

A Control Mathematical Model for the Survival of a Biological Species Affected by Two Toxicants, One Emitted from External Sources as well as Formed by its Precursors and the Other Produced by Species itself

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Abstract: In this paper, a control mathematical model is proposed to study the simultaneous effect of two toxicants with different toxicities emitted into the environment on the biological species. One of the toxicants is emitted into the environment by external sources and their concentration is augmented by precursor and the other toxicant is produced by the population itself through its various actions. The model is formulated by using system of non-linear ordinary differential equations. In the analysis, all the feasible equilibrium points of the system have been obtained. The conditions for local and non-linear stability of the non-trivial equilibrium points have been carried out using a suitable Lyapunov function. Also, a region of attraction has been found for non-linear asymptotic stability of the equilibrium point. . It has been shown that the density of the biological species decreases with the increase in the total emission rate of pollutant in the environment. The analysis of the non linear stability shows that the system settles at much lower density of the biological species when the concentration of the pollutants in the environment and in the uptake phase of the species is high. But the control measures (environmental taxes) imposed on the pollutant emitting industries and people, control concentration of the pollutants in the environment and due to this, the equilibrium point shifts in such a way that the density of the biological species is more near to the density when eco-system is pollution free.

Keywords: Biological species, Toxicity, Uptake phase, Precursor, Control measure.

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1. Introduction

The pollution of our environment is one of the biggest hazards that humanity faces today. Due to rapid growth of human population and consequently over exploitation of natural resources; development in the sector of agriculture, industry and urbanization; and tremendous increase in

number of vehicles have dumped various hazardous substance (toxicants) in our environment. The example of this kind of problem is oxides of sulphur may be emitted from industrial complexes and oxides of carbon may be formed by burning wastes in households of human populations. In recent years some investigations have been conducted to study the effects of toxicants (pollutants) on biological species¹⁻³. Hallam and co-workers⁴ proposed models to study the effect of a toxicant on the growth rate of biological population, but the effect of uptake concentration relative to the environmental concentration and carrying capacity was not considered. Freedman and Shukla⁵ in their model considered that the intrinsic growth rate of biological species decreases as the uptake concentration of the toxicant increases, while the carrying capacity of the species decreases as the concentration of the toxicant in the environment increases. Shukla and Dubey⁶⁻⁷ proposed the model to study the simultaneous effects of two toxicants with different toxic concentrations on a biological population, in their model they assume that both toxicants are emitted from external sources. In view of this, in this paper we have modeled and analyzed the simultaneous effects of two toxicants, each having different toxicity, one being emitted through external sources and its augmented by precursor, and other is produced by the population such as human being through its various actions. The effect of the both the pollutants on the biological species have also been considered as different. The properties of both of these are different therefore the concentrations of these in the environment and in the uptake phase of the biological species have been considered as different.

2. Mathematical Model

We consider a control model to study the simultaneous effect of two toxicants with different toxicities emitted into the environment on the biological population. It is assumed that the rate of emission of pollutants/toxicant into the environment is a constant and the amount of precursor produced by the species is proportional to the producing species. It is also assumed that the concentration of this toxicants in the environment decreases due to its assimilation, absorption, deposition, uptake etc. by biological species. It is further assumed that the some of the pollutants in the uptake phase are decomposed and the same re-enter in the environment. It is also assumed that the growth rate of biological species is getting affected by the concentration of both the toxicants with the same concentration but having different toxicities, growth rate is decreased more in case of more toxic pollutants. In view of the above assumption, we purpose a following

model governing the dynamics of the biological species, concentrations of the toxicants/pollutants and the control mechanism.

$$\frac{dN}{dt} = r(U_1, U_2)N - \frac{r_0 N^2}{K(C_1, C_2)} - k_1 \alpha_1 C_1 N - k_2 \alpha_2 C_2 N.$$

$$\frac{dP}{dt} = \lambda N - \lambda_0 P - \theta P.$$

$$\frac{dC_1}{dt} = Q + \pi \theta P - \delta_1 C_1 - \alpha_1 C_1 N + \pi_1 \nu_1 N U_1 - \mu_1 F.$$

$$\frac{dC_2}{dt} = \phi N - \delta_2 C_2 - \alpha_2 C_2 N + \pi_2 \nu_2 N U_2 - \mu_2 F.$$

$$\frac{dU_1}{dt} = (1 - k_1) \alpha_1 C_1 N - \phi_1 U_1 - \nu_1 N U_1.$$

$$\frac{dU_2}{dt} = (1 - k_2) \alpha_2 C_2 N - \phi_2 U_2 - \nu_2 N U_2.$$

$$\frac{dF}{dt} = \theta_1 (C - C_{p_1}) + \theta_2 (C - C_{p_2}) - \theta_0 F.$$

$$N(0) = N_0 \geq 0, P(0) = P_0 \geq 0, C_i(0) = C_{i0} \geq 0,$$

$U_i(0) = U_{i0} \geq 0, F(0) = F_0 \geq 0, 0 \leq k_i \leq 1, 0 \leq \pi_i \leq 1, 0 \leq \pi \leq 1$ for $i = 1, 2$
 $\alpha_i, \delta_i, \phi_i, \nu_i, \theta_i, \theta, \theta_0, Q, C_{p_i}, \lambda, \lambda_0, \mu_i$ ($i = 1, 2$) are all positive constants.

In the model (2.1) the function $r(U_1, U_2)$ represent the growth rate coefficient of biological species which decrease with increase of U_1 and U_2 and hence, r_0 is the growth rate coefficient in the pollution free environment.

$$(2.2a) \quad r(U_1, U_2) = r(0, 0) = r_0 > 0, \quad \frac{\partial r}{\partial U_1} < 0, \quad \frac{\partial r}{\partial U_2} < 0 \quad \text{for } U_1 \geq 0, U_2 \geq 0.$$

We assume that the toxicant with concentration C_2 is more toxic than the toxicant with concentration C_1 and therefore toxicant C_2 affect more to the growth rate of the biological species. Now we consider the two toxicants at some concentration $U_c > 0$ during the time t , the growth rate of the biological species in the more toxic case would be smaller than that for the less toxic case i.e.

$$(2.2b) \quad r(0, U_c) < r(U_c, 0) \quad \text{for } U_c > 0,$$

then we have

$$(2.2c) \quad r(U_1, U_c) < r(0, U_c) < r(U_c, 0) < r(0, 0) \text{ for } U_c > 0, U_1 > 0.$$

We also assume that the function $K(C_1, C_2)$ denotes the carrying capacity (i.e. the maximum population density which the environment can support). We assume that $K(C_1, C_2)$ decreases as C_1 and C_2 increase, hence we have

$K(0, 0) = K_0 > 0$, carrying capacity in the pollution free environment and

$$(2.2d) \quad \frac{\partial K}{\partial C_1} < 0, \quad \frac{\partial K}{\partial C_2} < 0 \text{ for } C_1 > 0 \text{ and } C_2 > 0.$$

Using the earlier assumption, that the toxicant with concentration C_2 is more toxic than the toxicant with concentration C_1 and therefore toxicant C_2 affect more to the carrying capacity of the biological species, the two toxicants at some concentration level C_T , we have

$$K(0, C_T) < K(C_T, 0) \text{ for } C_T > 0.$$

Thus, we get

$$K(C_1, C_T) < K(0, C_T) < K(C_T, 0) < K(0, 0) \text{ for } C_T > 0, C_1 > 0.$$

3. Equilibrium Analysis

The given model (2.1) has two non-negative real equilibria (Feasible equilibrium points) in $N - P - C_1 - C_2 - U_1 - U_2 - F$ space denoted by $E_0(0, 0, C_1, C_2, 0, 0, F)$ and $E^*(N^*, P^*, C_1^*, C_2^*, U_1^*, U_2^*, F^*)$.

