A(x, x, x, \dots, x, z) \leq (n-1)A(x, x, x, \dots, x, y) + A(y, y, y, \dots, y, z).$$

Definition¹ 2.2: The A -metric space (X, A) is said to be bounded if there exists a constant $r > 0$ such that $A(x, x, x, \dots, x, y) \leq r$ for all $x, y \in X$. Otherwise, X is unbounded.

Definition¹ 2.3: Given a point x_0 in A -metric space (X, A) and a positive real number r , the set $B(x_0, r) = \{y \in X : A(y, y, y, \dots, y, x_0) < r\}$ is called an open ball centered at x_0 with radius r . The set $\overline{B(x_0, r)} = \{y \in X : A(y, y, y, \dots, y, x_0) \leq r\}$ is called a closed ball centered at x_0 with radius r .

Definition¹ 2.4: A subset G in a A -metric space (X, A) is said to be an open set if for each $x \in G$ there exists an $r > 0$ such that $B(x, r) \subset G$. A subset $F \subset X$ is called closed if $X \setminus F$ is open.

Lemma¹ 2.3: In any A -metric space (X, A) , each open ball is an open set in X and each closed ball is also a closed set in X .

Theorem¹ 2.1: Let (X, A) be A -metric space, then

- (i) An arbitrary union and finite intersection of open balls $B(x, r) \in X$ is open.
- (ii) An arbitrary intersection and finite union of closed balls $B(x, r) \in X$ is closed.

Theorem¹ 2.2: The collection $\mathfrak{B} = \{B(x, r) : x \in X, r > 0\}$ of all balls in A -metric space (X, A) is a basis for a topology τ on X .

Definition¹ 2.5: Let (X, A) be A -metric space. A sequence $\{x_k\}$ in X is said to converge to a point $x \in X$ if $A(x_k, x_k, x_k, \dots, x_k, x) \rightarrow 0$ as $k \rightarrow \infty$. That is, for each $\varepsilon \geq 0$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$ we have $A(x_k, x_k, x_k, \dots, x_k, x) \leq \varepsilon$ and we write $\lim_{k \rightarrow \infty} x_k = x$.

Lemma¹ 2.4: Let (X, A) be A -metric space. If the sequence $\{x_n\}$ in X converges to a point x , then x is unique.

Lemma¹ 2.5: Let (X, A) be A -metric space. If the sequence $\{x_k\}$ in X is called a Cauchy sequence if $A(x_k, x_k, x_k, \dots, x_k, x_m) \rightarrow 0$ as $k, m \rightarrow \infty$. That is,

for each $\varepsilon \geq 0$, there exists $N \in \mathbb{N}$ such that for all $k, m \in \mathbb{N}$ we have $A(x_k, x_k, x_k, \dots, x_k, x_m) \leq \varepsilon$ and we write $\lim_{k \rightarrow \infty} x_k = x$.

Lemma¹ 2.6: Every convergent sequence in A -metric space is a Cauchy sequence.

Lemma¹ 2.7: The converse of Lemma 2.6. does not hold in general. A Cauchy sequence in an A_b -metric space does not need to be convergent.

Definition¹ 2.6: The A -metric space (X, A) is said to be complete if every Cauchy sequence in X is convergent.

Definition¹² 2.7: Let f and g be single-valued self-mappings on a set X . If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g , and w is called a point of coincidence of f and g .

Definition¹³ 2.8: Let f and g be weakly compatible self-mappings on a set X . Mappings f and g are said to be weakly compatible if $fx = gx$ implies $fgx = gfx$, $x \in X$.

Definition¹² 2.9: Let f and g be weakly compatible self-mappings on a set X . If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g .

Definition¹⁴ 2.10: Let Φ be the set of all function ϕ such that $\phi: [0, \infty) \rightarrow [0, \infty)$ is a non-decreasing function satisfying $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ for all $t \in (0, \infty)$. If $\phi \in \Phi$, then ϕ is called a ϕ -map. Further, if ϕ is a ϕ -map, then

- (i) $\phi(t) < t$ for all $t \in (0, \infty)$,
- (ii) $\phi(0) = 0$.

