

On the Approximation of Conjugate Function Belonging to Lip α and Weighted $(L_r, \xi(t))$ Class by Product Summability Method

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Abstract: In this paper, a new theorem on the degree of approximation of conjugate function belonging to Lip α and weighted $(L_r, \xi(t))$ class by product summability method of conjugate Fourier series has been proved.

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1. Introduction

The study of degree of approximation of function belonging to Lip α classes using Cesàro and Nörlund method studied by Alexits¹, Sahney and Goel² Qureshi and Neha³ Qureshi⁴⁻⁶, Chandra⁷, Khan⁸ Leindler et al⁹⁻¹⁵. But till now nothing seems to have been done in the direction of this work. Therefore, in this paper, we established a new theorem on degree of approximation of conjugate function belonging to Lip α and weighted $(L_r, \xi(t))$ class by product summability method of conjugate Fourier series.

2. Preliminaries

Let $f(x)$ be periodic with period 2π and integrable in the sense of Lebesgue. The Fourier series of $f(x)$ is given by

$$(2.1) \quad f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) .$$

The conjugate series of (2.1) is given by

$$(2.2) \quad \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x) ,$$

with n th partial sum $S_n(f; x)$. A signal (function) $f \in Lip \alpha$ if,

$$f(x+t) - f(x) = O(|t|^\alpha) \text{ for } 0 < \alpha \leq 1,$$

A signal (function) $f \in Lip(\alpha, r)$ if,

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{\frac{1}{r}} = O(|t|^\alpha), 0 < \alpha \leq 1, r \geq 1.$$

Given a positive increasing function $\xi(t)$ and an integer $r \geq 1$, $f \in Lip(\xi(t), r)$ if,

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{\frac{1}{r}} = O(\xi(t)),$$

and that $f \in W(L_r, \xi(t))$ if,

$$\left(\int_0^{2\pi} |\{f(x+t) - f(x)\} \sin^\beta x| dx \right)^{\frac{1}{r}} = O(\xi(t)), \beta \geq 0.$$

If $\beta = 0$ then $W(L_r, \xi(t))$ coincides with $Lip(\xi(t), r)$ and if $\xi(t) = t^\alpha$ then $Lip(\xi(t), r)$ class coincides with the class $Lip(\alpha, r)$ and if $r \rightarrow \infty$ then $Lip(\alpha, r)$ class reduce to the class $Lip \alpha$.

Norm defined by

$$(2.3) \quad \|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx \right)^{\frac{1}{r}}, \quad r \geq 1.$$

Degree of approximation $E_n(f)$ is given by

$$(2.4) \quad E_n(f) = \min \|f - T_n\|_r,$$

where $T_n(x)$ is a trigonometric polynomial of degree 'n'

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its nth partial sum $\{s_n\}$, The (C, 1) transform is defined as the nth partial sum of (C, 1) summability and is given by

$$t_n = \frac{s_0 + s_1 + s_2 + s_3 + \dots + s_n}{n+1},$$

$$t_n = \frac{1}{n+1} \sum_{k=0}^n s_k \rightarrow s, \text{ as } n \rightarrow \infty$$

Then the series $\sum_{n=0}^{\infty} u_n$ is summable to definite number 's' by (C, 1) method.

If,

$$(E, q) = \frac{1}{(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} s_k \rightarrow s \text{ as } n \rightarrow \infty.$$

Then the infinite series $\sum_{n=0}^{\infty} u_n$ with partial sum 's_n' is said to be summable

by (E, q) method to a definite number 's'. A product of (C, 1) transform of (E, q) transform defines (C, 1) (E, q) transform and denotes by $C_n^1 E_n^q$.

Thus if,

$$C_n^1 E_n^q = (C, 1)(E, q) = \frac{1}{n+1} \sum_{k=0}^n E_k^q \rightarrow s \text{ as } n \rightarrow \infty,$$

where E_n^q denotes the (E, q) transform, then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by (C, 1) (E, q)

means or summable (C,1) (E,q) to a definite number 's'.

