Generalized Degenerated Bernoulli Numbers and Polynomials

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Abstract: The Generalized degenerate Bernoulli numbers $B_m^{\alpha}(\lambda)$ can be defined by means of the exponential generating function

$$f(t) = \left(1 + \lambda t \right)^{1/\lambda} - 1.$$ 

As further applications we derive several identities, recurrences, and congruences involving the Generalized Bernoulli numbers, Generalized degenerate Bernoulli numbers and polynomials.

Keywords: Bernoulli polynomial, Bernoulli number, degenerate Bernoulli polynomial, degenerate Bernoulli number, Generalized degenerate Bernoulli polynomial.

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1. Introduction

Carlitz\textsuperscript{1} defined the generalized degenerate Bernoulli numbers $B_m^{\alpha}(\lambda)$ by means of the generating function
we have, $B_m^{\alpha}(0) = B_m^{\alpha}$, the ordinary generalized Bernoulli number. In \( r \) Carlitz proved many properties of $B_m^{\alpha}(\lambda)$. He also pointed out that $B_m^{\alpha}(\lambda)$ is a polynomials in $\lambda$ with degree $\leq m$, we have

\[
B_0^{\alpha}(\lambda) = 1 \\
B_1^{\alpha}(\lambda) = -\frac{\alpha}{2} + \frac{\alpha \lambda}{2} \\
\text{and so on.}
\]

Carlitz\(^2\) also defined the generalized degenerate Bernoulli polynomials $B_m^{\alpha}(\lambda, x)$ for $\lambda \neq 0$ by means of the generating function.

\[
\left( \frac{(t)^\alpha}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \right) = \sum_{m=0}^{\infty} B_m^{\alpha}(\lambda, x) \frac{t^m}{m!}
\]

where $\lambda \mu = 1$. These are polynomials in $\lambda$ and $x$ with rational coefficients. We often write $B_m^{\alpha}(\lambda)$ for $B_m^{\alpha}(\lambda, 0)$, and refer to the polynomial $B_m^{\alpha}(\lambda)$ as a generalized degenerate Bernoulli number. The first few are

\[
B_0^{\alpha}(\lambda, x) = 1 \\
B_1^{\alpha}(\lambda, x) = x - \frac{\alpha}{2} + \frac{\alpha \lambda}{2} \\
\text{and so on}
\]

Clearly, we have

\[
B_m^{1}(\lambda, x) = B_m^{\alpha}(\lambda, x)
\]

2. A Recurrence Relation of $B_m^{\alpha}(\lambda, x)$

In this section, we derive the following recurrence relation for $B_m^{\alpha}(\lambda, x)$
\begin{equation}
B_m^\alpha(\lambda, x) = \sum_{k=0}^{m\choose k} B_k^\alpha(\lambda) \left( \frac{x}{\lambda} \right)_{m-k}
\end{equation}

**Proof:** We know that the generating function of generalized degenerate Bernoulli polynomial

\[ \frac{(t)^\alpha (1 + \lambda t)^{\frac{x}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x) \frac{t^m}{m!} \]

By (1.1) we get

\[ \sum_{m=0}^{\infty} B_m^\alpha(\lambda) \frac{t^m}{m!} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x) \frac{t^m}{m!} \]

By the Binomial expansion

\[ [(1 + x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + ...] \]

\[ \sum_{m=0}^{\infty} B_m^\alpha(\lambda) \frac{t^m}{m!} \sum_{m=0}^{\infty} \left( \frac{x}{\lambda} \right)_m \frac{t^m}{m!} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x) \frac{t^m}{m!} \]

By the Cauchy product rule

\[ B_m^\alpha(\lambda, x) = \sum_{k=0}^{m\choose k} B_k^\alpha(\lambda) \left( \frac{x}{\lambda} \right)_{m-k} \]

where \( \left( \frac{x}{\lambda} \right)_m = [x(x - \lambda)(x - 2\lambda)....(x - (m - 1)\lambda] \)