Further, unless otherwise stated, ϕ will mean the ϕ -map.

Lemma 2.8: Let (X, A) be an A -metric space and let $\{x_n\}$ be a sequence in it such that $\lim_{n \rightarrow \infty} A(x_{n+1}, x_{n+1}, x_{n+1}, \dots, x_{n+1}, x_n) = 0$. If $\{x_n\}$ is not Cauchy sequence, then there exists an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \rightarrow \infty$:

$$A(x_{m_k}, x_{m_k}, x_{m_k}, \dots, x_{m_k}, x_{n_k}), A(x_{m_k}, x_{m_k}, x_{m_k}, \dots, x_{m_k}, x_{n_k+1}), A(x_{m_k-1}, \dots, x_{m_k-1}, x_{n_k})$$

$$A(x_{m_k-1}, \dots, x_{m_k-1}, x_{n_k+1}), A(x_{m_k+1}, \dots, x_{m_k+1}, x_{n_k+1}), \dots$$

Proof: Suppose that the sequence $\{x_n\}$ is not Cauchy. Then, there exists $\varepsilon > 0$ and subsequences $\{x_{m_k}\}, \{x_{n_k}\}$ such that for every $k \in \mathbb{N}$ and $n_k > m_k > k$ the following is satisfied:

$$A(x_{m_k}, \dots, x_{m_k}, x_{n_k}) \geq \varepsilon \text{ and } A(x_{m_k}, \dots, x_{m_k}, x_{n_k-1}) < \varepsilon.$$

Then, using Lemma 2.4 and (2.3), we have,

$$\begin{aligned} \varepsilon &\leq A(x_{m_k}, \dots, x_{m_k}, x_{n_k}) \\ &= A(x_{n_k}, x_{n_k}, \dots, x_{m_k}) \\ &\leq (n-1)A(x_{n_k}, x_{n_k}, \dots, x_{n_k-1}) + A(x_{m_k}, \dots, x_{m_k}, x_{n_k-1}) \\ &< (n-1)A(x_{n_k}, x_{n_k}, \dots, x_{n_k}, x_{n_k-1}) + \varepsilon \end{aligned}$$

And

$$\varepsilon \leq \lim_{k \rightarrow \infty} A(x_{m_k}, \dots, x_{m_k}, x_{n_k}) \leq \varepsilon$$

Therefore

$$\lim_{k \rightarrow \infty} A(x_{n_k}, \dots, x_{n_k}, x_{m_k}) = \lim_{k \rightarrow \infty} A(x_{m_k}, \dots, x_{m_k}, x_{n_k}) = \varepsilon.$$

Further, as

$$|A(x_{n_k}, \dots, x_{n_k}, x_{m_k}) - A(x_{n_k+1}, \dots, x_{n_k+1}, x_{m_k})| \leq (n-1)A(x_{n_k+1}, \dots, x_{n_k+1}, x_{n_k}).$$

We obtain that

$$\lim_{k \rightarrow \infty} A(x_{n_k+1}, \dots, x_{n_k+1}, x_{m_k}) = \lim_{k \rightarrow \infty} A(x_{m_k}, \dots, x_{m_k}, x_{n_k+1}) = \varepsilon.$$

Analogously, it can be proved that $A(x_{m_k-1}, \dots, x_{m_k-1}, x_{n_k}), A(x_{m_k-1}, \dots, x_{m_k-1}, x_{n_k-1}), A(x_{m_k+1}, \dots, x_{m_k+1}, x_{n_k+1}), \dots$ tends to ε .

3. Main Result

Now we prove the following theorems.