Notations:

$$\varphi(t) = f(x+t) - f(x-t) - 2f(x),$$

$$K_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^n \left| \frac{1}{(1+q)^k} \sum_{k=0}^n \left\{ \binom{k}{\nu} q^{k-\nu} \cdot \frac{\sin\left(\nu + \frac{1}{2}\right)t}{\sin \frac{t}{2}} \right\} \right|.$$

3. Main theorem

If f is 2π -periodic function, Lebesgue integrable on $[0, 2\pi]$ and belong to $W(L_r, \xi(t))$ class, then its degree of approximation is given by

$$(3.1) \quad \|C_n^1 E_n^q - f\|_r = O\left[(n+1)^{\beta + \frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right],$$

provided $\xi(t)$ satisfies the following condition:

$$(3.2) \quad \left\{ \frac{\xi(t)}{t} \right\}, \text{ be a decreasing sequence,}$$

$$(3.3) \quad \left\{ \int_0^{\frac{\pi}{n+1}} \left(\frac{t|\varphi(t)|}{\xi(t)} \right)^r \sin^{\beta r} t \, dt \right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right)$$

and

$$(3.4) \quad \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{t^{-\delta} |\varphi(t)|}{\xi(t)} \right)^r \sin^{\beta r} t \, dt \right\}^{\frac{1}{r}} = O\left\{(n+1)^{\delta}\right\},$$

where δ is an arbitrary number such that $s(1-\delta)-1>0$, $\frac{1}{r}+\frac{1}{s}=1$, condition (3.3) and (3.4) holds uniformly in x and $C_n^1 E_n^q$ is $(C,1)$ (E,q) means of the conjugate Fourier series (2.2).

4. Lemmas

Following lemmas are required for the proof of our main theorem:

Lemaa 4.1: $|K_n(t)| = O(n+1)$ for $0 \leq t \leq \frac{1}{n+1}$

Proof: For $0 \leq t \leq \frac{\pi}{n+1}$, $\sin nt \leq n \sin t$

$$\begin{aligned} K_n(t) &\leq \frac{1}{2\pi(n+1)} \sum_{k=0}^n \left| \frac{1}{(1+q)^k} \sum_{k=0}^n \left\{ \binom{k}{\nu} q^{k-\nu} \cdot \frac{(2\nu+1) \sin \frac{1}{2}t}{\sin \frac{t}{2}} \right\} \right| \\ &\leq \frac{1}{2\pi(n+1)} \left| \sum_{k=0}^n \left[\frac{1}{(1+q)^k} (2k+1) \sum_{\nu=0}^k \binom{k}{\nu} q^{k-\nu} \right] \right| \\ &= \frac{1}{2\pi(n+1)} \sum_{k=0}^n (2k+1) \\ &= O(n+1). \end{aligned}$$

Lemma 4.2: $|K_n(t)| = O\left(\frac{1}{t}\right)$ for $\frac{\pi}{n+1} \leq t \leq \pi$.

Proof: For $\frac{\pi}{n+1} \leq t \leq \pi$, by applying Jordan's lemma, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$ and $\sin(nt) \leq 1$.

$$K_n(t) \leq \frac{1}{2\pi(n+1)} \left| \sum_{k=0}^n \left[\frac{1}{(1+q)^k} (2k+1) \sum_{\nu=0}^k \binom{k}{\nu} q^{k-\nu} \cdot \frac{1}{\left(\frac{t}{\pi}\right)} \right] \right|$$

$$\begin{aligned}
&\leq \frac{1}{2\pi(n+1)} \sum_{k=0}^n \left[\frac{1}{(1+q)^k} (2k+1) \right] \\
&= \frac{1}{2t(n+1)} \sum_{k=0}^n 1 \\
&= O\left(\frac{1}{t}\right).
\end{aligned}$$

5. Proof of the Main Theorem

Following Titchmarsh¹⁶ and using Riemann-Lebesgue theorem, $s_n\{f;x\}$ of the series (2.2) is given by

$$s_n\{f;x\} - f(x) = \frac{1}{2\pi} \int_0^\pi \varphi(t) \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin \frac{t}{2}} dt.$$