For $E_0(0, 0, C_1, C_2, U_1, U_2, F)$,

where

$$C_1 = \frac{Q(\delta_2\theta_0 + \mu_2\theta_2) + \mu_1\delta_2(\theta_1C_{p_1} + \theta_2C_{p_2})}{(\delta_1\delta_2\theta_0 + \delta_1\mu_2\theta_2 + \delta_2\mu_1\theta_1)},$$

$$C_2 = \frac{-Q\mu_2\theta_1 + \mu_2\delta_1(\theta_1C_{p_1} + \theta_2C_{p_2})}{(\delta_1\delta_2\theta_0 + \delta_1\mu_2\theta_2 + \delta_2\mu_1\theta_1)},$$

provided $\mu_2\delta_1(\theta_1C_{p_1} + \theta_2C_{p_2}) > Q\theta_1\mu_2$.

$$F = \frac{Q\delta_2\theta_1 - \delta_1\delta_2(\theta_1C_{p_1} + \theta_2C_{p_2})}{(\delta_1\delta_2\theta_0 + \delta_1\mu_2\theta_2 + \delta_2\mu_1\theta_1)},$$

provided $\delta_1\delta_2(\theta_1C_{p_1} + \theta_2C_{p_2}) < Q\theta_1\delta_2$.

The interior equilibrium $E^*(N^*, P^*, C_1^*, C_2^*, U_1^*, U_2^*, F^*)$ are the positive solution of the following system of equations.

$$(3.2a) \quad N = \frac{[r(U_1, U_2) - k_1\alpha_1C_1 - k_2\alpha_2C_2]K(C_1, C_2)}{r_0},$$

provided $[r(U_1, U_2) - k_1\alpha_1C_1 - k_2\alpha_2C_2] > 0$.

$$(3.2b) \quad P = \frac{\lambda N}{(\lambda_0 + \theta)},$$

$$(3.2c) \quad C_1 = \frac{(\phi_1 + \nu_1 N)}{f(N)} \begin{vmatrix} Q + \pi\theta\left(\frac{\lambda N}{\lambda_0 + \theta}\right) & 0 & \mu_1 \\ \phi N & f_2(N) & \mu_2 \\ \theta_1C_{p_1} + \theta_2C_{p_2} & \theta_2(\phi_2 + \nu_2 N) & -\theta_0 \end{vmatrix} = h_1(N), \text{ (assuming)}$$

$$(3.2d) \quad C_2 = \frac{(\phi_2 + \nu_2 N)}{f(N)} \begin{vmatrix} f_1(N) & Q + \pi\theta\left(\frac{\lambda N}{\lambda_0 + \theta}\right) & \mu_1 \\ 0 & \phi N & \mu_2 \\ \theta_1(\phi_1 + \nu_1 N) & \theta_1C_{p_1} + \theta_2C_{p_2} & -\theta_0 \end{vmatrix} = h_2(N), \text{ (assuming)}$$

$$(3.2e) \quad U_1 = \frac{(1 - k_1)\alpha_1 N}{f(N)} \begin{vmatrix} Q + \pi\theta\left(\frac{\lambda N}{\lambda_0 + \theta}\right) & 0 & \mu_1 \\ \phi N & f_2(N) & \mu_2 \\ \theta_1C_{p_1} + \theta_2C_{p_2} & \theta_2(\phi_2 + \nu_2 N) & -\theta_0 \end{vmatrix} = g_1(N), \text{ (assuming)}$$

$$(3.2f) \quad U_2 = \frac{(1 - k_2)\alpha_2 N}{f(N)} \begin{vmatrix} f_1(N) & Q + \pi\theta\left(\frac{\lambda N}{\lambda_0 + \theta}\right) & \mu_1 \\ 0 & \phi N & \mu_2 \\ \theta_1(\phi_1 + \nu_1 N) & \theta_1C_{p_1} + \theta_2C_{p_2} & -\theta_0 \end{vmatrix} = g_2(N), \text{ (assuming)}$$

$$(3.2g) \quad F = \frac{1}{f(N)} \begin{vmatrix} f_1(N) & 0 & Q + \pi\theta \left(\frac{\lambda N}{\lambda_0 + \theta} \right) \\ 0 & f_2(N) & \phi N \\ \theta_1(\phi_1 + \nu_1 N) & \theta_2(\phi_2 + \nu_2 N) & \theta_1 C_{p_1} + \theta_2 C_{p_2} \end{vmatrix},$$

where

$$f(N) = \begin{vmatrix} f_1(N) & 0 & \mu_1 \\ 0 & f_2(N) & \mu_2 \\ \theta_1(\phi_1 + \nu_1 N) & \theta_2(\phi_2 + \nu_2 N) & -\theta_0 \end{vmatrix},$$

or

$$f(N) = - \begin{vmatrix} f_1(N) & \mu_1 & 0 \\ 0 & \mu_2 & f_2(N) \\ \theta_1(\phi_1 + \nu_1 N) & -\theta_0 & \theta_2(\phi_2 + \nu_2 N) \end{vmatrix},$$

where

$$f_1(N) = \phi_1 \delta_1 + (\phi_1 \alpha_1 + \delta_1 \nu_1) N + \{1 - \pi_1(1 - k_1)\} \alpha_1 \nu_1 N^2 > 0,$$

$$f_2(N) = \phi_2 \delta_2 + (\phi_2 \alpha_2 + \delta_2 \nu_2) N + \{1 - \pi_2(1 - k_2)\} \alpha_2 \nu_2 N^2 > 0.$$

From the expansion of each determinant, it can be seen that it is a negative function of N and to make each interior equilibrium point to be positive, $f(N)$ should be negative function of N .

Rewrite equation (3.2a) as

$$(3.2h) \quad r_0 N = [r(U_1, U_2) - k_1 \alpha_1 C_1 - k_2 \alpha_2 C_2] K(C_1, C_2)$$

Substituting the values of C_1 , C_2 , U_1 and U_2 from equations (3.2c)-(3.2f) in the above equation (3.2h), we get

$$(3.2i) \quad r_0 N = [r(g_1(N), g_2(N)) - k_1 \alpha_1 h_1(N) - k_2 \alpha_2 h_2(N)] K(h_1(N), h_2(N)).$$

To show the existence of the internal equilibrium point $E^*(N^*, P^*, C_1^*, C_2^*, U_1^*, U_2^*, F^*)$, its sufficient to show that equation (3.2i) has a unique positive solution in N .

Let us consider a function $F(N)$ such that

$$(3.2j) \quad F(N) = r_0 N - [r(g_1(N), g_2(N)) - k_1 \alpha_1 h_1(N) - k_2 \alpha_2 h_2(N)] K(h_1(N), h_2(N))$$

Putting $N = 0$ and K_0 in equation (3.2j)

$$F(0) = -[r(0, 0) - k_1 \alpha_1 h_1(0) - k_2 \alpha_2 h_2(0)] K(h_1(0), h_2(0))$$

$$(3.2k) \quad F(0) = - \left[r_0 - \frac{\begin{vmatrix} -k_2 \alpha_2 \phi_2 \delta_1 \phi_1 & Q & \mu_1 \\ k_1 \alpha_1 \phi_1 \delta_2 \phi_2 & 0 & \mu_2 \\ k_1 \alpha_1 \phi_1 \theta_1 \phi_2 - k_2 \alpha_2 \phi_2 \theta_2 \phi_1 & \theta_1 C_{p_1} + \theta_2 C_{p_2} & -\theta_0 \end{vmatrix}}{\begin{vmatrix} \delta_1 \phi_1 & \mu_1 & 0 \\ 0 & \mu_2 & \delta_2 \phi_2 \\ \theta_1 \phi_1 & -\theta_0 & \theta_2 \phi_2 \end{vmatrix}} \right] K(h_1(0), h_2(0))$$