Theorem 3.1: Let (X, A) be an A -metric space. Suppose that the mapping $f, g: X \rightarrow X$ satisfy

$$(3.1) \quad A(fx_1, fx_2, \dots, fx_n) \leq \phi \left(\max \left\{ A(gx_1, gx_2, \dots, gx_n, fx_1), \right. \right. \\ \left. \left. A(gx_2, gx_2, \dots, gx_2, fx_2), \dots, A(gx_n, gx_n, \dots, gx_n, fx_n) \right\} \right),$$

for all $x_1, x_2, \dots, x_n \in X$. If the range of g contains the range of f , and one of $f(X)$ or $g(X)$ is complete subset of X , then f and g have a unique point of coincidence in X . Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof: Assume that f and g satisfy the condition (3.1). Let x_0 be an arbitrary point in X . Since the range of g contains the range of f , there is $x_1 \in X$ such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$, for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then f and g have a coincidence point. Thus we can assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) &= A(fx_{n-1}, fx_{n-1}, \dots, fx_{n-1}, fx_n) \\ &\leq \phi \left(\max \left\{ A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, fx_{n-1}), A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, fx_{n-1}), \right. \right. \\ &\quad \left. \left. \dots, A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, fx_{n-1}), A(gx_n, gx_n, \dots, gx_n, fx_n) \right\} \right) \\ &\leq \phi \left(\max \left\{ A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, fx_{n-1}), A(gx_n, gx_n, \dots, gx_n, fx_n) \right\} \right) \\ &= \phi \left(\max \left\{ A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n), A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) \right\} \right). \end{aligned}$$

If

$$\begin{aligned} &\max \left\{ A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n), A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) \right\} \\ &= A(gx_n, gx_n, \dots, gx_n, gx_{n+1}), \end{aligned}$$

then

$$A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) \leq \phi \left(A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) \right)$$

$$< A(gx_n, gx_n, \dots, gx_n, gx_{n+1}),$$

which leads to a contraction. This implies that

$$A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) \leq \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n)).$$

That is, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) &= A(fx_{n-1}, fx_{n-1}, \dots, fx_{n-1}, fx_n) \\ &\leq \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n)) \\ &\leq \phi^2(A(gx_{n-2}, gx_{n-2}, \dots, gx_{n-2}, gx_{n-1})) \\ &\vdots \\ &\leq \phi^n(A(gx_0, gx_0, \dots, gx_0, gx_1)). \end{aligned}$$

So we have $\lim_{n \rightarrow \infty} A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) = 0$. If $\{gx_n\} = \{fx_{n-1}\}$ is not Cauchy sequence in A -metric space (X, A) , then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that following sequences tend to ε when

$$k \rightarrow \infty; A(gx_{m_k+1}, \dots, gx_{m_k+1}, gx_{n_k+1}) \text{ and } A(gx_{m_k}, gx_{m_k}, \dots, gx_{m_k}, gx_{n_k}).$$

Putting now in (3.1) $x_1 = x_2 = \dots = x_{n-1} = x_{m_k}$, $x_n = x_{n_k}$, we obtain

$$\begin{aligned} A(gx_{m_k+1}, \dots, gx_{m_k+1}, gx_{n_k+1}) &= A(fx_{m_k}, fx_{m_k}, \dots, fx_{m_k}, fx_{n_k}) \\ &\leq \phi(\max\{A(gx_{m_k}, gx_{m_k}, \dots, gx_{m_k}, fx_{m_k}), A(gx_{m_k}, gx_{m_k}, \dots, gx_{m_k}, fx_{n_k}), \\ &\quad \dots, A(gx_{m_k}, gx_{m_k}, \dots, gx_{m_k}, fx_{m_k}), A(gx_{n_k}, \dots, gx_{n_k}, fx_{n_k})\}) \\ &= \phi(\max\{A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1}), A(gx_{n_k}, \dots, gx_{n_k}, gx_{n_k+1})\}). \end{aligned}$$

If

$$\begin{aligned} &\max\{A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1}), A(gx_{n_k}, \dots, gx_{n_k}, gx_{n_k+1})\} \\ &= A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1}) \end{aligned}$$

and since $A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1}) > 0$, we have

$$\begin{aligned} A(gx_{m_k+1}, \dots, gx_{m_k+1}, gx_{n_k+1}) &\leq \phi(A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1})) \\ &< A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1}). \end{aligned}$$