**Particular case:** It is interesting to note that (2.1) reduces to the well-known recurrence relation of degenerate Bernoulli polynomial for \( \alpha = 1. \)

\begin{equation}
B_m(\lambda, x) = \sum_{k=0}^{m\choose k} B_k(\lambda) \left( \frac{x}{\lambda} \right)_{m-k}
\end{equation}

3. **Properties of generalized Degenerate Bernoulli polynomial**
In this section, some of well-known properties of generalized Degenerate Bernoulli polynomials are derived from the generating function (1.2)

**Property-1**

\[(3.1) \quad B^\alpha_m(\lambda, x + y) = \sum_{k=0}^{\infty} \binom{m}{k} B^\alpha_k(\lambda, x) \left( \frac{y}{\lambda} \right)^{m-k}\]

**Proof:** Now put \( x \to x + y \) in (1.2)

\[\left(\frac{t^\alpha}{(1 + \lambda t)^\alpha}\right)^{\mu(x+y)} = \sum_{m=0}^{\infty} B^\alpha_m(\lambda, x + y) \frac{t^m}{m!}\]

\[\left(\frac{t^\alpha}{(1 + \lambda t)^\alpha}\right)^{\mu(x+y)} = \sum_{m=0}^{\infty} B^\alpha_m(\lambda, x + y) \frac{t^m}{m!}\]

By the equation (1.2)

\[\sum_{m=0}^{\infty} B^\alpha_m(\lambda, x) \frac{t^m}{m!} (1 + \lambda t)^{\mu y} = \sum_{m=0}^{\infty} B^\alpha_m(\lambda, x + y) \frac{t^m}{m!}\]

By the help of Binomial expansion

\[(1 + \lambda t)^{\mu y} = \sum_{m=0}^{\infty} \binom{\mu y}{m} \frac{t^m}{m!}\]

Therefore

\[\sum_{m=0}^{\infty} B^\alpha_m(\lambda, x) \frac{t^m}{m!} \sum_{m=0}^{\infty} \binom{\mu y}{m} \frac{t^m}{m!} = \sum_{m=0}^{\infty} B^\alpha_m(\lambda, x + y) \frac{t^m}{m!}\]

By the Cauchy product rule
\[
\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \left( \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}
\]

\[ c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \]

where

\[
B_m^\alpha(\lambda, x + y) = \sum_{k=0}^{m} \binom{m}{k} B_k^\alpha(\lambda, x) \left( \frac{y}{\lambda} \right)^{m-k}
\]

**Particular case:** When \( \alpha = 1 \), we get the ordinary Degenerate Bernoulli polynomial

\[ B_m^1(\lambda, x + y) = \sum_{k=0}^{m} \binom{m}{k} B_k^1(\lambda, x) \left( \frac{y}{\lambda} \right)^{m-k} \]

Here \( y = 1 \) then,

\[ B_m^\alpha(\lambda, x + 1) = \sum_{k=0}^{m} \binom{m}{k} B_k^\alpha(\lambda, x) \left( \frac{1}{\lambda} \right)^{m-k} \]

**Property-2**

\[ \frac{d}{dx} B_m^\alpha(\lambda, x) = \lambda^{-1} B_m^\alpha(\lambda, x) \]

**Proof:** By the generating function of generalized degenerate Bernoulli polynomials

\[ \frac{(t)^\alpha (1 + \lambda t)^\mu x}{(1 + \lambda t)^\mu - 1} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x) \frac{t^m}{m!} \]

Differentiate above equation with respect to \( x \)
\[
\frac{\mu(t)^{\alpha}(1 + \lambda t)^{\mu x}}{[1 + \lambda t]^{\mu} - 1} = \sum_{m=0}^{\infty} \frac{d}{dx} B_{m}^{\alpha}(\lambda, x) \frac{t^{m}}{m!}
\]

\[
\mu \sum_{m=0}^{\infty} B_{m}^{\alpha}(\lambda, x) \frac{t^{m}}{m!} = \sum_{m=0}^{\infty} \frac{d}{dx} B_{m}^{\alpha}(\lambda, x) \frac{t^{m}}{m!}
\]

