

A Finsler Space with Conservative Berwald Curvature

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Abstract: The present paper deals with a Finsler space with conservative Berwald curvature tensor. The condition characterizing such space is weaker than the condition characterizing a symmetric Finsler space^{1,2}. It is established that Berwald torsion tensor and deviation tensor of a Finsler space with conservative Berwaldcurvature tensor are conservative but the converse is true only when certain conditions are satisfied. Apart from establishing the symmetry of the covariant derivative $B_h H_{jk}$ of Ricci tensor in the indices h and k and certain identities, it is proved that the projective curvature tensor of a Finsler space with conservative Berwald curvature tensor is conservative only when the tensor $\dot{\partial}_m B_k H$ is symmetric.

Keywords: Finsler space, Berwaldcurvature tensor, deviation tensor, symmetric space, conservativeness, projective deviation tensor, projective curvature tensor.

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1. Introduction

In a manifold, a vector field is called conservative if its divergence is zero. If in a neighbourhood of a general point p of a manifold, the divergence of curvature is zero, then the bending is conservative i.e., rigid around point p . U. C. De and A. A. Shaikh³ studied the conservativeness of quasi- conformal curvature tensor in 1997. C.S. Bagewadi and N. B. Gatti⁴ studied conservative curvature tensor on an Einstein manifold. Several significant contributions towards spaces with conservative curvature tensors were made by several authors including C.S.Bagewadi and D. G. Prakasha⁵, C. S. Bagewadi and N. S. Basavarajappa⁶.

C. S. Bagewadi and M. C. Bharthi⁷, introduced the concept of conservative conditions on curvature tensor of a Kaehler manifold. The

aim of the present paper is to study a Finsler space with conservative Berwald curvature tensor.

2. Preliminaries

Consider F^n , a Finsler space of dimension n having F as a metric function satisfying the requisite conditions^{8,9}. Let g_{ij} , G_{jk}^i , H_j^i , H_{jk}^i , H_{jkh}^i , denote the components of corresponding metric tensor, Berwald's connection coefficients, components of curvature deviation tensor, components of torsion tensor and the components of Berwald curvature tensor respectively. Similarly, the components of projective deviation tensor, projective torsion tensor and projective curvature tensor are denoted by W_j^i , W_{jk}^i and W_{jkh}^i respectively.

The above mentioned components of different Berwald tensors satisfy following identities:

$$\begin{aligned}
 (2.1) \quad (a) \quad H_{kh}^i &= H_{jkh}^i y^j, & (b) \quad H_h^i &= H_{kh}^i y^k, \\
 (c) \quad H_{jkh}^i &= \dot{\partial}_j H_{kh}^i, & (d) \quad H_{kh}^i &= \frac{1}{3}(\dot{\partial}_k H_h^i - \dot{\partial}_h H_k^i), \\
 (e) \quad H_{jk}^i &= -H_{kj}^i, & (f) \quad H_i^i &= (n-1)H, \\
 (g) \quad H_k y^k &= (n-1)H, & (h) \quad H_{kh} y^k &= H_h.
 \end{aligned}$$

Bianchi identities^{8,9} satisfied by the components of Berwald curvature tensor are given by

$$\begin{aligned}
 (2.2) \quad (a) \quad H_{jkh}^i + H_{hjk}^i + H_{khj}^i &= 0, \\
 (b) \quad B_m H_{jkh}^i + B_h H_{jmk}^i + B_k H_{jhm}^i + H_{kh}^r G_{mjr}^i + H_{mk}^r G_{hjr}^i + H_{hm}^r G_{kjr}^i &= 0.
 \end{aligned}$$

The components of H- Ricci tensor are

$$(2.3) \quad H_{jk} = H_{jkr}^r.$$

Contracting the indices i and j in equation (2.2) (b), we get

$$(2.4) \quad H_{rkh}^r = (H_{hk} - H_{kh}).$$

The components of projective deviation tensor, projective torsion tensor and the projective curvature tensor satisfy the following identities:

$$(2.5) \quad (a) \quad W_{kh}^i = W_{jkh}^i y^j, \quad (b) \quad W_h^i = W_{kh}^i y^k,$$

$$(c) W_{jkh}^i = \dot{\partial}_j W_{kh}^i, \quad (d) W_{kh}^i = \frac{1}{3}(\dot{\partial}_k W_h^i - \dot{\partial}_h W_k^i).$$