Letting $k \rightarrow \infty$ we obtain

$$\varepsilon \leq \lim_{n \rightarrow \infty} \phi(A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1})) \leq 0.$$

a contradiction. Analogously, if

$$\begin{aligned} \max\{A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1}), A(gx_{n_k}, \dots, gx_{n_k}, gx_{n_k+1})\} \\ = A(gx_{n_k}, \dots, gx_{n_k}, gx_{n_k+1}), \end{aligned}$$

we got a contradiction. So, it follows that $\{gx_n\} = \{fx_{n+1}\}$ is Cauchy sequence. By the completeness of $g(X)$ (or $f(X)$), we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. Suppose that $gp \neq fp$. By (3.1), we have

$$\begin{aligned} A(gx_n, gx_n, \dots, gx_n, fp) &= A(fx_{n-1}, fx_{n-1}, \dots, fx_{n-1}, fp) \\ &\leq \phi\left(\max\{A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n), \dots, A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n), \right. \\ &\quad \left. A(gp, gp, \dots, gp, fp)\right\}) \\ &= \phi\left(\max\{A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, fp), A(gp, gp, \dots, gp, fp)\}\right). \end{aligned}$$

Case 1:

$$\begin{aligned} \max\{A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n), A(gp, gp, \dots, gp, fp)\} \\ = A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n). \end{aligned}$$

We obtain that

$$\begin{aligned} A(gx_n, gx_n, \dots, gx_n, fp) &\leq \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n)) \\ &< A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n) \end{aligned}$$

By letting $n \rightarrow \infty$, we have $A(gp, gp, \dots, gp, fp) = 0$ and so $gp = fp$.

Case 2:

$$\begin{aligned} & \max \{ A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n), A(gp, gp, \dots, gp, fp) \} \\ &= A(gp, gp, \dots, gp, fp). \end{aligned}$$

We obtain that

$$A(gx_n, gx_n, \dots, gx_n, fp) \leq \phi(A(gp, gp, \dots, gp, fp)).$$

By letting $n \rightarrow \infty$, we have

$$A(gp_n, gp_n, \dots, gp_n, fp) \leq \phi(A(gp, gp, \dots, gp, fp)) < A(gp, gp, \dots, gp, fp),$$

which leads to a contradiction. Therefore $gp = fp$. We now show that f and g have a unique point of coincidence. Suppose that $fl = gl$ for some $l \in X$. By applying (3.1), it follows that

$$\begin{aligned} & A(gp, gp, \dots, gp, gl) = A(fp, fp, \dots, fp, fl) \\ & \leq \phi(\max \{ A(gp, gp, \dots, gp, fp), A(gp, gp, \dots, gp, fp), \dots, \\ & A(gp, gp, \dots, gp, fp), A(gl, gl, \dots, gl, fl) \}) = 0. \end{aligned}$$

Therefore $gp = gl$. This implies that f and g have a unique point of coincidence. By Proposition¹⁵, we can conclude that f and g have a unique common fixed point.

Corollary 3.1: Let (X, A) be an A -metric space. Suppose that the mappings $f, g : X \rightarrow X$ satisfy

$$\begin{aligned} & A(fx_1, fx_2, \dots, fx_n) \leq k \max \{ A(gx_1, gx_1, \dots, gx_1, fx_1), A(gx_2, gx_2, \dots, gx_2, fx_2) \\ & \dots, A(gx_n, gx_n, \dots, gx_n, fx_n) \}, \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in X$ where $0 \leq k < 1$. If the range of g contains the range of f and one of $f(X)$ or $g(X)$ is complete subspace of X , then f and g have a unique point of coincidence in X . Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof: Putting $\phi(t) = kt$, $t \geq 0$, $0 \leq k < 1$ in (3.1), the result follows.