From the equation (3.2a), we see that for N to be positive, we have

$$[r(U_1, U_2) - k_1 \alpha_1 C_1 - k_2 \alpha_2 C_2] > 0,$$

$$[r(g_1(N), g_2(N)) - k_1 \alpha_1 h_1(N) - k_2 \alpha_2 h_2(N)] > 0.$$

For $N = 0$, we get

$$[r(g_1(0), g_2(0)) - k_1 \alpha_1 h_1(0) - k_2 \alpha_2 h_2(0)] > 0$$

$$r(0, 0) - \frac{k_1 \alpha_1 \phi_1}{f(0)} \begin{vmatrix} Q & 0 & \mu_1 \\ 0 & \phi_2 \delta_2 & \mu_2 \\ \theta_1 C_{p_1} + \theta_2 C_{p_2} & \theta_2 \phi_2 & -\theta_0 \end{vmatrix} - \frac{k_2 \alpha_2 \phi_2}{f(0)} \begin{vmatrix} \phi_1 \delta_1 & Q & \mu_1 \\ 0 & 0 & \mu_2 \\ \theta_1 \phi_1 & \theta_1 C_{p_1} + \theta_2 C_{p_2} & -\theta_0 \end{vmatrix} > 0$$

$$r_0 + \frac{1}{f(0)} \begin{vmatrix} 0 & Q & \mu_1 \\ \phi_2 \delta_2 k_1 \alpha_1 \phi_1 & 0 & \mu_2 \\ \theta_2 \phi_2 k_1 \alpha_1 \phi_1 & \theta_1 C_{p_1} + \theta_2 C_{p_2} & -\theta_0 \end{vmatrix} - \frac{k_2 \alpha_2 \phi_2}{f(0)} \begin{vmatrix} \phi_1 \delta_1 & Q & \mu_1 \\ 0 & 0 & \mu_2 \\ \theta_1 \phi_1 & \theta_1 C_{p_1} + \theta_2 C_{p_2} & -\theta_0 \end{vmatrix} > 0$$

$$(3.2l) \quad r_0 - \frac{\begin{vmatrix} -k_2 \alpha_2 \phi_2 \delta_1 \phi_1 & Q & \mu_1 \\ \phi_2 \delta_2 k_1 \alpha_1 \phi_1 & 0 & \mu_2 \\ \theta_2 \phi_2 k_1 \alpha_1 \phi_1 - k_2 \alpha_2 \phi_2 \theta_1 \phi_1 & \theta_1 C_{p_1} + \theta_2 C_{p_2} & -\theta_0 \end{vmatrix}}{\begin{vmatrix} \delta_1 \phi_1 & \mu_1 & 0 \\ 0 & \mu_2 & \delta_2 \phi_2 \\ \theta_1 \phi_1 & -\theta_0 & \theta_2 \phi_2 \end{vmatrix}} > 0.$$

$K(C_1, C_2)$ being carrying capacity is always positive and from equation (3.2l), equation (3.2k) becomes

$$(3.2m) \quad F(0) < 0.$$

Again,

$$\begin{aligned} F(K_0) &= r_0 K_0 - [r(g_1(K_0), g_2(K_0)) - k_1 \alpha_1 h_1(K_0) - k_2 \alpha_2 h_2(K_0)] K(h_1(K_0), h_2(K_0)) \\ F(K_0) &= r_0 K_0 - [r(g_1(K_0), g_2(K_0))] K(h_1(K_0), h_2(K_0)) \\ &\quad + [k_1 \alpha_1 h_1(K_0) + k_2 \alpha_2 h_2(K_0)] K(h_1(K_0), h_2(K_0)) > 0. \end{aligned}$$

Since $r_0 K_0$ is the maximum value of $r(U_1, U_2) K(C_1, C_2)$, therefore the first two terms make a positive term and the third is already positive. Hence

$$(3.2m) \quad F(K_0) > 0.$$

From equations (3.2m) and (3.2n), it is clear that there exists a root N^* in the interval $0 < N^* < K_0$ such that $F(N^*) = 0$.

For Uniqueness of N^* , the necessary and sufficient condition is $F'(N) > 0$ in the interval $0 < N^* < K_0$.

From (3.2j), we get

$$\begin{aligned} (3.2o) \quad F'(N) &= r_0 - K(h_1(N), h_2(N)) \left[\frac{\partial r}{\partial g_1} \frac{dg_1}{dN} + \frac{\partial r}{\partial g_2} \frac{dg_2}{dN} - k_1 \alpha_1 \frac{dh_1}{dN} - k_2 \alpha_2 \frac{dh_2}{dN} \right] \\ &\quad - [r(g_1(N), g_2(N)) - k_1 \alpha_1 h_1(N) - k_2 \alpha_2 h_2(N)] \left[\frac{\partial K}{\partial h_1} \frac{dh_1}{dN} + \frac{\partial K}{\partial h_2} \frac{dh_2}{dN} \right] > 0. \end{aligned}$$

Since $\frac{\partial r}{\partial g_1}, \frac{\partial r}{\partial g_2}, \frac{\partial K}{\partial h_1}$ and $\frac{\partial K}{\partial h_2}$ are negative, the above condition is automatically satisfied if $\frac{dg_1}{dN}, \frac{dg_2}{dN}, \frac{dh_1}{dN}$ and $\frac{dh_2}{dN}$ are positive.

Thus, the condition for unique and positive N^* is $F'(N) > 0$. Once N^* has been determined $P^*, C_1^*, C_2^*, U_1^*, U_2^*$ and F^* can be found from equations (3.2b)-(3.2g).

In view of the above, we have the following conditions:

Let $F(N)$ be given by equation (3.2j).

$$\text{If (i)} \quad r(U_1, U_2) - k_1 \alpha_1 C_1 - k_2 \alpha_2 C_2 > 0,$$

$$\text{(ii)} \quad F'(N) > 0 \text{ for } 0 < N < K_0,$$

then there exists a unique interior equilibrium $E^*(N^*, P^*, C_1^*, C_2^*, U_1^*, U_2^*, F^*)$ for the model (2.1).

Now taking as a particular case suppose that when only C_1 affects the population and $C_2 = U_2 = 0$ also ($\phi = 0$), then to prove the existence of the equilibrium, from equation (3.2j), we have

$$(3.2p) \quad F_{p_1}(N) = r_0 N - [r(g_1(N), 0) - k_1 \alpha_1 h_1(N)] K(h_1(N), 0),$$

Putting $N = 0$ and K_0 in equation (3.2p),

$$F_{p_1}(0) < 0 \quad \text{and} \quad F_{p_1}(K_0) > 0.$$

From the above inequalities, it is clear that there exists a root N_1^* in the interval $0 < N_1^* < K_0$ such that $F_{p_1}(N_1^*) = 0$.

For N_1^* is unique iff $F_{p_1}'(N) > 0$. i.e.

$$\frac{dF_{p_1}}{dN} = r_0 - \{r(g_1(N), 0) - k_1 \alpha_1 h_1(N)\} \frac{\partial K}{\partial h_1} \frac{dh_1}{dN} - K(h_1(N), 0) \left\{ \frac{\partial r}{\partial g_1} \frac{dg_1}{dN} - k_1 \alpha_1 \frac{dh_1}{dN} \right\} > 0.$$

Similarly, suppose that only C_2 affects the population and $Q = 0$, $C_1 = U_1 = 0$, then to prove the existence of equilibrium point.

From equation (3.2j), we get

$$(3.2p_1) \quad F_{p_2}(N) = r_0 N - [r(0, g_2(N)) - k_2 \alpha_2 h_2(N)] K(0, h_2(N)).$$

It can be easily proved that

$$F_{p_2}(0) < 0 \quad \text{and} \quad F_{p_2}(K_0) > 0.$$

In view of the above inequalities, it is clear that there exists a root N_2^* in the interval $0 < N_2^* < K_0$ such that $F_{p_2}(N_2^*) = 0$.

N_2^* is unique iff $F_{p_2}'(N) > 0$. i.e.

$$\begin{aligned} \frac{dF_{p_2}}{dN} = r_0 - \{r(0, g_2(N)) - k_2 \alpha_2 h_2(N)\} \frac{\partial K}{\partial h_2} \frac{dh_2}{dN} \\ - K(0, h_2(N)) \left\{ \frac{\partial r}{\partial g_2} \frac{dg_2}{dN} - k_2 \alpha_2 \frac{dh_2}{dN} \right\} > 0. \end{aligned}$$

Again, now suppose that no pollutant is emitted in the environment i.e. $C_1 = C_2 = U_1 = U_2 = 0$, then to prove the existence of equilibrium.