The projective deviation tensor, the projective torsion tensor and the projective curvature tensor are defined in terms of the Berwald deviation tensor, the Berwald torsion tensor and the Berwald curvature tensor respectively as following:

$$(2.6) \quad (a) W_h^i = H_h^i - \delta_h^i H - \frac{y^j}{(n+1)}(\dot{\partial}_r H_h^r - \dot{\partial}_r H),$$

$$(b) W_{kh}^i = H_{kh}^i + \frac{y^j}{(n+1)}H_{rhk}^r + \frac{1}{(n^2-1)}\{\delta_k^i(nH_h + H_{hr}y^r) - \delta_h^i(nH_k + H_{kr}y^r)\},$$

$$(c) W_{jkh}^i = H_{jkh}^i + \frac{y^j}{(n+1)}(\delta_j^i H_{r h k}^r + y^j \dot{\partial}_j H_{r h k}^r) + \frac{1}{(n^2-1)}\left\{\delta_k^i(nH_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr}) - \delta_h^i(nH_{jk} + H_{kj} + y^r \dot{\partial}_j H_{kr})\right\}.$$

The covariant operator B_h commutes with the directional differential operator $\dot{\partial}_k$ in the following way:

$$(2.7) \quad \dot{\partial}_k B_h T_j^i - B_h \dot{\partial}_k T_j^i = T_j^r G_{rkh}^i - T_r^i G_{jkh}^r,$$

where T_j^i are the components of an arbitrary tensor of type (1,1) and $\dot{\partial}_j G_{kh}^i = G_{jkh}^i$ are the components of a symmetric tensor i.e.

$$(2.8) \quad G_{jkh}^i = G_{khj}^i = G_{hjk}^i.$$

Also, G_{jkh}^i satisfy

$$(2.9) \quad y^j G_{jkh}^i = y^k G_{jkh}^i = y^k G_{jkh}^i = 0.$$

Definition: A Finsler space F^n is called symmetric if

$$(2.10) \quad B_m H_{jkh}^i = 0, \quad H_{jkh}^i \neq 0.$$

Definition: A Finsler space F^n is called projectively symmetric if

$$(2.11) \quad B_m W_{jkh}^i = 0, \quad W_{jkh}^i \neq 0.$$

Definition: A Finsler space F^n with conservative Berwald curvature tensor is characterized by

$$(2.12) \quad B_r H_{jkh}^r = 0, \quad H_{jkh}^i \neq 0.$$

Definition: A Finsler space F^n ($n > 2$) with conservative projective curvature tensor is defined as

$$(2.13) \quad B_r W_{jkh}^r = 0, \quad W_{jkh}^i \neq 0.$$

3. Finsler Space with Conservative Berwald Curvature Tensor

Consider a Finsler space F^n with conservative Berwald curvature tensor characterized by (2.12).

Transvecting (2.12) by y^j and using (2.1) (a), we get

$$(3.1) \quad B_r H_{kh}^r = 0.$$

Again transvecting (3.1) by y^k and using (2.1) (h), we have

$$(3.2) \quad B_r H_h^r = 0.$$

Thus, we conclude:

Theorem 3.1: *The Berwald torsion tensor and the Berwald deviation tensor of a Finsler space with conservative Berwald curvature tensor are conservative.*

Now, we will prove some identities.

Transvecting the Bianchi's second identity given by (2.2) (b) by y^j and using (2.1) (a) and (2.9), we get

$$(3.3) \quad B_m H_{kh}^i + B_h H_{mk}^i + B_k H_{hm}^i = 0.$$

On contracting the indices i and m and using (3.1), we have

$$(3.4) \quad B_h H_k = B_k H_h$$

where $H_k = H_{kr}^r$.

Differentiating (3.4) partially with respect to y^j and using the commutation formula (2.7), we get

$$(3.5) \quad B_h H_{jk} = B_k H_{jh}.$$

This shows that (3.4) implies (3.5). Transvecting (3.5) by y^j , we immediately find (3.4). Thus, the conditions (3.4) and (3.5) are equivalent.