Theorem 3.1. Let (X, A) be an A -metric space. Suppose that the mappings $f, g: X \rightarrow X$ satisfy

$$A(fx_1, fx_2, \dots, fx_n) \leq \max \left\{ \phi(A(gx_1, gx_1, \dots, gx_1, fx_1)), \phi(A(gx_2, gx_2, \dots, gx_2, fx_2)), \dots, \phi(A(gx_n, gx_n, \dots, gx_n, fx_n)) \right\}.$$

for all $x_1, x_2, \dots, x_n \in X$. If the range of g contains the range of f , and one of $f(X)$ or $g(X)$ is complete subspace of X , then f and g have a unique point of coincidence in X . Moreover, f and g are weakly compatible, then f and g have a unique common fixed point. The proof is very similar to the proof of Theorem 3.1, so we omitted it.

Theorem 3.2: Let (X, A) be an A -metric space. Suppose that the mapping $f, g: X \rightarrow X$ satisfy

$$(3.2) \quad A(fx_1, fx_2, \dots, fx_{n-1}, fx_n) \leq \phi(A(gx_1, gx_2, \dots, gx_n)), \quad \text{for all } x_1, x_2, \dots, x_n \in X,$$

where ϕ satisfies $\lim_{s \rightarrow t+} \phi(s) < t$ for all $t > 0$. If the range of g contains the range of f , and one of $f(X)$ or $g(X)$ is complete subspace of X , then f and g have a unique point of coincidence in X . Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X . Since the range of g contains the range of f , there is $x_1 \in X$ such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $g_n = fx_{n+1}$, then f and g have a point of coincidence. Thus we can suppose that $gx_n \neq g_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
& A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) = A(fx_{n-1}, fx_{n-1}, \dots, fx_n) \\
& \leq \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n)) \\
& \leq \phi^2(A(gx_{n-2}, gx_{n-2}, \dots, gx_{n-2}, gx_{n-1})) \\
& \cdot \\
& \cdot \\
& \cdot \\
& \leq \phi^n(A(gx_0, gx_0, \dots, gx_0, gx_1))
\end{aligned}$$

This implies that $\lim_{n \rightarrow \infty} A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) = 0$. If $\{gx_n\} = \{fx_{n-1}\}$ is not Cauchy sequence in A -metric space (X, A) , then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \rightarrow \infty$; $A(gx_{m_k+1}, \dots, gx_{m_k+1}, gx_{n_k+1})$ and $A(gx_{m_k}, \dots, gx_{m_k}, gx_{n_k})$.

Putting now in (3.2) $x_1 = x_2 = \dots = x_{n-1} = x_{m_k}$, $x_n = x_{n_k}$ and since $A(gx_{m_k}, \dots, gx_{m_k}, gx_{n_k}) > 0$ we obtain

$$A(gx_{m_k+1}, \dots, gx_{m_k+1}, gx_{n_k+1}) = A(fx_{m_k}, \dots, fx_{m_k}, fx_{n_k}) \leq \phi(A(gx_{m_k}, \dots, gx_{m_k}, gx_{n_k})).$$

Letting $k \rightarrow \infty$ and using the assumption of the mapping ϕ we obtain

$$\varepsilon \leq \lim_{k \rightarrow \infty} \phi(A(gx_{m_k}, \dots, gx_{m_k}, gx_{n_k}))$$

$$\lim_{t \rightarrow \varepsilon^+} A(gx_{m_k}, \dots, gx_{m_k}, gx_{n_k}) \phi(A(gx_{m_k}, \dots, gx_{m_k}, gx_{n_k})) = \lim_{t \rightarrow \varepsilon^+} \phi(t) < \varepsilon.$$

a contradiction. Therefore, the sequences $\{gx_n\} = \{fx_{n-1}\}$ is Cauchy sequence. By the completeness of $g(X)$ (or $f(X)$), we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. By (3.2) we have

$$\begin{aligned}
& A(gp, gp, \dots, gp, fp) \leq (n-1)A(gp, gp, \dots, gp, gx_{n+1}) + A(gx_{n+1}, gx_{n+1}, \dots, gx_{n+1}, fp) \\
& \leq (n-1)A(gp, gp, \dots, gp, gx_{n+1}) + \phi(A(gx_n, gx_n, \dots, gx_n, gp)) \\
& \leq (n-1)A(gp, gp, \dots, gp, gx_{n+1}) + A(gx_n, gx_n, \dots, gx_n, gp).
\end{aligned}$$

