

Inventory Problem Revisited

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Abstract: The paper is basically a review work. The purpose of the paper is to highlight most meaningful approach to Inventory problem. In fact, it establishes that inventory problem is a continuous or discrete optimal control problem. To be more precise, it is to be treated as a continuous- discrete optimal control problem. The paper reviews all types of dynamic inventory problems, their nature and the methodologies for their solutions. In the process of sequential development of realistic approaches to inventory problems, the very purpose of inventory problems is challenged. It no longer remains a cost minimization or profit maximization of inventory at hand. Rather it reduces to a control theoretic optimization of net cost or profit expressed as a functional, with respect to the given control. The control may be taken as the current inventory in case of a discrete model and updating of the current inventory in case of a continuous or a continuous-discrete model. Lastly the paper suggests some open problems, based on which, some additional contributions to the literature of Inventory models could be made in future.

Keywords: Inventory models, continuous, discrete, optimization.

1. Critical Literature Survey

1.1. General Inventory models: Inventory is the stock of any item/ items, which is used in an organization. Normally the models are taken to be continuous. There are two types of Inventory models- (1) EOQ (Economic order quantity) and (2) EPQ (Economic production quantity). A recent book on such single item continuous inventory models is given by J. B. G. Frenk, 2015¹. The objective of inventory problem is usually to keep enough inventories to meet customer demand, so that total cost of all different types involved in the process is minimized. It is a straight forward minimization problem to calculate the minimum cost with respect to the amount of

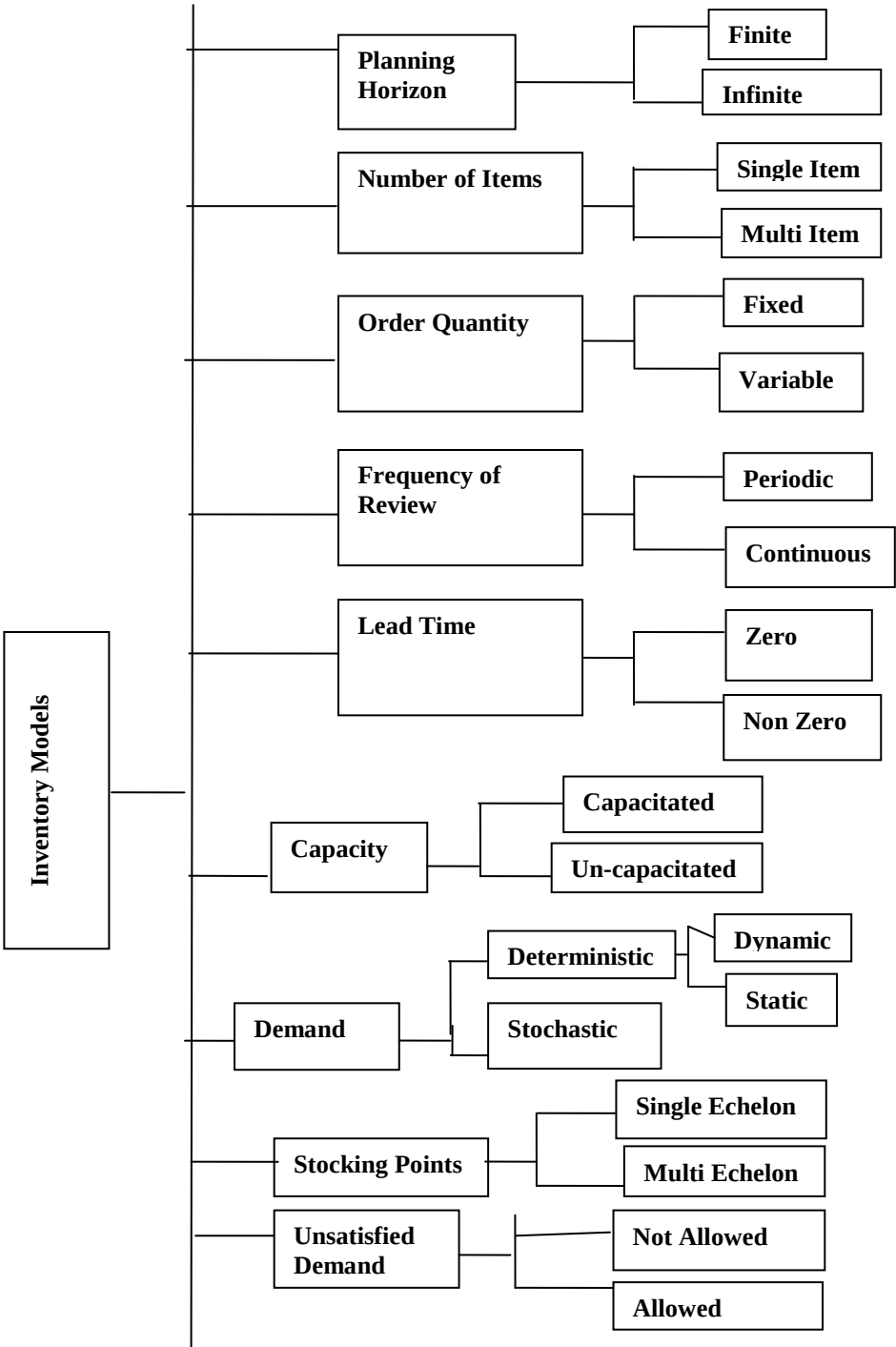
inventory at hand. From practical point of view such models are not interesting unless profit is involved in the process. Once the profit is taken into consideration, the problem becomes maximization of net profit realized, with respect to the inventory at hand.