Equation (3.2j) becomes

$$(3.2q) \quad F_{p_0}(N) = r_0 N - r_0 K_0 ,$$

from equation (3.2i) and (3.2q), we get

$$F_{p_0}(N) = 0 \text{ has a unique root } N_0^* = K_0 .$$

Since we have assumed that at the same concentration level, C_2 is more toxic than C_1 , from equations (3.2j), (3.2p), (3.2p₁) and (3.2q), we have

$$F(N) > F_{p_1}(N) > F_{p_2}(N) > F_{p_0}(N) ,$$

and for $U_1 = U_2 = U_c > 0$ and $C_1 = C_2 = C_T > 0$, thus we have

$$N^* > N_2^* > N_1^* > N_0^* = K_0 .$$

From the above analysis, it is concluded that when each of the two toxicants affects the biological species in the ratio of their effectiveness, the equilibrium level of the population is less than in case of more toxic case, but when both the toxicants affect the population simultaneously the equilibrium density decreases. Further, the amount of decrease being dependent on their toxicities, emission rate and washout rates.

Now consider the last equation of the model (2.1), we have

$$0 = \theta_1(C - C_{p_1}) + \theta_2(C - C_{p_2}) - \theta_0 F$$

$$F = \frac{\theta_1}{\theta_0}(C - C_{p_1}) + \frac{\theta_2}{\theta_0}(C - C_{p_2})$$

$$F = \frac{\theta_1}{\theta_0}(h_1(N) - C_{p_1}) + \frac{\theta_2}{\theta_0}(h_2(N) - C_{p_2})$$

Differentiating above equations with respect to F, we get

$$(3.2r) \quad \begin{cases} 1 = \frac{\theta_1}{\theta_0} \frac{dh_1}{dN} \frac{dN}{dF} + \frac{\theta_2}{\theta_0} \frac{dh_2}{dN} \frac{dN}{dF}, \\ 1 = \left(\frac{\theta_1}{\theta_0} \frac{dh_1}{dN} + \frac{\theta_2}{\theta_0} \frac{dh_2}{dN} \right) \frac{dN}{dF}. \end{cases}$$

From equation (3.2o), we have $\frac{dh_1}{dN} > 0$ and $\frac{dh_2}{dN} > 0$.

From equation (3.2r), we have $\frac{dN}{dF} > 0$.

From the above equation, it is clear that with the increase of the control measure, the population of the biological species increases. Thus the control measure has a positive impact in the eco-system and the biological species may be saved from going to extinction.

4. Stability Analysis

Here we shall discuss the local and global stability of equilibrium points. The local stability can be studied from the variational matrices corresponding to each equilibrium point and for the global stability, suitable lyapunov functions are found in the interior of some region Ω .

4a. Local Stability via Eigen Value Method

To study the local stability behavior of equilibria, we compute the variational matrices corresponding to each equilibrium points. Let M_0 and M^* be the variational matrices corresponding to equilibrium points E_0 and E^* respectively.

The variational matrix corresponding to the point $E_0(0,0,C_1,C_2,0,0,F)$, we get

$$M_0 = \begin{bmatrix} r_0 - k_1\alpha_1C_1 - k_2\alpha_2C_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & -(\lambda_0 + \theta) & 0 & 0 & 0 & 0 & 0 \\ -\alpha_1C_1 & \pi\theta & -\delta_1 & 0 & 0 & 0 & -\mu_1 \\ \phi - \alpha_2C_2 & 0 & 0 & -\delta_2 & 0 & 0 & -\mu_2 \\ (1-k_1)\alpha_1C_1 & 0 & 0 & 0 & -\phi_1 & 0 & 0 \\ (1-k_2)\alpha_2C_2 & 0 & 0 & 0 & 0 & -\phi_2 & 0 \\ 0 & 0 & \theta_1 & \theta_2 & 0 & 0 & -\theta_0 \end{bmatrix},$$

and the variational matrix corresponding to the point $E^*(N^*,P^*,C_1^*,C_2^*,U_1^*,U_2^*,F^*)$,

$$M^* = \begin{bmatrix} \frac{-r_0N^*}{K(C_1^*,C_2^*)} & 0 & a_{12} & a_{13} & \frac{\partial r(U_1^*,U_2^*)}{\partial U_1}N^* & \frac{\partial r(U^*,U^*)}{\partial U_2}N^* & 0 \\ \lambda & -(\lambda_0 + \theta) & 0 & 0 & 0 & 0 & 0 \\ -\alpha_1C_1^* + \pi_1v_1U_1^* & \pi\theta & -\delta_1 - \alpha_1N^* & 0 & \pi_1v_1N^* & 0 & -\mu_1 \\ \phi - \alpha_2C_2^* + \pi_2v_2U_2^* & 0 & 0 & -\delta_2 - \alpha_2N^* & 0 & \pi_2v_2N^* & -\mu_2 \\ (1-k_1)\alpha_1C_1^* - v_1U_1^* & 0 & (1-k_1)\alpha_1N^* & 0 & -\phi_1 - v_1N^* & 0 & 0 \\ (1-k_2)\alpha_2C_2^* - v_2U_2^* & 0 & 0 & (1-k_2)\alpha_2N^* & 0 & -\phi_2 - v_2N^* & 0 \\ 0 & 0 & \theta_1 & \theta_2 & 0 & 0 & -\theta_0 \end{bmatrix}.$$

where

$$a_{12} = \frac{r_0 N^{*2}}{[K(C_1^*, C_2^*)]^2} \frac{\partial K(C_1^*, C_2^*)}{\partial C_1} - k_1 \alpha_1 N^*,$$

$$a_{13} = \frac{r_0 N^{*2}}{[K(C_1^*, C_2^*)]^2} \frac{\partial K(C_1^*, C_2^*)}{\partial C_2} - k_2 \alpha_2 N^*.$$

From the matrix M_0 , it is clear that $E_0(0, 0, C_1, C_2, 0, 0, F)$ is a saddle point with stable manifold locally in the $P-U_1-U_2$ space and unstable manifold locally in the N -direction if $r_0 - k_1 \alpha_1 C_1 - k_2 \alpha_2 C_2 > 0$.

The stability behavior of E^* is not obvious from M^* . However, in the following theorem we find sufficient condition for E^* to be locally asymptotically stable.

The following theorem gives the criteria for the local stability of E^* which can be proved by constructing a suitable Lyapunov function.

4b. Local Stability via Lyapunov Method

Theorem 4.1: *If the following inequalities hold*

$$(4.1a) \quad \left[\frac{r_0 N^*}{[K(C_1^*, C_2^*)]^2} \left(\frac{-\partial K(C_1^*, C_2^*)}{\partial C_1} + k_1 \alpha_1 \right) + a_3 (\alpha_1 C_1^* - \pi_1 \nu_1 U_1^*) \right]^2 < \frac{1}{3} \frac{a_3 r_0 \delta_1}{K(C_1^*, C_2^*)},$$

$$(4.1b) \quad \left[\frac{r_0 N^*}{[K(C_1^*, C_2^*)]^2} \left(\frac{-\partial K(C_1^*, C_2^*)}{\partial C_2} + k_2 \alpha_2 \right) + a_4 (\alpha_2 C_2^* - \pi_2 \nu_2 U_2^* - \phi) \right]^2 < \frac{1}{3} \frac{a_4 r_0 \delta_2}{K(C_1^*, C_2^*)},$$

$$(4.1c) \quad \frac{-\left(\frac{\partial r(U_1^*, U_2^*)}{\partial U_1} \right) (1 - k_1) \alpha_1 (\pi \theta)^2}{[(1 - k_1) \alpha_1 C_1^* - \nu_1 U_1^*] \delta_1 \pi_1 \nu_1} < \frac{1}{3} \frac{r_0}{K(C_1^*, C_2^*)} \frac{(\lambda_0 + \theta)^2}{\lambda^2},$$

$$(4.1d) \quad [-a_3 \mu_1 \theta_2 + a_4 \mu_2 \theta_1]^2 < \frac{1}{2} a_3 a_4 \mu_2 \theta_2 \delta_1 \theta_0,$$

where

$$a_3 = \frac{-\left(\frac{\partial r(U_1^*, U_2^*)}{\partial U_1}\right)(1-k_1)\alpha_1}{\left[(1-k_1)\alpha_1 C_1^* - v_1 U_1^*\right]\pi_1 v_1}, \text{ provided } (1-k_1)\alpha_1 C_1^* > v_1 U_1^*,$$

$$a_4 = \frac{-\left(\frac{\partial r(U_1^*, U_2^*)}{\partial U_2}\right)(1-k_2)\alpha_2}{\left[(1-k_2)\alpha_2 C_2^* - v_2 U_2^*\right]\pi_2 v_2}, \text{ provided } (1-k_2)\alpha_2 C_2^* > v_2 U_2^*,$$

then $E^*(N^*, P^*, C_1^*, C_2^*, U_1^*, U_2^*, F^*)$ is locally asymptotically stable.

