

On Recurrent Light Like Hyper Surfaces of Indefinite Almost Hyperbolic Hermitian Manifold with Semi-Symmetric Metric Connection

Sushil Shukla

Department of Mathematics
Uma Nath Singh Institute of Engineering and Technology
Veer Bahadur Singh Purvanchal University, Jaunpur 222001, India
Email: sushilcws@gmail.com

P. N. Pandey

Department of Mathematics
University of Allahabad, Prayagraj-211002, India
Email: pnpiaps@gmail.com

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Abstract: The object of present paper is to study the properties of recurrent lightlike hypersurfaces of indefinite almost hyperbolic Hermitian manifold with semi-symmetric metric connection.

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1. Introduction

The notion of semi-symmetric metric connection on a semi-Riemannian manifold was introduced by Yano¹. A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called semi-symmetric connection if its torsion tensor T satisfies

$$(1.1) \quad \bar{T}(\bar{X}, \bar{Y}) = \pi(\bar{Y})J\bar{X} - \pi(\bar{X})J\bar{Y},$$

where J is a $(1, 1)$ -type tensor field and π is a 1-form associated with a unit vector field ζ , which is called the characteristic vector field, by

$$\pi(\bar{X}) = \bar{g}(\bar{X}, \varsigma),$$

Let us consider a differential manifold M_{2n} of class C^∞ endowed with a tensor field of type $(1, 1)$ F such that for an arbitrary vector field X .

$$(1.2) \quad \bar{\bar{X}} = X,$$

where $\bar{X} \stackrel{\text{def}}{=} F(X)$.

Then F is called an almost hyperbolic Hermitian structure, and the differential manifold M_{2n} is called almost hyperbolic Hermitian manifold.

On almost hyperbolic Hermitian manifold M_{2n} , there exists a symmetric metric tensor g such that,

$$(1.3) \quad g(\bar{X}, \bar{Y}) + g(X, Y) = 0.$$

2. Lightlike Hypersurfaces

Let (M, g) be a lightlike hypersurface of $\bar{M}$. The normal bundle $TM^\perp$ of M is a subbundle of the tangent bundle TM of M , of rank 1, and coincides with the radical distribution $\text{Rad}(TM) = TM \cap TM^\perp$. Denote by $F(M)$ the algebra of smooth functions on M and by $T(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M .

A complementary vector bundle $S(TM)$ of $\text{Rad}(TM)$ in TM is non-degenerate distribution on M , which is called a screen distribution on M , such that

$$TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM),$$

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. For any null section ξ of $\text{Rad}(TM)$, there exists a unique null section N of a unique lightlike vector bundle $tr(TM)$ in the orthogonal complement $S(TM)^\perp$ of $S(TM)$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall X \in T(S(TM)).$$

We call $tr(TM)$ and N the transversal vector bundle and the null transversal vector field of M with respect to the screen distribution $S(TM)$, respectively.

The tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM)$$

In the sequel, let X, Y, Z and W be the vector fields on M , unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given respectively by

$$(2.1) \quad \bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N,$$

$$(2.2) \quad \bar{\nabla}_X N = -A_N X + \tau(X)N,$$

$$(2.3) \quad \nabla_X PY = \nabla_X^* PY + C(X, PY)\xi.$$

$$(2.4) \quad \nabla_X \xi = -A_\xi^* X + \sigma(X)\xi,$$

where ∇ and ∇^* are the induced linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A_ξ^* are the shape operators on TM and $S(TM)$ respectively, and are 1-forms on TM .

For a lightlike hypersurface M of an indefinite almost Hermitian manifold $(\bar{M}, \bar{g})$, it is known² that $J(Rad(TM))$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1 such that $J(Rad(TM)) \cap J(tr(TM)) = 0$. Thus there exist two non-degenerate almost complex distributions D_0 and D on M with respect to J , i.e., $J(D_0) = D_0$ and $J(D) = D$, such that

$$S(TM) = J(Rad(TM)) \oplus J(tr(TM)) \oplus_{orth} D_0,$$

$$D = \{Rad(TM) \oplus_{orth} J(Rad(TM))\} \oplus_{orth} D_0$$

$$(2.5) \quad TM = D \oplus J(tr(TM)).$$

Consider two null vector fields U and V , and two 1-forms u and v such that

$$(2.6) \quad U = -JN, V = J\xi, u(X) = g(X, V), v(X) = g(X, U),$$

Denote by S the projection morphism of TM on D . Any vector field X of M is expressed as $X = SX + u(X)U$. Applying J to this form, we have

$$(2.7) \quad JX = FX + u(X)N,$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $F = JoS$. Applying J to (2.7) and using (1.2), (1.3) and (2.6), we have

$$(2.8) \quad F^2X = X - u(X)U.$$

As $u(U) = 1$ and $FU = 0$, the set (F, u, U) defines an indefinite almost contact structure on M and F is called the structure tensor field of M . But it is not an indefinite almost contact metric structure on M and satisfies