Therefore, using (2.2) the (E, q) transform (E_n^q) of $s_n\{f;x\}$ given by

$$E_n^q - f(x) = \frac{1}{2\pi(1+q)^k} \int_0^\pi \frac{\varphi(t)}{\sin \frac{t}{2}} \left\{ \sum_{k=0}^n \binom{n}{k} q^{n-k} \sin\left(k + \frac{1}{2}\right)t \right\} dt.$$

Now denoting $(C, 1)$ (E, q) transform of $s_n\{f;x\}$ as $C_n^1 E_n^q$, we write

$$\begin{aligned}
C_n^1 E_n^q - f(x) &= \frac{1}{2\pi(n+1)} \sum_{k=0}^n \left[\frac{1}{(1+q)^k} \int_0^\pi \frac{\varphi(t)}{\sin \frac{t}{2}} \left\{ \sum_{k=0}^n \binom{n}{k} q^{n-k} \sin\left(\nu + \frac{1}{2}\right)t \right\} dt \right] \\
&= \left[\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^\pi \right] \varphi(t) K_n(t) dt
\end{aligned}$$

$$(5.1) \quad = I_1 + I_2 \quad \text{say.}$$

Now,

$$|I_1| \leq \int_0^{\frac{\pi}{n+1}} |\varphi(t)| |K_n(t)| dt.$$

We have,

$$|\varphi(x+t) - \varphi(x)| \leq |f(u+x+t) - f(u+x)| + |f(u-x-t) - f(u-x)|.$$

Using Minkowski's inequality

$$\begin{aligned} & \left[\int_0^{2\pi} \left| \{\varphi(x+t) - \varphi(x)\} \sin^\beta x \right|^r dx \right]^{\frac{1}{r}} \\ & \leq \left[\int_0^{2\pi} \left| \{f(u+x+t) - f(u+x)\} \sin^\beta x \right|^r dx \right]^{\frac{1}{r}} \\ & \quad + \left[\int_0^{2\pi} \left| \{f(u+x-t) - f(u-x)\} \sin^\beta x \right|^r dx \right]^{\frac{1}{r}} \\ & = O(\xi(t)). \end{aligned}$$

Then $f \in W(L_r, \xi(t)) \Rightarrow \varphi(t) \in W(L_r, \xi(t))$.

Using Hölder's inequality and the fact that $\varphi(t) \in W(L_r, \xi(t))$, (3.3), lemma 4.1 and second mean value theorem for integrals, We have,

$$\begin{aligned} |I_1| & \leq \left[\int_0^{\frac{\pi}{n+1}} \left\{ \frac{t |\varphi(t)| \sin^\beta t}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[\int_0^{\frac{\pi}{n+1}} \left\{ \frac{\xi(t) |K_n(t)|}{t \sin^\beta t} \right\}^s dt \right]^{\frac{1}{s}} \\ & = O\left(\frac{1}{n+1}\right) \left[\int_0^{\frac{\pi}{n+1}} \left\{ \frac{(n+1)\xi(t)}{t^{1+\beta}} \right\}^s dt \right]^{\frac{1}{s}} \end{aligned}$$

$$\begin{aligned}
&= O \left\{ \xi \left(\frac{1}{n+1} \right) \right\} \left[\int_{\varepsilon}^{\frac{\pi}{n+1}} \left\{ \frac{(n+1)\xi(t)}{t^{(1+\beta)s}} \right\}^s dt \right]^{\frac{1}{s}} \text{ for some } 0 < \varepsilon < \frac{1}{n+1} \\
&= O \left\{ (n+1)^{\beta+\frac{1}{r}} \xi \left(\frac{1}{n+1} \right) \right\} \text{ since } \frac{1}{r} + \frac{1}{s} = 1.
\end{aligned}$$

Now Hölder's, $|\sin t| < 1$, $\sin t \geq \left(\frac{2t}{\pi}\right)$, lemma (4.2), (3.2), (3.4) and mean value theorem.