Proof: See the Appendix.

5. Region of Attraction for Global Stability

The following theorem gives the globally asymptotically behavior of the equilibrium point E^* . For this we first need a lemma which establishes the region of attraction for the system (2.1).

Lemma 5.1: *The set*

$$\Omega = \left\{ (N, P, C_1, C_2, U_1, U_2, F) : \begin{aligned} &0 \leq N \leq K_0; 0 \leq P + C_1 + U_1 \leq \frac{(Q + \lambda K_0)}{\phi_{10}}; \\ &0 \leq C_2 + U_2 \leq \frac{\phi K_0}{\phi_{20}}; \\ &0 \leq F \leq \frac{1}{\theta_0} \left[\frac{\theta_1 (Q + \lambda K_0)}{\phi_{10}} + \frac{\theta_2 (\phi K_0)}{\phi_{20}} \right] \end{aligned} \right\}$$

is a region of attraction for all solutions initiating in the region

$$R = \{(N, P, C_1, C_2, U_1, U_2, F) : N > 0, P > 0, C_1 > 0, C_2 > 0, U_1 > 0, U_2 > 0, F > 0\}$$

where $\phi_{10} = \min(\lambda_1, \delta_1, \phi_1)$ and $\phi_{20} = \min(\delta_2, \phi_2)$.

Proof: The proof is given in Appendix.

6. Global Stability Analysis via Lyapunov's Method

Theorem 6.1: *In addition to the assumptions*

$$\frac{\partial r}{\partial U_i} < 0; \text{ for } U_i > 0, \quad i = 1, 2$$

$$\frac{\partial K}{\partial C_i} < 0; \text{ for } C_i > 0, \quad i = 1, 2$$

let K_m, p_1, p_2, k_1, k_2 be positive constants such that, in the region Ω ,

$$\begin{aligned} K_m \leq K(C_1, C_2) \leq K_0 ; \quad 0 \leq -\frac{\partial r(U_1, U_2)}{\partial U_1} \leq p_1 ; \quad 0 \leq -\frac{\partial r(U_1, U_2)}{\partial U_2} \leq p_2 ; \\ 0 \leq -\frac{\partial K(C_1, C_2)}{\partial C_1} \leq k_1 ; \quad 0 \leq -\frac{\partial K(C_1, C_2)}{\partial C_2} \leq k_2 \end{aligned}$$

If the following inequalities hold

$$(6.1a) \quad \left[p_1 + \left((1-k_1)\alpha_1 C_1^* - v_1 U_1^* \right) \right]^2 < \frac{2}{5} \frac{r_0 \phi_1}{K_0},$$

$$(6.1b) \quad \left[p_2 + \left((1-k_2)\alpha_2 C_2^* - v_2 U_2^* \right) \right]^2 < \frac{2}{5} \frac{r_0 \phi_2}{K_0},$$

$$(6.1c) \quad \left[\left(r_0 N^* \frac{k_1}{k_m^2} + k_1 \alpha_1 \right) + a_3 \left(\alpha_1 C_1^* - \pi_1 v_1 U_1^* \right) \right]^2 < \frac{1}{5} a_3 \frac{r_0 \delta_1}{K_0},$$

$$(6.1d) \quad \left[\left(r_0 N^* \frac{k_2}{k_m^2} + k_2 \alpha_2 \right) + a_4 \left(\alpha_2 C_2^* - \pi_2 v_2 U_2^* - \phi \right) \right]^2 < \frac{1}{5} a_4 \frac{r_0 \delta_2}{K_0},$$

$$(6.1e) \quad \frac{(1-k_1)\alpha_1(\pi\theta)^2}{\delta_1 \pi_1 v_1} < \frac{1}{5} \frac{r_0(\lambda_0 + \theta)^2}{K_0 \lambda^2},$$

$$(6.1f) \quad \left[-a_3 \mu_1 \theta_2 + a_4 \mu_2 \theta_1 \right]^2 < \frac{1}{2} a_3 a_4 \mu_2 \theta_2 \delta_1 \theta_0,$$

where

$$a_3 = \frac{(1-k_1)\alpha_1}{\pi_1 v_1} \quad \text{and} \quad a_4 = \frac{(1-k_2)\alpha_2}{\pi_2 v_2},$$

then E^* is globally asymptotically stable with respect to all solutions initiating in the interior of the region Ω .

Proof: See the Appendix.

Summary

In this paper, an ecological model has been proposed and analyzed to study the simultaneous effect of two toxicants one is emitted by external sources with a constant rate (such as smoke or poisonous gas emitted by vehicular traffic or chimney) and augmented by its precursor, the other produced by the population itself through its various action. The existence and uniqueness of non-trivial equilibrium point has been discussed and stability behavior (both local and global stability) has been analyzed. It has

been shown that the density of the biological species decreases with the increase in the total emission rate of pollutant in the environment. The analysis of the non linear stability shows that the system settles at much lower density of the biological species when the concentration of the pollutants in the environment and in the uptake phase of the species is high. But the control measures (environmental taxes) imposed on the pollutant emitting industries and people, control concentration of the pollutants in the environment and due to this, the equilibrium point shifts in such a way that the density of the biological species is more near to the density when ecosystem is pollution free. It has also been noted that toxicants emitted into the environment by population dependant human actions are not desirable and it must be controlled.

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Appendix

Proof of Theorem 4.1:

First we linearize the system (2.1) about $E^*(N^*, B^*, P^*, C^*, U^*, F^*)$ by using the following transformations

$$N = N^* + n, P = P^* + p, C_1 = C_1^* + c_1, C_2 = C_2^* + c_2, U_1 = U_1^* + u_1, \\ U_2 = U_2^* + u_2, F = F^* + f,$$

where n, b, p, c, u and f are small perturbations around E^* . This results into

$$\begin{aligned} \frac{dn}{dt} = & -\frac{r_0 N^*}{K(C_1^*, C_2^*)} n + \left(\frac{r_0 N^{*2}}{[K(C_1^*, C_2^*)]^2} \frac{\partial K(C_1^*, C_2^*)}{\partial C_1} - k_1 \alpha_1 N^* \right) c_1 \\ & + \left(\frac{r_0 N^{*2}}{[K(C_1^*, C_2^*)]^2} \frac{\partial K(C_1^*, C_2^*)}{\partial C_2} - k_2 \alpha_2 N^* \right) c_2 \\ & + \left(\frac{\partial r(U_1^*, U_2^*)}{\partial U_1} N^* \right) u_1 + \left(\frac{\partial r(U_1^*, U_2^*)}{\partial U_2} N^* \right) u_2 \end{aligned}$$

$$(4.1e) \quad \begin{cases} \frac{dp}{dt} = \lambda n - (\lambda_0 + \theta) p, \\ \frac{dc_1}{dt} = -(\alpha_1 C_1^* - \pi_1 \nu_1 U_1^*) n + \pi \theta p - (\delta_1 + \alpha_1 N^*) c_1 + (\pi_1 \nu_1 N^*) u_1 - \mu_1 f, \\ \frac{dc_2}{dt} = (\phi - \alpha_2 C_2^* - \pi_2 \nu_2 U_2^*) n - (\delta_2 + \alpha_2 N^*) c_2 + (\pi_2 \nu_2 N^*) u_2 - \mu_2 f, \\ \frac{du_1}{dt} = ((1 - k_1) \alpha_1 C_1^* - \nu_1 U_1^*) n + ((1 - k_1) \alpha_1 N^*) c_1 - (\phi_1 + \nu_1 N^*) u_1, \\ \frac{du_2}{dt} = ((1 - k_2) \alpha_2 C_2^* - \nu_2 U_2^*) n + ((1 - k_2) \alpha_2 N^*) c_2 - (\phi_2 + \nu_2 N^*) u_2, \\ \frac{df}{dt} = \theta_1 c_1 + \theta_2 c_2 - \theta_0 f. \end{cases}$$