$$(2.9) \quad g(FX, FY) = g(X, Y) - u(X)v(Y) - u(Y)v(X).$$

Denote by $\bar{R}$, R and R^* the curvature tensor of the semi-symmetric metric connection $\bar{\nabla}$ on $\bar{M}$ and the induced linear connection ∇ and ∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formulas, we obtain two Gauss-Codazzi equations for M and $S(TM)$ such that

$$(2.10) \quad \begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ &\quad + \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) \\ &\quad - \tau(Y)B(X, Z) + B(T(X, Y), Z)\}N, \end{aligned}$$

$$(2.11) \quad \begin{aligned} \bar{R}(X, Y)N &= -\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X, Y] + \tau(X)A_N Y \\ &\quad - \tau(Y)A_N X + \{B(Y, A_N X) - B(X, A_N Y) + 2d\tau(X, Y)\}N, \end{aligned}$$

$$(2.12) \quad \bar{R}(X, Y)PZ = R^*(X, Y)PZ + C(X, PZ)A_\xi^* Y - C(Y, PZ)A_\xi^* X$$

$$\begin{aligned}
 & +\{(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \sigma(X)C(Y, PZ) \\
 & - \sigma(Y)C(X, PZ) + C(T(X, Y), PZ)\}\xi, \\
 (2.13) \quad & R(X, Y)\xi = -\nabla_X^*(A_\xi^*Y) + \nabla_Y^*(A_\xi^*X) + A_\xi^*[X, Y] - \sigma(X)A_\xi^*Y \\
 & + \sigma(Y)A_\xi^*X + \{C(Y, A_\xi^*X) - C(X, A_\xi^*Y) - 2d\sigma(X, Y)\}\xi.
 \end{aligned}$$

3. Semi-Symmetric Metric Connection

Let $(\bar{M}, \bar{g}, J)$ be an indefinite almost hyperbolic Hermitian manifold with a semi symmetric metric connection $\bar{\nabla}$. Using (1.1), (2.1) and (2.7), we obtain

$$(3.1) \quad (\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y),$$

$$(3.2) \quad T(X, Y) = \pi(Y)FX + \pi(X)FY,$$

$$(3.3) \quad B(X, Y) - B(Y, X) = \pi(Y)u(X) - \pi(X)u(Y),$$

where T is the torsion tensor with respect to ∇_r and η is a 1-form such that

$$\eta(X) = \bar{g}(X, N).$$

Let $a = \theta(N)$ and $b = \theta(\xi)$. As $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, so B is independent of the choice of $S(TM)$ and satisfies

$$(3.4) \quad B(X, \xi) = 0, B(\xi, X) = -b u(X).$$

Local second fundamental forms are related to their shape operators by

$$(3.5) \quad B(X, Y) = g(A_\xi^*X, Y), \bar{g}(A_\xi^*X, N) = 0$$

$$(3.6) \quad C(X, PY) = g(A_N X, PY), \bar{g}(A_N X, N) = 0.$$

Applying $\bar{\nabla}_X$ to $\bar{g}(N, \xi)$ and using (2.2), (2.4) and $B(X, \xi) = 0$, we get $\tau = \sigma$ and

$$(3.7) \quad \bar{\nabla}_X \xi = -A_\xi^* X - \tau(X)\xi.$$

Taking $X = \xi$ to (3.5) and using $B(\xi, X) = -bu(X)$ and $S(TM)$ is non-degenerate, so

$$(3.8) \quad A_\xi^* \xi - bV = 0.$$

Applying $\bar{\nabla}_X$ to (2.6) and (2.7) and using (2.1), (2.2), (2.6), (2.7), (3.1), (3.6) and (3.7), we have

$$(3.9) \quad B(X, U) = C(X, V),$$

$$(3.10) \quad \nabla_X U = F(A_N X) + \tau(X)U,$$

$$(3.11) \quad \nabla_X V = F(A_\xi^* X) - \tau(X)V,$$

$$(3.12) \quad (\nabla_X F)Y = u(Y)A_N X - B(X, Y)U.$$

4. Recurrent Hypersurfaces

Structure tensor field F of M is said to be recurrent³⁻⁶ if there exists a 1-form ω on TM such that

$$(4.1) \quad (\nabla_X F)Y = \omega(X)FY.$$

A lightlike hypersurface M of an indefinite almost hyperbolic Hermitian manifold $\bar{M}$ is called recurrent if it admits a recurrent structure tensor field F .

Theorem 4.1: *Let M be a recurrent lightlike hypersurface of an indefinite almost hyperbolic Hermitian manifold $\bar{M}$ with a semi-symmetric metric connection. Then*

- (1) F is parallel with respect to the induced connection ∇ on M .
- (2) D and $J(\text{tr}(TM))$ are parallel distributions on M , and
- (3) M is locally a product manifold $C_0 \otimes M^\#$, where C_0 is a null curve tangent to $J(\text{tr}(TM))$ and $M^\#$ is a leaf of the distribution D .

Proof: (1) From (4.1) and (3.12), we get

$$(4.2) \quad \omega(X)FY = u(Y)A_N X - B(X, Y)U.$$

Replacing Y by ξ and using $B(X, \xi) = 0$, and the fact that $F\xi = -V$, we get $\omega(X)V = 0$. Taking the scalar product with U , we get $\omega = 0$. Which follows that $\nabla_X F = 0$. Thus F is parallel with respect to ∇ .

