

On $\mathcal{G}$ -Curvature Tensor of a Generalized Randers Space

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Abstract: The purpose of the present paper is to find the relation between $\mathcal{G}$ -curvatures with respect to Cartan connection $\mathcal{C}\Gamma$ of a Finsler space $F^n = (M^n, L)$ and a Finsler space $F^{*n} = (M^n, L^*)$ whose metric L^* is derived from the metric L of F^n by $L^{*2}(x, y) = L^2(x, y) + (b_i y^i)^2$, where $b_i(x, y)$ is an h -vector. The Finsler space F^{*n} is called as generalized Randers space. The expression for $\mathcal{G}$ -curvature of F^{*n} are obtained when F^n is S-3-like and S-4-like.

Keywords: Finsler metric, Randers space, h -vector, $\mathcal{G}$ -curvature tensor, S-3-like Finsler space and S-4-like Finsler space.

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1. Introduction

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space, where M^n is an n -dimensional differentiable manifold and $L(x, y)$ is the fundamental metric function. Matsumoto¹ introduced a Finsler space (M^n, L^*) whose fundamental metric function L^* is defined as:

$$(1.1) \quad L^{*2}(x, y) = L^2(x, y) + (b_i y^i)^2.$$

where the vector b_i is a function of positional coordinates x^i only. If $L(x, y)$ is a metric function of a Riemannian space then $L^*(x, y)$ reduces to the metric function of a Randers space²⁻⁵. Such a Finsler metric was first introduced by Randers⁶ from the stand point of general theory of relativity and was applied to the theory of electron microscope by Ingarden⁷. The geometrical properties of such space have been used for various works. In all these works the vector b_i is assumed to be the function of positional coordinates only.

Izumi⁸ while studying a conformal transformation⁹ of a Finsler space, introduced the h -vector b_i which is $\mathcal{G}$ covariantly constant with respect to Cartan connection $C\Gamma$ and $LC_{hij} b^h = \rho h_{ij}$. The h -vector b_i is not only a function of positional coordinates x^i but also a function of directional arguments y^i . In fact, $L(\partial b_i / \partial y^j) = \rho h_{ij}$. Here $b_i(x, y)$ is an h -vector in (M^n, L) .

Let b_i be an h -vector in the Finsler space (M^n, L) and (M^n, L^*) be another Finsler space whose fundamental metric function $L^*(x, y)$ is defined by

$$(1.2) \quad L^{*2}(x, y) = L^2(x, y) + \beta^2(x, y),$$

where $\beta(x, y) = b_i y^i$. Let us call the Finsler space $F^{n*} = (M^n, L^*)$ as generalized Randers space. To distinguish the geometrical objects of F^{n*} from those of the Finsler space F^n , we shall put a $*$ sign on the corresponding objects of F^n .

For an h -vector b_i we have the following lemmas⁸:

Lemma 1: *If b_i is an h -vector then the function ρ and $l_i^* = b_i - \rho l_i$ are independent of y .*

Lemma 2: *The magnitude of an h -vector b_i is independent of y .*

2. Preliminaries

Let b_i be a vector field in the Finsler space (M^n, L) . If b_i satisfy the conditions¹⁰,

$$(2.1) \quad (i) \quad b_i|_j = 0 \quad (ii) \quad LC_{ij}^h b_h = \rho h_{ij},$$

then the vector field b_i is called an h -vector. Here $|_j$ denotes the covariant differentiation with respect to Cartan connection $C\Gamma$, C_{hij} is the Cartan C-tensor, h_{ij} is the angular metric tensor and ρ is a function given by

$$(2.2) \quad \rho = (n-1)^{-1} LC^i b_i,$$

where $C^i = C_{jk}^i g^{jk}$ and $g^{jk} g_{ki} = \delta_i^j$.