$$\begin{aligned}
(5.3) \quad |I_2| &\leq \left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{ \frac{t^{-\delta} |\varphi(t)| \sin^{\beta} t}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[\int_{\frac{\pi}{n+1}}^{\pi} \left\{ \frac{\xi(t) |K_n(t)|}{t^{-\delta} \sin^{\beta} t} \right\}^s dt \right]^{\frac{1}{s}} \\
&= O \left\{ (n+1)^{\delta} \right\} \left[\int_{\frac{1}{\pi}}^{n+1} \left\{ \frac{\xi \left(\frac{1}{y} \right)}{y^{\delta-1-\beta}} \right\}^s \frac{dy}{y^2} \right]^{\frac{1}{s}} \\
&= O \left\{ (n+1)^{\delta} \xi \left(\frac{1}{n+1} \right) \right\} \left[\int_{\frac{1}{\pi}}^{n+1} \frac{dy}{y^{s(\delta-1-\beta)+2}} \right]^{\frac{1}{s}} \\
&= O \left\{ (n+1)^{\delta} \xi \left(\frac{1}{n+1} \right) \right\} \left[(n+1)^{s(\delta-1-\beta)+1} - \pi^{s(\delta-1-\beta)+1} \right]^{\frac{1}{s}} \\
&= O \left\{ (n+1)^{\delta} \xi \left(\frac{1}{n+1} \right) \right\} \left[(n+1)^{(1+\beta-\delta)-\frac{1}{s}} \right]^{\frac{1}{s}} \\
&= O \left\{ (n+1)^{\beta+\frac{1}{r}} \xi \left(\frac{1}{n+1} \right) \right\} \because \frac{1}{r} + \frac{1}{s} = 1.
\end{aligned}$$

Now collecting (5.1), (5.2) and (5.3) we get

$$\begin{aligned}
 |C_n^1 E_n^q - f(x)| &= O \left\{ (n+1)^{\beta + \frac{1}{r}} \xi \left(\frac{1}{n+1} \right) \right\} \\
 \|C_n^1 E_n^q - f(x)\|_r &= O \left\{ \int_0^{2\pi} O \left\{ (n+1)^{\beta + \frac{1}{r}} \xi \left(\frac{1}{n+1} \right) \right\}^r dx \right\}^{\frac{1}{r}} \\
 &= O \left\{ (n+1)^{\beta + \frac{1}{r}} \xi \left(\frac{1}{n+1} \right) \int_0^{2\pi} dx \right\}^{\frac{1}{r}} \\
 &= O \left\{ (n+1)^{\beta + \frac{1}{r}} \xi \left(\frac{1}{n+1} \right) \right\}.
 \end{aligned}$$

This completes the proof the main theorem.

6. Corollary

Following corollary becomes particular case of our main theorem.

Corollary 6.1: If $\beta=0$ and $\xi(t)=t^\alpha$, then the degree of approximation of a function $f \in \text{Lip}(\alpha, r)$, $0 < \alpha \leq 1$, is given by

$$\|C_n^1 E_n^q - f(x)\|_r = O \left\{ \frac{1}{(n+1)^{\alpha + \frac{1}{r}}} \right\}.$$

Corollary 6.2: If $r \rightarrow \infty$ in corollary 6.1, then for $0 < \alpha < 1$, is given by

$$\|C_n^1 E_n^q - f(x)\|_r = O \left\{ \frac{1}{(n+1)^\alpha} \right\}.$$

7. Applications

The approximations have important in new dimensions due to their wide applications in signal analysis¹⁷, in general and in Digital Signal Processing in particular, in view of the classical Shannon sampling theorem. Analysis of signals or time functions is of great importance, because it conveys information or attributes of some phenomenon. The engineers and scientists use properties of Fourier approximation for designing digital filters. Especially, Psarakis and Moustakides¹⁷ presented a new L_2 based method for

designing the Finite Impulse Response (FIR) digital filters and get corresponding optimum approximations having improved performance. Signals are treated as functions of one variable and images are represented by functions of two variables.

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