References	EPQ/ EOQ	EPQ/ EOQ	EPQ/ EOQ	EPQ/ EOQ	EPQ/EO Q	EPQ/ EOQ
Harris (1913)	EOQ	No	No	Yes	No	No
Grubbstrom (1980)	EOQ	No	No	No	Yes	No
Zhang (1996)	EOQ	No	yes	No	No	No
Chung (2002)	EPQ	No	No	No	Yes	No
Teng and Chang (2005)	EPQ	Yes	No	No	No	No
Huang (2007)	EPQ	No	No	No	Yes	No
Liao (2007)	EPQ	No	No	No	Yes	const ant
Teng (2009)	EOQ	No	No	No	Yes	No
Feng et al. (2013)	EPQ	No	No	No	Yes	No
Chen et al. (2014)	EPQ	No	No	No	Yes	const ant
Chung (2014)	EPQ	No	No	No	Yes	const ant
Mahata (2014)	EPQ	No	No	No	Yes	No
Taleizadeh (2014)	EOQ	No	Yes	No	Yes	const ant
Majumder et al. (2015)	EPQ	No	No	No	Yes	const ant
Zia and Taleizadeh (2015)	EOQ	No	Yes	No	Yes	No
Wu et al. (2016)	EOQ	No	No	Yes	Yes	Linea r
Chakraborty et al. (2016)	EPQ	No	No	No	Yes	Yes
Li et al. (2017)	EOQ	Yes	Yes	Yes	Yes	Time Varyi ng
Wu et al. (2018)	EOQ	No	Yes	Yes	Yes	Linea r
Shaikh et al. (2018)	EPQ	Yes	No	No	Yes	Yes
Majumder et al. (2019)	EPQ	Yes	No	No	Yes	Subst itute
Yu-Chung Tsao1 et.al(2019	EPQ	Yes	Yes	Yes	Yes	const ant

Surprisingly the latest information as given by Yu Chung Tsao, et. Al. July 2019² shows that the consideration of profit in Inventory problem is rarely considered. (See the above comparison chart)