By taking $n \rightarrow \infty$, we have $A(gp, gp, \dots, gp, fp) = 0$ and so $gp = fp$. We now show that f and g have a unique point of coincidence. Suppose that $fq = gq$ for some $q \in X$. Assume that $gp = gq$. By applying (3.2), it follows that

$$\begin{aligned} A(gp, gp, \dots, gp, gq) &= A(fp, fp, \dots, fp, fq) \\ &\leq \phi(A(gp, gp, \dots, gp, gq)) < A(gp, gp, \dots, gp, gq), \end{aligned}$$

which leads to a contradiction. Therefore $gp = gq$. This implies that f and g have a unique point of coincidence. By proposition¹⁹, we can conclude that f and g have a unique common fixed point. By setting g to be the identity function on X , we immediately have the following corollary. This result extends and generalizes Boyd-Wong theorem from the metric space to A -metric spaces. We do not need semi-continuity of the comparison function, we only use $\phi \in \Phi$ with $\lim_{A \rightarrow t^+} \phi(s) < t$, $t > 0$.

Corollary 3.2: Let (X, A) be a complete A -metric space. Suppose that the mapping $f: X \rightarrow X$ satisfies $A(fx_1, fx_2, \dots, fx_n) \leq \phi(A(x_1, x_2, \dots, x_n))$ for all $x_1, x_2, \dots, x_n \in X$. then f has a unique fixed point. Theorem 3.2. Let (X, A) be a A -metric space. Suppose that the mappings $f, g: X \rightarrow X$ satisfy

$$\begin{aligned} (3.3) \quad A(fx_1, fx_2, \dots, fx_n) &\leq k_1 \phi(A(gx_1, gx_2, \dots, gx_n, fx_1)) + k_2 \phi(A(gx_2, \dots, gx_n, fx_2)) \\ &+ \dots + k_{n-1} \phi(A(gx_{n-1}, gx_n, \dots, gx_n, fx_{n-1})) + k_n \phi(A(gx_n, \dots, gx_n, fx_n)) \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in X$, $k_1 + k_2 + \dots + k_n < 1$.

If the range of g contains the range of f , and one of $f(X)$ or $g(X)$ is complete subspace of X , then f and g have a unique point of coincidence in X . Moreover f and g are weakly compatible, then f and g have a unique common fixed point.

Proof: Assume that f and g satisfy the condition (3.3). Let x_0 be an arbitrary point in X . Since the range of g contains the range of f , there is $x_1 \in X$ such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is

$n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$ then f and g have a point of coincidence. Thus we can suppose that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) &= A(fx_{n-1}, fx_{n-1}, \dots, fx_{n-1}, fx_n) \\ &\leq k_1 \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, fx_{n-1})) + k_2 \phi(A(gx_{n-1}, \dots, gx_{n-1}, fx_{n-1})) \\ &\quad + \dots + k_{n-1} \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, fx_{n-1})) + k_n \phi(A(gx_n, \dots, gx_n, fx_n)) \\ &= k_1 \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n)) + k_2 \phi(A(gx_{n-1}, \dots, gx_{n-1}, gx_n)) \\ &\quad + \dots + k_{n-1} \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n)) + k_n \phi(A(gx_n, \dots, gx_n, gx_{n+1})) \\ &< (k_1 + k_2 + \dots + k_{n-1}) \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n)) \\ &\quad + k_n \phi(A(gx_n, \dots, gx_n, gx_{n+1})). \end{aligned}$$