Consider the positive definite function V around E^* , defined as

$$(4.1f) \quad V = \frac{1}{2} a_1 \frac{n^2}{N^*} + \frac{1}{2} a_2 p^2 + \frac{1}{2} a_3 c_1^2 + \frac{1}{2} a_4 c_2^2 + \frac{1}{2} a_5 u_1^2 + \frac{1}{2} a_6 u_2^2 + \frac{1}{2} a_7 f^2$$

where $a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 are positive constants, we can show that the derivative of V with respect to t along the linearized system (4.1e) is negative definite under the conditions of theorem (4.1).

Hence, V is a Lyapunov function with respect to E^* , therefore E^* is locally asymptotically stable.

Proof of Lemma 5.1:

The first equation of the model (2.1), we have

$$\frac{dN}{dt} = r(U_1, U_2) N - \frac{r_0 N^2}{K(C_1, C_2)} - k_1 \alpha_1 C_1 N - k_2 \alpha_2 C_2 N,$$

$$\frac{dN}{dt} \leq r(U_1, U_2) N - \frac{r_0 N^2}{K(C_1, C_2)},$$

$$\frac{dN}{dt} \leq r_0 N - \frac{r_0 N^2}{K_0}.$$

Hence, $\lim_{t \rightarrow \infty} \sup N(t) \leq K_0$, i.e.

$$(5.1a) \quad 0 \leq N \leq K_0.$$

Adding the second, third and fourth equations of the model (2.1), we have

$$\begin{aligned}
\frac{dP}{dt} + \frac{dC_1}{dt} + \frac{dU_1}{dt} &= \lambda N - \lambda_0 P - \theta P + Q + \pi \theta P - \delta_1 C_1 - \alpha_1 C_1 N \\
&\quad + \pi_1 \nu_1 N U_1 - \mu_1 F + (1 - k_1) \alpha_1 C_1 N - \phi_1 U_1 - \nu_1 N U_1, \\
&= \lambda N + Q - (\lambda_0 + \theta - \pi \theta) P - \delta_1 C_1 - \phi_1 U_1 \\
&\quad - k_1 \alpha_1 C_1 N - (1 - \pi_1) \nu_1 N U_1 - \mu_1 F \\
&\leq Q + \lambda N - \lambda_1 P - \delta_1 C_1 - \phi_1 U_1,
\end{aligned}$$

where $\lambda_1 = \lambda_0 + \theta - \pi_1 \theta$.

$$\frac{d}{dt}(P + C + U_1) \leq Q + \lambda K_0 - \phi_{10}(P + C_1 + U_1).$$

Using inequality (5.1a)

$$(5.1b) \quad 0 \leq P + C_1 + U_1 \leq \frac{(Q + \lambda K_0)}{\phi_{10}},$$

where $\phi_{10} = \min(\lambda_1, \delta_1, \phi_1)$.

Adding fifth and sixth equations of the model (2.1), we have

$$\begin{aligned}
\frac{dC_2}{dt} + \frac{dU_2}{dt} &= \phi N - \delta_2 C_2 - \alpha_2 C_2 N + \pi_2 \nu_2 N U_2 - \mu_2 F \\
&\quad + (1 - k_2) \alpha_2 C_2 N - \phi_2 U_2 - \nu_2 N U_2, \\
&\leq \phi N - \delta_2 C_2 - \phi_2 U_2.
\end{aligned}$$

Using the result $\lim_{t \rightarrow \infty} \sup N(t) \leq K_0$, we get

$$\begin{aligned}
&\leq \phi K_0 - \phi_{20}(C_2 + U_2) \\
(5.1c) \quad &0 \leq C_2 + U_2 \leq \frac{\phi K_0}{\phi_{20}}.
\end{aligned}$$

From the last equation of the model (2.1),

$$\frac{dF}{dt} = \theta_1(C_1 - C_{p_1}) + \theta_2(C_2 - C_{p_2}) - \theta_0 F$$

$$\Rightarrow \frac{dF}{dt} \leq C_1 \theta_1 + C_2 \theta_2 - \theta_0 F$$

$$\Rightarrow \frac{dF}{dt} \leq \theta_1 \left(\frac{Q + \lambda K_0}{\phi_{10}} \right) + \theta_2 \left(\frac{\phi K_0}{\phi_{20}} \right) - \theta_0 F$$

$$\Rightarrow F(t) \leq \frac{1}{\theta_0} \left[\theta_1 \left(\frac{(Q + \lambda K_0)}{\phi_{10}} \right) + \theta_2 \left(\frac{\phi K_0}{\phi_{20}} \right) \right],$$

and hence,

$$(5.1d) \quad 0 \leq F \leq \frac{1}{\theta_0} \left[\theta_1 \left(\frac{(Q + \lambda K_0)}{\phi_{10}} \right) + \theta_2 \left(\frac{\phi K_0}{\phi_{20}} \right) \right],$$

where

$$\phi_{10} = \min(\lambda_1, \delta_1, \phi_1) \quad \text{and} \quad \phi_{20} = \min(\delta_2, \phi_2).$$

Proof of Theorem 6.1:

To prove the assertion, we will construct a positive definite function W around E^*

$$(6.1g) \quad W(N, P, C_1, C_2, U_1, U_2, F) = a_1 \left(N - N^* - N^* \log \frac{N}{N^*} \right) + \frac{1}{2} a_2 (P - P^*)^2 \\ + \frac{1}{2} a_3 (C_1 - C_1^*)^2 + \frac{1}{2} a_4 (C_2 - C_2^*)^2 + \frac{1}{2} a_5 (U_1 - U_1^*)^2 \\ + \frac{1}{2} a_6 (U_2 - U_2^*)^2 + \frac{1}{2} a_7 (F - F^*)^2.$$

where $a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 are positive constants to be chosen such that it becomes a Lyapunov function and its domain contains the region of attraction as (5.1) defined above.

On differentiating W with respect to t , we get

$$\dot{W} = a_1 \left(\frac{N - N^*}{N} \right) \frac{dN}{dt} + a_2 (P - P^*) \frac{dP}{dt} + a_3 (C_1 - C_1^*) \frac{dC_1}{dt} + a_4 (C_2 - C_2^*) \frac{dC_2}{dt} \\ + a_5 (U_1 - U_1^*) \frac{dU_1}{dt} + a_6 (U_2 - U_2^*) \frac{dU_2}{dt} + a_7 (F - F^*) \frac{dF}{dt}.$$