From (2.1) we get

$$(2.3) \quad \frac{\partial b_i}{\partial y^j} = L^{-1} \rho h_{ij}.$$

Differentiating $\beta = b_i y^i$ partially with respect to y^i and using (2.3), we have

$$\frac{\partial \beta}{\partial y^i} = b_i.$$

Differentiation of (1.2) with respect to y^i yields

$$(2.4) \quad l_i^* = \left(\frac{L}{L^*} \right) l_i + \left(\frac{\beta}{L^*} \right) b_i.$$

After differentiating equation (2.4) with respect to y^j and using (2.3), we get

$$(2.5) \quad l_i^* l_j^* + h_{ij}^* = \sigma h_{ij} + l_i l_j + b_i b_j,$$

$$\text{where } \sigma = \left(1 + \frac{\beta \rho}{L} \right).$$

From (2.5), we may deduce

$$(2.6) \quad h_{ij}^* = \sigma h_{ij} + A_0 \beta^2 l_i l_j + A_0 L^2 b_i b_j - A_0 \beta L (l_i b_j + l_j b_i),$$

$$\text{where } A_0 = \left(\frac{1}{L^2 + \beta^2} \right).$$

Transvection of (2.6) with h^{*ip} yields the following

$$h^{*ip} \left[\sigma h_{ij} + A_0 \beta^2 l_i l_j + A_0 L^2 b_i b_j - A_0 \beta L (l_i b_j + l_j b_i) \right] = h^{*ip} h_{ij}^*$$

which implies

$$(2.7) \quad h^{*ip} \left[\sigma h_{ij} + A_0 \beta^2 l_i l_j + A_0 L^2 b_i b_j - A_0 \beta L (l_i b_j + l_j b_i) \right] = \delta_j^p.$$

Again transvecting both sides of the above equation by $h^{jk} b_k$, we get

$$(2.8) \quad h^{*ip} b_i b^m = \frac{b^p b^m}{\sigma + \frac{L^2 b^2 + \beta^2 - \beta L \mu}{A_0}},$$

where $\mu = h^{ij} (l_i b_j + l_j b_i)$.

Transvecting (2.7) by h^{jk} and applying (2.8), we get

$$(2.9) \quad h^{*m p} = \beta h^{m p} - \frac{L^2 \beta b^p b^m}{A_0 + L^2 \beta b^2}.$$

Thus, we have:

Theorem 2.1: *The covariant and contravariant components of the angular metric tensor of the generalized Randers space F^{*n} are given by (2.6) and (2.9) respectively.*

From equation (2.5) and $g_{ij} = h_{ij} + l_i l_j$ one gets

$$(2.10) \quad g_{ij}^* = \sigma g_{ij} + (1 - \sigma) l_i l_j + b_i b_j.$$

From equation (2.10) we obtain the contravariant components of the fundamental metric tensor (g_{ij}^*) of F^{*n} in the following form:

$$(2.11) \quad g^{*ij} = \sigma^{-1} g^{ij} - A_1 (l^i b^j + l^j b^i) + A_2 l^i l^j + A_3 b^i b^j,$$

where
$$A_1 = \frac{(1 - \sigma)\beta}{L} A_3,$$

$$A_2 = (1 - \sigma)(b^2 + \sigma) A_3$$

and

$$A_3 = \frac{1}{\left\| \frac{\sigma \{ (1 - \sigma) \beta^2 - L^2 \}}{L^2} - b^2 \right\|}.$$

Theorem 2.2: *The covariant components g_{ij}^* and the contravariant components g^{*ij} of the metric tensor of $F^{*n}(g_{ij}^*)$ are given by (2.10) and (2.11) respectively.*

From equation (2.10) and $C_{ijk} = (\frac{1}{2}) \partial g_{ij} / \partial y^k$, we get

$$(2.12) \quad C_{ijk}^* = \sigma C_{ijk} + \frac{\rho}{2L} [h_{ij} m_k + h_{jk} m_i + h_{ki} m_j],$$

where $m_i = b_i - \frac{\beta}{L} l_i$.

Let us assume that

$$(2.13) \quad C_{ijk}^* = \sigma C_{ijk}$$

holds. Then, from equations (2.12) and (2.13), we get

$$(2.14) \quad \rho(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) = 0.$$

This implies at least one of the following conditions:

$$(2.15) \quad (a) \quad \rho = 0 \quad (b) \quad h_{ij}m_k + h_{jk}m_i + h_{ki}m_j = 0,$$

If (2.15a) is true, equation (2.3) implies $\frac{\partial b_i}{\partial y^j} = 0$, i.e. the vector b_i is independent of y^j , which contradicts our assumption that b_i are functions of (x^j, y^j) . Hence (2.15 a) cannot be true. Therefore, we have (2.15(b)), which, in view of Walker's Lemma, implies $m_i = 0$ for $h_{ij} \neq 0$. Since $m_i = b_i - \frac{\beta}{L} l_i$, $m_i = 0$ implies $b_i = \frac{\beta}{L} l_i$. Using this result in (2.1(ii)), we get $\beta C_{ij}^r l_r = \rho h_{ij}$. But $C_{ij}^r l_r = C_{ijr} l^r = 0$. Hence $\rho h_{ij} = 0$, which implies $\rho = 0$ for $h_{ij} \neq 0$. Again, we get a contradiction. Thus, we have:

Theorem 2.3: *There exists no generalized Randers space F^{*n} whose torsion tensor is proportional to the torsion tensor of the Finsler space F^n .*

3. Connection Coefficients

Corresponding to γ_{jk}^i of F^n , we define connection coefficients γ_{jk}^{*i} of F^{*n} as follows:

$$(3.1) \quad \gamma_{jk}^{*i} = \frac{1}{2} g^{*ih} \left(\frac{\partial g_{jh}^*}{\partial y^k} + \frac{\partial g_{kh}^*}{\partial y^j} - \frac{\partial g_{jk}^*}{\partial y^h} \right).$$

By virtue of (2.11), the connection coefficients γ_{jk}^{*i} in F^{*n} can be expressed as

$$(3.2) \quad \gamma_{jk}^{*i} = \sigma g^{*ih} \gamma_{jhh} + \frac{\rho}{2L} g^{*ih} [Q_{jhh} + Q_{khj} - Q_{jkh}],$$

where $Q_{jhh} = h_{jh} m_k + h_{hk} m_j + h_{kj} m_h$ and $\gamma_{jhh} = C_{jhh} + C_{khj} - C_{jkh}$.

Theorem 3.1: The connection coefficients γ_{jk}^{*i} of F^{*n} are given by (3.2).

Multiplying (3.2) by $y^j y^k$ and using $\gamma_{jk}^j y^j y^k = 2G^i$, we get

$$(3.3) \quad G^{*i} = \sigma g^{*ih} G_h + \frac{\rho}{2L} g^{*ih} y^j y^k [Q_{jhh} + Q_{khj} - Q_{jkh}],$$

where $\gamma_{jhh} y^j y^k = 2G_h$.

Differentiating (3.3) partially with respect to y^m and y^p , we get G_{mp}^{*i} (Berwald connection coefficients).

4. $\mathcal{G}$ – Curvature Tensor

Definition 4.1: The $\mathcal{G}$ – curvature tensor S_{hijk} of $F^n = (M^n, L)$ with respect to Cartan connection $C\Gamma$ is defined by¹

$$(4.1) \quad S_{hijk} = C_{hkm} C_{ij}^m - C_{hjm} C_{ik}^m.$$

From (2.1), (2.11), (2.12) and (2.14), we get

$$(4.2) \quad \begin{aligned} C_{ij}^{*h} = & C_{ij}^h + C_0 (h_{ij} m^h + h_j^h m_i + h_i^h m_j) - C_1 [h_{ij} l^h C_2 + m_i m_j l^h] \\ & + C_3 [C_2 h_{ij} b^h + m_i m_j b^h], \end{aligned}$$

where

$$C_0 = \frac{\rho}{2L\sigma}, \quad C_1 = \frac{(1-\sigma)\beta\rho}{L^2} A_3, \quad C_2 = \frac{1}{2} \left(b^2 - \frac{\beta^2}{L^2} \right) + \sigma$$

and A_3 being defined in (2.11).

From equations (2.12) and (4.2), we have

$$(4.3) \quad \begin{aligned} C_{hkm}^* C_{ij}^{*m} = & C_{hkm} C_{ij}^m + \mu_1 h_{ij} h_{hk} + R_1 h_{hk} m_i m_j + R_2 h_{ij} m_h m_k \\ & + R_0 \text{cycl}(C_{ijk} m_h) + R_0^2 A_3 \text{cycl}(h_{jh} m_i m_k), \end{aligned}$$

where

$$\mu_1 = \left(b^2 - \frac{\beta^2}{L^2}\right) \left(\frac{\rho^2 \sigma}{L^2} A_3 + \frac{\rho^2}{4L^2 \sigma} + \frac{\rho^2}{4L^2} A_3\right) + \frac{\rho^2 \sigma^2}{L^2} A_3, \quad R_0 = \frac{\rho}{2L},$$

$$R_1 = \frac{\rho^2}{2L^2} A_3 (\sigma + 1) + \frac{\rho^2}{2L^2 \sigma} \quad \text{and} \quad R_2 = \frac{\rho^2}{2L^2 \sigma} + \frac{\rho^2}{2L^2} A_3 \left\{ \sigma + \frac{1}{2} \left(b^2 - \frac{\beta^2}{LH^2}\right) \right\}.$$