This leads to:

Theorem 3.2: *The Ricci tensor H_{jk} and the vector H_k of a Finsler space with conservative Berwald curvature tensor satisfy $B_h H_{jk} = B_k H_{jh}$ and $B_h H_k = B_k H_h$ equivalently.*

The commutation formula (2.7) for Berwald torsion tensor H_{kh}^i is given by

$$(3.6) \quad \dot{\partial}_j B_m H_{kh}^i - B_m H_{jkh}^i = H_{kh}^r G_{jmr}^i - H_{rh}^i G_{jmk}^r - H_{kr}^i G_{jmh}^r.$$

Contracting the indices i and m , and using (2.12) and (3.1), we have

$$(3.7) \quad H_{kh}^r G_{jsr}^s + H_{hs}^r G_{jkr}^s + H_{sk}^r G_{jhr}^s = 0.$$

Rotating k, h, j cyclically two times, we get

$$(3.8) \quad H_{jk}^r G_{hsr}^s + H_{ks}^r G_{hjr}^s + H_{sj}^r G_{hkr}^s = 0$$

and

$$(3.9) \quad H_{hj}^r G_{ksr}^s + H_{js}^r G_{khr}^s + H_{sh}^r G_{kjr}^s = 0.$$

Adding (3.7), (3.8), and (3.9) and using (2.1) (e), we have

$$(3.10) \quad H_{kh}^r G_{jsr}^s + H_{jk}^r G_{hsr}^s + H_{hj}^r G_{ksr}^s = 0.$$

Transvecting (3.10) by y^k and using (2.1) (b), we get

$$(3.11) \quad H_h^r G_{jsr}^s = H_j^r G_{hsr}^s.$$

Also, transvecting (3.7) by y^k and using (2.1) (b) gives

$$(3.12) \quad H_h^r G_{jsr}^s = H_s^r G_{jhr}^s.$$

Thus, we have:

Theorem 3.3: *In a Finsler space with conservative Berwald curvature tensor identities (3.7), (3.10), (3.11) and (3.12) hold:*

Transvecting (3.4) by y^h and using (2.1) (g), we have

$$(3.13) \quad y^h B_h H_k = (n-1) B_k H.$$

Partial differentiation of (3.13) partially with respect to y^m gives

$$(3.14) \quad B_m H_k + y^h \dot{\partial}_m B_h H_k = (n-1) \dot{\partial}_m B_k H.$$

In view of the commutation formula (2.7) and equation (2.9), (3.14) implies

$$(3.15) \quad B_m H_k + y^h B_h H_{mk} = (n-1) \dot{\partial}_m B_k H.$$

If we assume the tensor $\dot{\partial}_m B_k H$ to be symmetric, (3.4) and (3.15) gives us

$$(3.16) \quad y^h B_h H_{mk} = y^h B_h H_{km}.$$

Taking skew- symmetric part of (3.5) with respect to the indices j and k, we get

$$(3.17) \quad B_h H_{jk} - B_h H_{kj} = B_k H_{jh} - B_j H_{kh}.$$

Transvecting (3.17) by y^k , and using (3.4), we have

$$(3.18) \quad y^k B_h H_{jk} = y^k B_k H_{jh}.$$

In view of (3.5) and (3.16), (3.18) implies

$$(3.19) \quad y^k B_h H_{jk} = y^k B_j H_{hk}.$$

Differentiating (2.6) (b) covariantly with respect to y^m , we get

$$(3.20) \quad B_m W_{kh}^i = B_m H_{kh}^i + \frac{y^j}{(n+1)} B_m (H_{kh} - H_{hk}) + \frac{1}{(n^2-1)} \left\{ \delta_k^i (n B_m H_h + y^r B_m H_{hr}^r) - \delta_h^i (n B_m H_k + y^r B_m H_{kr}^r) \right\}.$$

In view of (3.1) and (3.16), contraction of the indices i and m in (3.20) implies

$$B_r W_{kh}^r = \frac{1}{(n^2-1)} \{ (n B_k H_h + y^r B_k H_{hr}^r) - (n B_h H_k + y^r B_h H_{kr}^r) \},$$

which, in view of (3.4) and (3.19), gives

$$(3.21) \quad B_r W_{kh}^r = 0.$$

Thus, the following theorem holds:

Theorem 3.4: *In a Finsler space with conservative Berwald curvature tensor, the symmetry of the tensor $\dot{\partial}_m B_k H$ implies the conservativeness of the projective torsion tensor.*

The commutation formula (2.7) for the components of projective torsion tensor is given as

$$(3.22) \quad \dot{\partial}_j B_m W_{kh}^i - B_m W_{jkh}^i = W_{kh}^r G_{jmr}^i - W_{rh}^i G_{jmk}^r - W_{kr}^i G_{jmh}^r.$$

On contracting the indices i and m in (3.22) and using (2.9) & (3.21), we have

$$(3.23) \quad B_r W_{jkh}^r = W_{rh}^s G_{jsk}^r + W_{kr}^s G_{jsh}^r - W_{kh}^r G_{jsr}^s.$$

Using the definition of W_{kh}^i in terms of H_{kh}^i from (2.6) (b) & (3.7), we conclude

$$(3.24) \quad W_{rh}^s G_{jsk}^r + W_{kr}^s G_{jsh}^r - W_{kh}^r G_{jsr}^s = 0.$$

Using (3.24) in (3.23), we get

$$(3.25) \quad B_r W_{jkh}^r = 0.$$

Thus, we have the following theorem:

Theorem 3.5: *In a Finsler space with conservative Berwald curvature tensor, the projective curvature tensor is also conservative provided the tensor $\dot{\partial}_h B_k H$ is symmetric.*

4. Finsler Space with Conservative Berwald Tensors

Let us consider a Finsler space F^n with conservative Berwald torsion tensor. Such space is characterized by (3.1). Transvecting (3.1) by y^k , we find (3.2). Therefore, the deviation tensor of a Finsler space with conservative Berwald torsion tensor is conservative.

Using commutation formula (2.7) for Berwald torsion tensor, we get

$$(4.1) \quad \dot{\partial}_j B_m H_{kh}^i - B_m H_{jkh}^i = H_{kh}^r G_{jmr}^i - H_{rh}^i G_{jmk}^r - H_{kr}^i G_{jmh}^r.$$

Contracting the indices m and i in (4.1) and using (3.1), we have

$$(4.2) \quad B_r H_{jkh}^r = H_{rh}^s G_{jsk}^r + H_{kr}^s G_{jsh}^s - H_{kh}^r G_{jsr}^s.$$

This shows that a Finsler space with conservative Berwald torsion tensor is not conservative. However, the space is conservative if the identity

$$(4.3) \quad H_{rh}^s G_{jsk}^r + H_{kr}^s G_{jsh}^s - H_{kh}^r G_{jsr}^s = 0,$$

is satisfied.

Identities (3.4) and (3.5) also hold in a Finsler space with conservative Berwald torsion tensor.

Thus, we have:

Theorem 4.1: *In a Finsler space, the conservativeness of Berwald torsion tensor implies the conservativeness of the Berwald deviation tensor. While the conservativeness of Berwald torsion tensor need not imply the conservativeness of Berwald curvature tensor. However, it implies the conservativeness of Berwald curvature tensor if the identity (4.3) holds.*

Now, we consider a Finsler space F^n with conservative Berwald deviation tensor given by (3.2).

The commutation formula (2.7) for Berwald deviation tensor is given by

$$(4.3) \quad \dot{\partial}_j B_m H_k^i - B_m \dot{\partial}_j H_k^i = H_k^r G_{jmr}^i - H_r^i G_{jmk}^r.$$

Taking skew symmetric part of (5.3) with respect to indices j and k, we have

$$(4.4) \quad \dot{\partial}_j B_m H_k^i - B_m \dot{\partial}_j H_k^i - \dot{\partial}_k B_m H_j^i + B_m \dot{\partial}_k H_j^i = H_k^r G_{jmr}^i - H_j^r G_{kmr}^i.$$

Contracting the indices i and m in (5.4) and using (3.2) and (2.1) (d), we conclude

$$(4.5) \quad 3B_r H_{jk}^r = H_j^r G_{ksr}^s - H_k^r G_{jsr}^s.$$

Thus, we have the following theorem:

Theorem 4.2: *In a Finsler space, the conservativeness of Berwald deviation tensor implies the conservativeness of Berwald torsion tensor provided the identity $H_j^r G_{ksr}^s = H_k^r G_{jsr}^s$ holds.*

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