Substituting the values of $\dot{N}, \dot{P}, \dot{C}_1, \dot{C}_2, \dot{U}_1, \dot{U}_2, \dot{F}$ from (2.1), we have

$$\dot{W} = a_1 (N - N^*) \left[r(U_1, U_2) - \frac{r_0 N}{K(C_1, C_2)} k_1 \alpha_1 C_1 - k_2 \alpha_2 C_2 \right] + a_2 (P - P^*) [\lambda N - \lambda_0 P - \theta P] \\ + a_3 (C_1 - C_1^*) [Q + \pi \theta P - \delta_1 C_1 - \alpha_1 N C_1 + \pi_1 \nu_1 N U_1 - \mu_1 F]$$

$$\begin{aligned}
& +a_4(C_2 - C_2^*)[\phi N - \delta_2 C_2 - \alpha_2 N C_2 + \pi_2 \nu_2 N U_2 - \mu_2 F] \\
& +a_5(U_1 - U_1^*)[(1 - k_1)\alpha_1 N C_1 - \phi_1 U_1 - \nu_1 U_1 N] \\
& +a_6(U_2 - U_2^*)[(1 - k_2)\alpha_2 N C_2 - \phi_2 U_2 - \nu_2 U_2 N] \\
& +a_7(F - F^*)[\theta_1(C_1 - C_{p_1}) + \theta_2(C_2 - C_{p_2}) - \theta_0 F].
\end{aligned}$$

After some algebraic manipulations, it can be written as

$$\begin{aligned}
(6.1h) \quad \dot{W} = & -a_1 \frac{r_0}{K(C_1, C_2)} (N - N^*)^2 - a_2 (\lambda_0 + \theta) (P - P^*)^2 - a_3 (\delta_1 + \alpha_1 N) (C_1 - C_1^*)^2 \\
& - a_4 (\delta_2 + \alpha_2 N) (C_2 - C_2^*)^2 - a_5 (\phi_1 + \nu_1 N) - a_6 (\phi_2 + \nu_2 N) (U_2 - U_2^*)^2 \\
& - a_7 \theta_0 (F - F^*)^2 + (N - N^*) (C_1 - C_1^*) [-a_1 (r_0 N^* \xi_1(C_1, C_2) + k_1 \alpha_1) \\
& - a_3 (\alpha_1 C_1^* - \pi_1 \nu_1 U_1^*)] + (N - N^*) (C_2 - C_2^*) [-a_1 (r_0 N^* \xi_2(C_1^*, C_2) + k_2 \alpha_2) \\
& - a_4 (\alpha_2 C_2^* - \pi_2 \nu_2 U_2^* - \phi) + (N - N^*) (U_1 - U_1^*) [a_1 \eta_1(U_1, U_2) \\
& + a_5 ((1 - k_1) \alpha_1 C_1^* - \nu_1 U_1^*)] + (N - N^*) (U_2 - U_2^*) [a_1 \eta_2(U_1^*, U_2) \\
& + a_6 ((1 - k_2) \alpha_2 C_2^* - \nu_2 U_2^*)] + (C_1 - C_1^*) (U_1 - U_1^*) [a_3 \pi_1 \nu_1 N \\
& + a_5 (1 - k_1) \alpha_1 N] + (C_2 - C_2^*) (U_2 - U_2^*) [a_4 \pi_2 \nu_2 N + a_6 (1 - k_2) \alpha_2 N] \\
& + (N - N^*) (P - P^*) a_3 \lambda + (P - P^*) (C_1 - C_1^*) a_3 \pi \theta \\
& + (C_1 - C_1^*) (F - F^*) [-\mu_1 a_3 + a_7 \theta_1] + (C_2 - C_2^*) (F - F^*) [-\mu_2 a_4 + a_7 \theta_2].
\end{aligned}$$

where

$$(6.1i) \quad \left\{ \begin{aligned} \eta_1(U_1, U_2) &= \begin{cases} \left\{ \frac{r(U_1, U_2) - r(U_1^*, U_2)}{(U_1 - U_1^*)} \right\} & ; \quad U_1 \neq U_1^* \\ \frac{\partial r(U_1, U_2)}{\partial U_1} & ; \quad U_1 = U_1^* \end{cases}, \\ \eta_2(U_1^*, U_2) &= \begin{cases} \left\{ \frac{r(U_1^*, U_2) - r(U_1^*, U_2^*)}{(U_2 - U_2^*)} \right\} & ; \quad U_2 \neq U_2^* \\ \frac{\partial r(U_1^*, U_2)}{\partial U_2} & ; \quad U_2 = U_2^* \end{cases}, \end{aligned} \right.$$

$$(6.1j) \quad \left\{ \begin{array}{l} \xi(C_1, C_2) = \left\{ \begin{array}{l} \frac{\frac{1}{K(C_1, C_2)} - \frac{1}{K(C_1^*, C_2)}}{C_1 - C_1^*} \\ - \frac{1}{[K(C_1^*, C_2)]^2} \frac{\partial K(C_1^*, C_2)}{\partial C_1} \end{array} \right\} \quad ; \quad C_1 \neq C_1^* \\ \xi(C_1^*, C_2) = \left\{ \begin{array}{l} \frac{\frac{1}{K(C_1^*, C_2)} - \frac{1}{K(C_1^*, C_2^*)}}{C_2 - C_2^*} \\ - \frac{1}{[K(C_1^*, C_2^*)]^2} \frac{\partial K(C_1^*, C_2^*)}{\partial C_2} \end{array} \right\} \quad ; \quad C_2 \neq C_2^* \end{array} \right. ,$$

For $\dot{W}$ to be negative definite the following inequalities must be satisfied:

$$\left[a_1(r_0 N^* \xi_1(C_1, C_2) + k_1 \alpha_1) + a_3(\alpha_1 C_1^* - \pi_1 \nu_1 U_1^*) \right]^2 < 4 a_1 \frac{r_0}{5K(C_1, C_2)} a_3 \left(\frac{\delta_1}{4} \right),$$

$$\left[a_1(r_0 N^* \xi_2(C_1, C_2) + k_2 \alpha_2) + a_4(\alpha_2 C_2^* - \pi_2 \nu_2 U_2^* - \phi) \right]^2 < 4 a_1 \frac{r_0}{5K(C_1, C_2)} a_4 \left(\frac{\delta_2}{4} \right),$$

$$\left[a_1 \eta_1(U_1, U_2) + a_5((1-k_1)\alpha_1 C_1^* - \nu_1 U_1^*) \right]^2 < 4 a_1 \frac{r_0}{5K(C_1, C_2)} a_5 \left(\frac{\phi_1}{2} \right),$$

$$\left[a_1 \eta_2(U_1, U_2) + a_6((1-k_2)\alpha_2 C_2^* - \nu_2 U_2^*) \right]^2 < 4 a_1 \frac{r_0}{5K(C_1, C_2)} a_6 \left(\frac{\phi_2}{2} \right),$$

$$\left[a_3 \pi_1 \nu_1 N + a_5((1-k_1)\alpha_1 N) \right]^2 < 4 a_3 a_5 \left(\frac{\delta_1}{4} + \alpha_1 N \right) \left(\frac{\phi_1}{2} + \nu_1 N \right),$$

$$\left[a_4 \pi_2 \nu_2 N + a_6((1-k_2)\alpha_2 N) \right]^2 < 4 a_4 a_6 \left(\frac{\delta_2}{2} + \alpha_2 N \right) \left(\frac{\phi_2}{2} + \nu_2 N \right),$$

$$[a_2 \lambda]^2 < 4 a_1 \frac{r_0}{5K(C_1, C_2)} a_2 \frac{(\lambda_0 + \theta)}{2},$$

$$(a_3 \pi \theta)^2 < 4 a_3 \left(\frac{\delta_1}{4} \right) a_2 \frac{(\lambda_0 + \theta)}{2},$$

$$[-\mu_1 a_3 + a_7 \theta_1]^2 < 4 a_3 \left(\frac{\delta_1}{4} \right) \frac{1}{2} a_7 \theta_0,$$

$$[-\mu_2 a_4 + a_7 \theta_2]^2 < 4 a_4 \left(\frac{\delta_2}{4} \right) \frac{1}{2} a_7 \theta_0.$$

Since

$$\begin{aligned} K_m \leq K(C) \leq K_0; \quad 0 \leq -\frac{\partial r(U_1, U_2)}{\partial U_1} \leq p_1; \quad 0 \leq -\frac{\partial r(U_1, U_2)}{\partial U_2} \leq p_2; \\ 0 \leq -\frac{\partial K(C_1, C_2)}{\partial C_1} \leq k_1; \quad 0 \leq -\frac{\partial K(C_1, C_2)}{\partial C_2} \leq k_2 \end{aligned}$$

from the mean value theorem,

$$(6.1h) \quad \left\{ \begin{array}{l} |\xi_1(C_1, C_2)| \leq \frac{k_1}{K_m^2}, \\ |\xi_2(C_1^*, C_2)| \leq \frac{k_2}{K_m^2}, \\ |\eta_1(U_1, U_2)| \leq p_1, \\ |\eta_2(U_1^*, U_2)| \leq p_2. \end{array} \right.$$

for some positive constants K_m, p_1, p_2, k_1 and k_2 in the region Ω .