1.2. Problem with multi-item inventory models: Even if EOQ or EPQ is a profit maximization problem, the difficulty arises when a realistic multi-item inventory model is to be developed. In fact, a multi-item inventory problem is preferred to a single item inventory problem, as it is observed that mere presence of one inventory creates demand on the other and vice versa. So for meaningful interpretation of a multi-item inventory, such joint interacting demands are to be accommodated. But in almost all the multi-item inventory models considered so far, such interacting demand terms are absent²⁻¹⁷. A detailed survey of continuous inventory models from 2012- 2015 is available in Larissa Janssenet.al. 2019¹⁸. Anyway the absence of interacting demands makes the analysis of a single item and a multi-item inventory parallel. As a matter of fact, in multi-item inventory the costs are obtained from the two single item inventories separately in terms of the respective inventories and final objective remains to minimize their sum. Once such demands involving interacting inventories are sowed in the model, the above process of optimization fails, it leads to a control theoretic optimization problem, which is more difficult to execute. Actually the multi-item inventory problem reduces to an optimal control problem to maximize the net profit. Moreover the maximization no longer depends on the amount of inventory at hand; rather it depends on the amount of continuous supply to the inventory, which is taken as the control in the optimal control problem. Thus for a realistic approach to continuous multi-item inventory, the very notion of inventory problem is challenged.

The pioneering work in this connection is given by D. K. Bhattacharya, 2005¹⁹, where a necessary condition is obtained for the existence of a steady state optimal solution for a two item inventory of deteriorating items. The most important point is that the condition is expressed purely in terms of the parameters of the model. Finally the condition is verified numerically under special choice of parameters. It is shown that under suitable choice of parameters, the condition remains true, so that the steady state may be taken as optimal. Under other choice of parameters the condition is found to be false, so that the corresponding steady state is non-optimal. It is a sub-optimal problem.



Following this approach, the first and the only one model is taken up by A. El-Gohary et.al.2008²⁰. It is also a two item inventory of deteriorating items. But it is a cost minimization problem, and not a profit maximizing one. The objective function includes the sum of the holding costs of inventory, the holding costs of one item due to the presence of other and the production costs. The problem is represented as an optimal control problem with two state variables and two control variables, which are the inventory levels and production rates respectively. The production rates are themselves the rates of continuous supply to inventory levels. The authors consider different types of demand rates, constant, linear, logistic and periodic. But no theoretical optimal solution, not even steady state optimal solution is attempted. Pontryagin's maximum principle²¹⁻²⁵ is applied to obtain the necessary conditions of optimality. These are given by system of ordinary differential equations, which are finally solved numerically. The limitations are (i) profit is not considered (iii) one fundamental assumption is an inequality satisfied by individual holding costs and Interacting holding costs. But interacting holding costs are not meaningful; they make the calculations complicated unnecessarily.

1.3. Discrete Inventory Models: Anyway, all the inventory models considered so far are continuous. But plenty of discrete inventory models are also available in the literature. A discrete time system is a more natural manner to describe inventory dynamics. The classical EOQ model is considered as a continuous-time approach whereas the lot-sizing problem is considered as a discrete-time approach. For lot-sizing models a detailed review up to 2010 is given by Hafiz Ullah et.al. in 2010²⁶. The above figure illustrates the different types of lot sizing models. Discrete Inventory models address mainly the lot-sizing problems, beginning with Wagner and Whitin¹⁹, Scarf²⁰. Model of discrete system control is both a simulation model of inventory dynamics and an optimization model which can give optimal control according to the defined performance criterion. In order to find an optimal inventory control for various variants of the dynamic lot-sizing problems, dynamic programming algorithms given in D. P. Bertsekas²⁷ can be applied. For dynamic programming methods references are given in^{28,29}. The theoretical foundations for the model of the optimal control of discrete systems can be found in the work of Boltianski³⁰. Setchi and Thompson²⁵ have shown how to apply optimal control theory to management science. Such optimal control theory can also be applied for solution of such a problem. The study of discrete inventory model under the application of control theory is found in Konstantin Kostic³¹. The discrete system consists of the law of dynamics, control domain and performance