Equating the coefficients

\[
\frac{d}{dx} B_{m}^{\alpha}(\lambda, x) = \mu B_{m}^{\alpha}(\lambda, x) \text{ Where } \mu \lambda = 1
\]

\[
\frac{d}{dx} B_{m}^{\alpha}(\lambda, x) = \lambda^{-1} B_{m}^{\alpha}(\lambda, x)
\]

**Property-3**

(3.5)

\[
B_{m}^{\alpha}(\lambda, \alpha - x) = (-1)^{m} B_{m}^{\alpha}(\lambda, x)
\]

Proof: By equation (1.2)

\[
\frac{(t)^{\alpha}(1 + \lambda t)^{\mu x}}{[1 + \lambda t]^{\mu} - 1} = \sum_{m=0}^{\infty} B_{m}^{\alpha}(\lambda, x) \frac{t^{m}}{m!}
\]

Replace \(x \rightarrow \alpha - x\) in the above equation

\[
\frac{(t)^{\alpha}(1 + \lambda t)^{\mu(\alpha-x)}}{[1 + \lambda t]^{\mu} - 1} = \sum_{m=0}^{\infty} B_{m}^{\alpha}(\lambda, \alpha - x) \frac{t^{m}}{m!}
\]
\[
\frac{(t)^\alpha (1+\lambda t)^{-\mu x}}{(1+\lambda t)^{-\mu} - 1} = \sum_{m=0}^{\infty} B_m^{\alpha}(\lambda, \alpha - x) \frac{t^m}{m!}
\]

\[
\frac{(t)^\alpha (1+\lambda t)^{-\mu x}}{1-(1+\lambda t)^{-\mu}} = \sum_{m=0}^{\infty} B_m^{\alpha}(\lambda, \alpha - x) \frac{t^m}{m!}
\]

\[
-\frac{(t)^\alpha (1+\lambda t)^{-\mu x}}{(1+\lambda t)^{-\mu} - 1} = \sum_{m=0}^{\infty} B_m^{\alpha}(\lambda, \alpha - x) \frac{t^m}{m!}
\]

\[
\sum_{m=0}^{\infty} B_m^{\alpha}(\lambda, x) \frac{(-1)^m}{m!} = \sum_{m=0}^{\infty} B_m^{\alpha}(\lambda, \alpha - x) \frac{t^m}{m!}
\]

\[
\sum_{m=0}^{\infty} B_m^{\alpha}(\lambda, x) \frac{(-1)^m}{m!} = \sum_{m=0}^{\infty} B_m^{\alpha}(\lambda, \alpha - x) \frac{t^m}{m!}
\]

Equating the coefficients

\[
B_m^{\alpha}(\lambda, \alpha - x) = (-1)^m \cdot B_m^{\alpha}(\lambda, x)
\]

**Property-4**

\[
(3.6) \quad B_m^{\alpha}(\lambda, x + 1) - B_m^{\alpha}(\lambda, x) = m B_{m-1}^{\alpha-1}(\lambda, x)
\]
Proof: By equation (1.2)

\[
\frac{(t)\alpha(1 + \lambda t)\mu_x}{(1 + \lambda t)\mu - 1} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x) \frac{t^m}{m!}
\]

(3.7)

Replace \( x \rightarrow x + 1 \) in the above equation.

\[
\frac{(t)\alpha(1 + \lambda t)\mu^{x+1}}{(1 + \lambda t)\mu - 1} = \sum_{m=0}^{\infty} B_m^\alpha(\lambda, x + 1) \frac{t^m}{m!}
\]

(3.8)

Subtracting (3.8)-(3.7)

\[
\frac{(t)\alpha(1 + \lambda t)\mu^{x+1}}{(1 + \lambda t)\mu - 1} = \sum_{m=0}^{\infty} \left[ B_m^\alpha(\lambda, x + 1) - B_m^\alpha(\lambda, x) \right] \frac{t^m}{m!}
\]

\[
\frac{(t)\alpha(1 + \lambda t)\mu^{x+1}}{(1 + \lambda t)\mu - 1} = \sum_{m=0}^{\infty} \left[ B_m^\alpha(\lambda, x + 1) - B_m^\alpha(\lambda, x) \right] \frac{t^m}{m!}
\]