Now we have,

$$A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) < \frac{k_1 + k_2 + \dots + k_{n-1}}{1 - k_n} \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n)).$$

Let $r = \frac{k_1 + k_2 + \dots + k_{n-1}}{1 - k_n} < 1$. Then

$$\begin{aligned} A(gx_n, gx_n, \dots, gx_n, gx_{n+1}) &< r \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n)) \\ &< \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n)) \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &< \phi^n(A(gx_0, gx_0, \dots, gx_0, gx_1)). \end{aligned}$$

This implies that $\lim_{n \rightarrow \infty} A(gx_n, \dots, gx_n, gx_{n+1}) = 0$. If $\{gx_n\} = \{fx_{n-1}\}$ is not Cauchy sequence in A -metric space (X, A) , then there exist $a_n > 0$. and two

sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to when

$$k \rightarrow \infty; A(gx_{m_k+1}, \dots, gx_{m_k+1}, gx_{n_k+1}) \text{ and } A(gx_{m_k}, \dots, gx_{m_k}, gx_{n_k}).$$

Putting now in (3.3) $x_1 = x_2 = \dots = x_{n-1} = x_{m_k}$, $x_n = x_{n_k}$ and using the fact that

$A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1}) > 0$ and $A(gx_{n_k}, \dots, gx_{n_k}, gx_{n_k+1}) > 0$ we obtain

$$\begin{aligned} A(gx_{m_k+1}, \dots, gx_{m_k+1}, gx_{n_k+1}) &= A(fx_{m_k}, \dots, fx_{m_k}, fx_{n_k}) \\ &\leq k_1 \phi(A(gx_{m_k}, \dots, gx_{m_k}, fx_{m_k})) + k_2 \phi(A(gx_{m_k}, \dots, gx_{m_k}, fx_{m_k})) \dots \\ &\quad + k_{n-1} \phi(A(gx_{m_k}, \dots, gx_{m_k}, fx_{m_k})) + k_n \phi(A(gx_{n_k}, \dots, gx_{n_k}, fx_{n_k})) \\ &= k_1 \phi(A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1})) + k_2 \phi(A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1})) + \dots \\ &\quad + k_{n-1} \phi(A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1})) + k_n \phi(A(gx_{n_k}, \dots, gx_{n_k}, gx_{n_k+1})) \\ &< k_1 (A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1})) + k_2 (A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1})) + \dots \\ &\quad + k_{n-1} (A(gx_{m_k}, \dots, gx_{m_k}, gx_{m_k+1})) + k_n (A(gx_{n_k}, \dots, gx_{n_k}, gx_{n_k+1})). \end{aligned}$$

Letting $k \rightarrow \infty$ we obtain $\varepsilon \leq 0$. A contradiction. So, the sequences $\{gx_n\} = \{fx_{n-1}\}$ is Cauchy sequence. By the completeness of $g(X)$ (or $f(X)$), we obtain that $\{gx_n\}$ is convergent to some $q \in g(X)$. So, there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. Suppose that $gp \neq fp$. By (3.3), we have

$$\begin{aligned} A(gx_n, \dots, gx_n, fp) &= A(fx_{n-1}, \dots, fx_{n-1}, fp) \\ &\leq k_1 \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n)) + k_2 \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, gx_n)) + \\ &\quad \dots + k_{n-1} \phi(A(gx_{n-1}, gx_{n-1}, \dots, gx_{n-1}, fx_n)) + k_n \phi(A(gp, gp, \dots, gp, fp)). \end{aligned}$$

Letting $n \rightarrow \infty$ we have

$$A(gp, \dots, gp, fp) \leq k_n \phi(A(gp, gp, \dots, gp, fp))$$

$$< k_n A(gp, \dots, gp, fp)$$

$$< A(gp, \dots, gp, fp).$$

We got a contradiction. So, $gp = fp$. This proves that f and g have a unique point of coincidence is as in theorem (22) so we omitted it.

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