criterion. It is primarily a simulation model of the inventory dynamics, but the performance criterion enables various order strategies to be compared. The dynamics of state variables depends on the inflow and outflow rates. This paper explicitly defines flow regulators for the four patterns of the inventory: discrete inflow – continuous/discrete outflow and continuous inflow-continuous/discrete outflow. Khmelnitsky and Tzur³² have analyzed a parallelism of continuous-time and discrete-time production planning problems. They consider the lot-sizing problem “as the discrete counterpart of the EOQ, since it is merely a special case of the dynamic demand case”. There are numerous meta-heuristics algorithms for dynamic lot-sizing problems; as found in Zoller and Robrade³³ and Jans and Degraeve³⁴. An earlier review on this topic is available in³⁵. Two more recent papers on lot sizing models are given by Zia and Taleizadeh (2015)³⁶ and Li et. Al. (2017)³⁷.

1.4. Discrete multi-item Inventory: So far as multi-item discrete inventory models are concerned, a parallel discrete version of the continuous multi-item inventory¹⁹ is considered by Yuzhen Wang et. Al. 2019³⁸. It is a control theoretic optimal control model for a discrete two item inventory of deteriorating items. The steady state analysis is made and modified Pontryagin’ maximum principle is used for optimal solution. Finally numerical analysis is considered. But the only limitation is that although the control is taken to be a discrete function, but finally under the assumptions of the model it reduces to a constant.

1.5. Continuous-discrete Inventory control models: Coming back to single item inventory model, whether continuous or discrete, it may be thought of as a control theoretic model; the control may be taken as the current inventory in case of a discrete model and updating of the current inventory, in case of continuous model. A more realistic approach may be made to modify such single item continuous/discrete Inventory control models to continuous-discrete inventory control models. The purpose behind such a modification is found in the pioneering work of D. K. Bhattacharya, 2011³⁹.

The paper formulates a single item inventory model of perishable goods based on the continuous updating of the inventory as a parameter. It is taken as a multi-cohort model. It is noted that, from the realistic point of view, the additional supply of inventories (control parameter) for a continuous inventory model is to be updated at regular intervals of time to have the best result. The maximum net profit is determined out of the projected profit coming from the price of the additional inventory less the amount of cost

incurred for updating the inventory during a given cycle. As the maximization is referred to as a parameter, so it relates to an optimal control model. Finally the decay from one cohort to the other takes place in a continuous manner, whereas updating at the end of each cohort takes place in a discrete manner. Hence the problem reduces ultimately to a continuous-discrete optimal control problem. Such continuous-discrete models are of two types- myopic and non-myopic^{40,41}. In this paper, it is treated as a myopic/separable problem.

2. Conclusion

From the realistic point of view, the basic idea behind a continuous or discrete inventory model may be reviewed in a completely different manner. Optimal control theoretic approach is more befitting to all types of Inventory problems, may be continuous, discrete or continuous-discrete.

3. Open Problems

- (1) Single item and Multi-item continuous Inventory models are to be framed by accommodating possible demand and cost functions; Stability analysis is to be performed on the models; net profit function is to be written properly with suitable choice of control variables. Finally as far as possible, steady state optimal control and corresponding optimal inventories are to be found out analytically by applying Pontryagin's maximum principle for system of differential equations. Numerical analysis is to be done in both cases, whether analytical solution is applicable or not. For appropriateness Cobb-Douglass type of demand functions or sigmoid type of demand functions may be tried.
- (2) Single item and Multi-item discrete Inventory models are to be considered parallel to case (i). Further in this case, dynamic programming approach may be applied. For analysis of single and multi-item inventories, Pontryagin's maximum principle and its modifications may be applied separately. Numerical analysis is to be tried in all cases. In this case also Cobb-Douglass type or sigmoid type of demand functions may be considered.
- (3) Continuous-discrete Inventory models are to be framed with general type of demand functions. Further the models may be analyzed separately in myopic and non-myopic cases.

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