(3.9)

\[
\frac{1}{(1 + \lambda t)\mu - 1} = \sum_{m=0}^{\infty} \left[ B_m^\alpha(\lambda, x + 1) - B_m^\alpha(\lambda, x) \right] \frac{t^m}{m!}
\]

(3.10)

\[
t \sum_{m=0}^{\infty} B_m^{\alpha-1}(\lambda, x) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left[ B_m^\alpha(\lambda, x + 1) - B_m^\alpha(\lambda, x) \right] \frac{t^m}{m!}
\]
\[
\sum_{m=0}^{\infty} B_m^{\alpha-1}(\lambda, x) \frac{t^{m+1}}{m!} = \sum_{m=0}^{\infty} \left[ B_m^{\alpha}(\lambda, x+1) - B_m^{\alpha}(\lambda, x) \right] \frac{t^m}{m!}
\]

Equating the coefficient of \( \frac{t^m}{m!} \)

\[
B_m^{\alpha}(\lambda, x + 1) - B_m^{\alpha}(\lambda, x) = m B_{m-1}^{\alpha-1}(\lambda, x)
\]

Property-4

\[
B_m^{\alpha-1}(\lambda, x) = \frac{1}{m+1} \sum_{k=0}^{m+1} \binom{m+1}{k} B_k^{\alpha}(\lambda, x) \left( \frac{1}{\lambda} \right)_{m+1-k}
\]

(3.9)

Proof:- By equation (3.6)

\[
B_m^{\alpha}(\lambda, x + 1) = \sum_{k=0}^{m} \binom{m}{k} B_k^{\alpha}(\lambda, x) \left( \frac{1}{\lambda} \right)_{m-k}
\]

\[
B_m^{\alpha}(\lambda, x + 1) = \sum_{k=0}^{m-1} \binom{m}{k} B_k^{\alpha}(\lambda, x) \left( \frac{1}{\lambda} \right)_{m-k} + \binom{m}{m} B_m^{\alpha}(\lambda, x) \left( \frac{1}{\lambda} \right)_{m-m}
\]

\[
B_m^{\alpha}(\lambda, x + 1) - B_m^{\alpha}(\lambda, x) = \sum_{k=0}^{m-1} \binom{m}{k} B_k^{\alpha}(\lambda, x) \left( \frac{1}{\lambda} \right)_{m-k}
\]

But we know that by (3.5)

\[
B_m^{\alpha}(\lambda, x + 1) - B_m^{\alpha}(\lambda, x) = m B_{m-1}^{\alpha-1}(\lambda, x)
\]

(3.10)

\[
m B_{m-1}^{\alpha-1}(\lambda, x) = \sum_{k=0}^{m-1} \binom{m}{k} B_k^{\alpha}(\lambda, x) \left( \frac{1}{\lambda} \right)_{m-k}
\]
Now put $m \to m+1$ in the above equation

$$B_{m}^{\alpha-1}(\lambda, x) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_{k}^{\alpha}(\lambda, x) \left( \frac{1}{\lambda} \right)_{m+1-k}$$

(3.11)

Now put $x=0$ in equation (3.10), then

$$mB_{m-1}^{\alpha-1}(\lambda, 0) = \sum_{k=0}^{m-1} \binom{m}{k} B_{k}^{\alpha}(\lambda, 0) \left( \frac{1}{\lambda} \right)_{m-k}$$

By definition

$$B_{m}^{\alpha}(\lambda, 0) = B_{m}^{\alpha}(\lambda)$$

Therefore

$$mB_{m-1}^{\alpha-1}(\lambda) = \sum_{k=0}^{m-1} \binom{m}{k} B_{k}^{\alpha}(\lambda) \left( \frac{1}{\lambda} \right)_{m-k}.$$  

(3.12)

**References**