Keeping the maximum values of the variable quantities $\xi_1(C_1, C_2), \xi_2(C_1^*, C_2), \eta_1(U_1, U_2)$ and $\eta_2(U_1^*, U_2)$ on left hand sides from equation (6.11) and rewriting the above inequalities as

$$(6.1l) \quad \left[a_1 \left(r_0 N^* \frac{k_1}{K_m^2} + k_1 \alpha_1 \right) + a_3 \left(\alpha_1 C_1^* - \pi_1 \nu_1 U_1^* \right) \right]^2 < \frac{r_0 a_1 a_3 \delta_1}{5K(C_1, C_2)},$$

$$(6.1m) \quad \left[a_1 \left(r_0 N^* \frac{k_2}{K_m^2} + k_2 \alpha_2 \right) + a_4 \left(\alpha_2 C_2^* - \pi_2 \nu_2 U_2^* - \phi \right) \right]^2 < \frac{r_0 a_1 a_4 \delta_2}{5K(C_1, C_2)},$$

$$(6.1n) \quad \left[a_1 p_1 + a_5 \left((1 - k_1) \alpha_1 C_1^* - \nu_1 U_1^* \right) \right]^2 < \frac{2}{5} \frac{r_0 a_1 a_5 \phi_1}{K(C_1, C_2)},$$

$$(6.1o) \quad \left[a_3 \pi_1 \nu_1 N + a_5 \left((1 - k_1) \alpha_1 N \right) \right]^2 < 4a_3 a_5 \left(\frac{\delta_1}{4} + \alpha_1 N \right) \left(\frac{\phi_1}{2} + \nu_1 N \right),$$

$$(6.1p) \quad \left[a_4 \pi_2 \nu_2 N + a_6 \left((1 - k_2) \alpha_2 N \right) \right]^2 < 4a_4 a_6 \left(\frac{\delta_2}{2} + \alpha_2 N \right) \left(\frac{\phi_2}{2} + \nu_2 N \right),$$

$$(6.1q) \quad a_2 [\lambda]^2 < 2a_1 \frac{r_0 (\lambda_0 + \theta)}{5K(C_1, C_2)},$$

$$(6.1r) \quad a_3 (\pi \theta)^2 < a_2 \delta_1 \frac{(\lambda_0 + \theta)}{2},$$

$$(6.1s) \quad [-\mu_1 a_3 + a_7 \theta_1]^2 < \frac{1}{2} a_3 a_7 \delta_1 \theta_0,$$

$$(6.1t) \quad [-\mu_2 a_4 + a_7 \theta_2]^2 < \frac{1}{2} a_4 a_7 \delta_2 \theta_0.$$

we rewrite the inequality (6.1p) and (6.1q) as

$$\left[a_3 \pi_1 \nu_1 N - a_5 (1 - k_1) \alpha_1 N \right]^2 + 4a_3 a_5 \pi_1 \nu_1 (1 - k_1) \alpha_1 N^2 < 4a_3 a_5 \left(\frac{\delta_1}{4} + \alpha_1 N \right) \left(\frac{\phi_1}{2} + \nu_1 N \right).$$

$$\left[a_4 \pi_2 \nu_2 N - a_6 (1 - k_2) \alpha_2 N \right]^2 + 4a_4 a_6 \pi_2 \nu_2 (1 - k_2) \alpha_2 N^2 < 4a_4 a_6 \left(\frac{\delta_2}{2} + \alpha_2 N \right) \left(\frac{\phi_2}{2} + \nu_2 N \right).$$

In order to reduce the above inequality, we choose the value of constants

$$(6.1u) \quad a_5 = 1, \quad a_6 = 1, \quad a_3 = \frac{(1 - k_1) \alpha_1}{\pi_1 \nu_1} \quad \text{and} \quad a_4 = \frac{(1 - k_2) \alpha_2}{\pi_2 \nu_2}.$$

If we take $a_1 = 1$ then from (6.1r), we get

$$(6.1v) \quad a_2 < 2 \frac{r_0 (\lambda_0 + \theta)}{5K(C_1, C_2) [\lambda]^2}.$$

Putting the value of a_3 from (6.1u) in inequality (6.1s), we get

$$(6.1w) \quad \frac{2(1-k_1)\alpha_1(\pi\theta)^2}{\pi_1\nu_1\delta_1(\lambda_0+\theta)} < a_2.$$

In view of (6.1v) and (6.1w), a_2 can be chosen in such a way that the following inequality holds

$$\frac{(1-k_1)\alpha_1(\pi\theta)^2}{\pi_1\nu_1\delta_1} < \frac{1}{5} \frac{r_0(\lambda_0+\theta)^2}{K(C_1, C_2)[\lambda]^2}.$$

If we choose $a_7 = \frac{a_4\mu_2}{\theta_2}$, we find that the inequality (6.1u) is automatically satisfied. Such choice of a_7 suggest that we have considered a system in which the concentration level of pollutants is much more than the harmful limit and therefore the control measures (efforts) have been taken to reduce the concentration of pollutants in the environment.

Using the above result, last but one inequality (6.1t) becomes

$$[-\mu_1 a_3 \theta_2 + \mu_2 \theta_1 a_4]^2 < \frac{1}{2} a_3 a_4 \theta_0 \mu_2 \theta_2 \delta_1.$$

Further on keeping the minimum value of the variables on right hand sides, we get the remaining inequalities.

$$\left[p_1 + \left((1-k_1)\alpha_1 C_1^* - \nu_1 U_1^* \right) \right]^2 < \frac{2}{5} \frac{r_0 \phi_1}{K_0},$$

$$\left[p_2 + \left((1-k_2)\alpha_2 C_2^* - \nu_2 U_2^* \right) \right]^2 < \frac{2}{5} \frac{r_0 \phi_2}{K_0},$$

$$\left[\left(r_0 N^* \frac{k_1}{k_m^2} + k_1 \alpha_1 \right) + a_3 \left(\alpha_1 C_1^* - \pi_1 \nu_1 U_1^* \right) \right]^2 < \frac{1}{5} a_3 \frac{r_0 \delta_1}{K_0},$$

$$\left[\left(r_0 N^* \frac{k_2}{k_m^2} + k_2 \alpha_2 \right) + a_4 \left(\alpha_2 C_2^* - \pi_2 \nu_2 U_2^* - \phi \right) \right]^2 < \frac{1}{5} a_4 \frac{r_0 \delta_2}{K_0},$$

$$\frac{(1-k_1)\alpha_1(\pi\theta)^2}{\delta_1\pi_1\nu_1} < \frac{1}{5} \frac{r_0(\lambda_0 + \theta)^2}{K_0\lambda^2},$$

$$[-a_3\mu_1\theta_2 + a_4\mu_2\theta_1]^2 < \frac{1}{2}a_3a_4\mu_2\theta_2\delta_1\theta_0,$$

where

$$a_3 = \frac{(1-k_1)\alpha_1}{\pi_1\nu_1} \text{ and } a_4 = \frac{(1-k_2)\alpha_2}{\pi_2\nu_2},$$

which are same as mentioned in the Theorem 6.1. Hence W is Lyapunov function with respect to E^* whose domain contains Ω and therefore E^* is non-linearly stable and hence the theorem.

The above theorem imply that under certain condition, the population would settle down to a much lower equilibrium level than its initial independent carrying capacity, the magnitude of which would depend upon the toxicity, emission and washout rate of the toxicant and would be much less than the case of a single toxicant having same characteristics as each of them.

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