From (4.3) and (4.1), we obtain:

Theorem 4.1: *The $\mathcal{G}$ -curvature tensor (S_{hijk}^*) of the generalized Randers space F^{*n} has the form*

$$(4.4) \quad S_{hijk}^* = \sigma S_{hijk} + h_{ij} e_{hk} + h_{hk} (e_{ij} + 1) - h_{ik} e_{hj} - h_{hj} e_{ik},$$

$$\text{where } e_{ij} = \frac{1}{2} \mu_1 h_{ij} + \mu_2 m_i m_j \quad \text{and} \quad \mu_2 = \frac{\rho^2}{L^2} A_3 \left\{ \sigma + \frac{1}{4} + \left(b^2 - \frac{\beta^2}{L^2}\right) \right\}.$$

Now we shall consider some special cases of $\mathcal{G}$ -curvature tensor S_{hijk} :

Case 1: S_3 -like Finsler space

Definition 4.2: A Finsler space $F^n (n \geq 4)$ with fundamental function $L(x, y)$ is called S_3 -like¹¹⁻¹³ if the $\mathcal{G}$ -curvature tensor S_{hijk} is of the form

$$(4.5) \quad S_{hijk} = \frac{S}{(n-1)(n-2)} (h_{hj} h_{ik} - h_{hk} h_{ij}),$$

where the scalar S is a function of positional co-ordinates only.

From equation (2.6), (4.4) and (4.5) we have

Theorem 4.3: *The $\mathcal{G}$ -curvature tensor S_{hijk}^* of a S_3 -like Finsler space $F^{*n} (n \geq 4)$ has the form*

$$(4.6) \quad S_{hijk}^* = \frac{S}{(n-1)(n-2)} \{ \sigma^2 (h_{hj} h_{ik} - h_{hk} h_{ij}) + P_{hijk} \},$$

$$\text{where} \quad P_{hijk} = \sigma (h_{ik} - h_{ij}) \{ \beta^2 A_0 l_h l_k + AL b_h b_k - \beta L (l_h b_k + l_k b_h) \}.$$

Theorem 4.4: *The generalized Randers space $F^{*n} (n \geq 4)$ of an S_3 -like Finsler space F_n is S_3 -like if $P_{hijk} = 0$.*

Case 2: S4-like Finsler space

Definition 4.3: A Finsler space $F_n (n \geq 5)$ with fundamental function $L(x, y)$ is called S4-like^{11, 12} if the $\mathcal{G}$ -curvature tensor S_{hijk} is of the form

$$(4.7) \quad L^2 S_{hijk} = h_{hj} M_{ik} + h_{ik} M_{hj} - h_{hk} M_{ij} - h_{ij} M_{hk},$$

where M_{ij} is a symmetric and indicatory tensor.

From equation (2.6), (4.4) and (4.7), we have

Theorem 4.5: The $\mathcal{G}$ -curvature tensor of a generalized Randers space of $F_n^* (n \geq 5)$ Can be expressed as :

$$(4.8) \quad L^2 S_{hijk}^* = [\sigma(h_{hj} M_{ik} + h_{ik} M_{hj} - h_{hk} M_{ij} - h_{ij} M_{hk}) + U_{hijk}],$$

$$\text{where } U_{hijk} = T_{(o)hijk} + T_{(1)hijk} + T_{(2)hijk} + T_{(3)hijk},$$

$$T_{(o)hijk} = A_o [\beta^2 l_h l_j + L^2 b_h b_j - \beta L (l_h b_j + l_j b_h)] M_{ik},$$

$$T_{(1)hijk} = A_o [\beta^2 l_i l_k + L^2 b_i b_k - \beta L (l_i b_k + l_k b_i)] M_{jh},$$

$$T_{(2)hijk} = A_o [\beta^2 l_h l_k + L^2 b_h b_k - \beta L (l_h b_k + l_k b_h)] M_{ij},$$

$$T_{(3)hijk} = A_o [\beta^2 l_i l_j + L^2 b_i b_j - \beta L (l_i b_j + l_j b_i)] M_{hk}.$$

Theorem 4.6: The generalized Randers space of $F_n^* (n \geq 5)$ of S4-like Finsler space is S4-like if $U_{hijk} = 0$.

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