(2) Taking scalar product with V to (4.2) with $\omega = 0$, we have

$$B(X, Y) = u(Y)u(A_N X).$$

Replacing Y by V and Y by FZ , $Z \in \Gamma(D_0)$ to above by turns, we get

$$(4.3) \quad B(X, V) = 0, B(X, FZ) = 0.$$

By equations (2.7), (3.1), (3.5) and (3.11), we have

$$(4.4) \quad \begin{cases} g(\nabla_X \xi, V) = -B(X, V), g(\nabla_X V, V) = 0, \\ g(\nabla_X Z, V) = B(X, FZ), \forall Z \in \Gamma(D_0). \end{cases}$$

From (4.4) and (4.3), we have

$$\nabla_X Y \in T(D), \forall X \in \Gamma(TM), \forall Y \in \Gamma(D).$$

Therefore D is a parallel distribution on M . Taking $Y = U$ to (4.2), we have

$$(4.5) \quad A_N X = B(X, U)U.$$

Applying F to (4.5), we have

$$(4.6) \quad F(A_N X) = 0.$$

From (4.6) and (3.10), we have

$$(4.7) \quad \nabla_X U = \tau(X)U.$$

Hence $J(tr(TM))$ is also a parallel distribution on M .

(3) As D and $J(tr(TM))$ are parallel distributions and satisfied (2.5), by the decomposition theorem⁷, M is locally a product manifold $C_0 \otimes M^\#$, where C_0 is a null curve tangent to $J(tr(TM))$ and $M^\#$ is a leaf of D .

5. Indefinite Complex Space Forms

An indefinite complex space form $\bar{M}(c)$ is a connected indefinite almost hyperbolic Hermitian manifold of constant holomorphic sectional curvature c such that

$$(5.1) \quad \begin{aligned} \bar{R}(\bar{X}, \bar{Y})\bar{Z} = & \frac{c}{4} \{ \bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} + \bar{g}(J\bar{Y}, \bar{Z})J\bar{X} \\ & - \bar{g}(J\bar{X}, \bar{Z})J\bar{Y} + 2\bar{g}(\bar{X}, J\bar{Y})J\bar{Z} \}. \end{aligned}$$

Now comparing the tangential and transversal components of the two equations (2.10) and (5.1), and using (2.7) and (3.2), we have

$$(5.2) \quad \begin{aligned} R(X, Y)Z = & B(Y, Z)A_N X - B(X, Z)A_N Y \\ & - \frac{c}{4} \{ g(Y, Z)X - g(X, Z)Y + \bar{g}(JY, Z)FX \\ & - \bar{g}(JX, Z)FY + 2\bar{g}(X, JY)FZ \}, \end{aligned}$$

$$(5.3) \quad \begin{aligned} \nabla_X(B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) \\ - \tau(Y)B(X, Z) - \pi(X)B(FY, Z) + \pi(Y)B(FX, Z) \\ = \frac{c}{4} \{ u(X)g(FY, Z) - u(Y)g(FX, Z) + 2u(Z)\bar{g}(X, JY) \}. \end{aligned}$$

Taking the scalar product with N to (2.12) and substituting (5.2) into the resulting equation and using (2.7), (3.2) and $\bar{g}(A_N, X, N)$, we have

$$(5.4) \quad \begin{aligned} \nabla_X(C)(Y, PZ) - (\nabla_Y C)(X, PZ) - \tau(X)C(Y, PZ) \\ + \tau(Y)C(X, PZ) - \pi(X)C(FY, PZ) + \pi(Y)C(FX, PZ) \end{aligned}$$

$$= \frac{c}{4} \{ \eta(X)g(Y, PZ) - \eta(Y)g(X, PZ) + \nu(X)g(FY, PZ) \\ - \nu(Y)g(FX, PZ) + 2\nu(PZ)\bar{g}(X, JY) \}.$$

Theorem 5.1: *Let M be a recurrent lightlike hypersurface of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric metric connection. Then $\bar{M}(c)$ is flat, i.e., $c=0$, and the 1-form τ is closed on M , i.e., $d\tau=0$.*

Proof: Taking the scalar product with U to (4.5) and using (3.6), we have

$$(5.5) \quad C(X, U) = 0.$$

Applying ∇_Y to this equation and using (4.7), we get

$$(5.6) \quad (\nabla_X C)(Y, U) = 0.$$

Replacing PZ by U to (5.4) and using (5.5) with (5.6), we have

$$(5.7) \quad \frac{c}{2} \{ \nu(Y)\eta(X) - \nu(X)\eta(Y) \} = 0.$$

Replacing X by ξ and Y by V and using $B(\xi, X) = -bu(X)$, we have $c=0$.

Putting (4.5) into (5.2) with $c=0$, we have

$$(5.8) \quad R(X, Y)U = 0.$$

By (4.7), we have

$$(5.9) \quad R(X, Y)U = 2d\tau(X, Y)U.$$

By (5.8) and (5.9), we have $d\tau=0$.

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