

Stability Analysis of Fractional-Order Hindmarsh-Rose Neuron Model with Time-Delays

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Abstract: In this paper, the authors considered a class of fractional-order Hindmarsh-Rose (HR) neuron model with time-delays and widely investigated the dynamical behaviors of considered fractional-order HR model with time-delays. Some new sufficient conditions are derived by using the proposed stability theory of linear fractional-order systems to guarantee the stability results of fractional-order HR model with time-delays. Numerical simulations are also presented as evidence for the effectiveness of our derived theoretical results.

Keywords: Hindmarsh-Rose neuron (HR) model - Stability - Fractional-order - Time delays.

1. Introduction

Recently, many mathematicians and researchers have focused their interest and attention to analysis of the dynamical properties of fractional differential equations due to wide range of applications in various fields of science and engineering such as dielectric, electromagnetic waves, viscoelastic systems , physics, biology, chemistry, medicine etc¹⁻⁶. In general, fractional calculus has proven to be a very suitable tool for the description of memory and hereditary properties of various materials and processes ⁶. Due to this merit, many of the real-world applications are mathematically modeled by using the fractional-order differential equations, which give more accurate results and have more complicated properties than the classical integer-order differential equations. Thus, the study of fractional-order dynamical systems and fractional-order differential equations are great deal of attention and significance in both theory and applications. Moreover, several fractional-order dynamical systems are

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proposed and extensively analysis the dynamical characteristics of considered systems in the existing literature, such as fractional-order Lorenz system, fractional-order Chen system, fractional-order Liu system⁷⁻⁹.

Nowadays, the analysis of dynamical spiking/bursting properties of neuron models or neuronal ensembles have become a hotspot in the area of research. As we know that bursts could provide a more reliable approach of information transfer¹⁰⁻¹². There are some neuronal models were presented in the literature such as Hodgkin-Huxley, FitzHugh-Nagumo, Morris-Lecar, Hindmarsh-Rose (HR)¹³⁻¹⁶. Many of the authors studied the dynamical properties of neuronal models in the form of integer-order differential equations such as stability, synchronization, bifurcation and Hopf bifurcation¹³⁻¹⁶. Moreover, the main advantages of fractional-order systems are that have more degrees of freedom and infinite memory. The neuronal models are incorporated in to the infinite memory by introducing fractional-order. On the other hand, time delay systems have been received much attention in the field of interdisciplinary subjects. The neuron model with timedelay may cause certain dynamical behaviors such as oscillation, divergences, chaos, instability, or other poor performance. Thus, the fractional-order neuron model with time delays have more complicated and interesting one. Many of researchers¹⁷⁻¹⁹ have focused their interest to analysis of the properties of fractional-order neuron model with time delays and provided some excellent results. HR model is a slow-fast system and its detailed description is given by X. Shi and Z. Wang¹⁷. The authors studied the problem of adaptive synchronization of time delay Hindmarsh-Rose neuron system via self-feedback and provided some sufficient conditions to ensure the synchronization based on adaptive control theory and Lyapunov stability theory. Analysis the dynamical behaviors such as stability, Hopfbifurcation and periodically firing of fractional-order Hindmarsh-Rose neuronal model were extensively studied by D. Jun, Z. Guang-jun, X. Yong, Y. Hong and W. Jue¹⁸. X. Yong, K. YanMei, L. Yong and W. Ying¹⁹ widely investigated the firing properties and synchronization rate of considered fractional-order Hindmarsh-Rose model neurons. However, the studies on fractional-order Hindmarsh-Rose neuron model are without consideration of time-delays. To the best of our knowledge, the problem of stability and Hopf bifurcation analysis of fractional-order HR neuron model with time delay has not been investigated in the literature.

Motivated by the above discussion, in this paper we consider the fractional-order HR neuron model with time delay and widely studied the stability and Hopf bifurcation properties of considered model. Some new sufficient conditions are derived to ensure the stability of HR neuron model

with time delays by using stability theorem for linear fractional order systems with multiple time delays. Finally, numerical simulations are proposed to show the effectiveness of our results.

2. Preliminaries

In this section, the description of fractional-order delayed Hindmarsh-Rose neuron (HR) model is given as well as some fundamental definitions, lemmas and Theorems of fractional calculus have been provided, which are used to demonstrate our main results. The Caputo fractional-order derivative is employed throughout this paper.

Definition 2.1: The Caputo fractional-order derivative of order μ for a function $h(t) \in C^{n+1}([t_0, +\infty), R)$ is defined as¹

$$(2.1) \quad {}_{t_0}D_t^\mu h(t) = \frac{1}{\Gamma(n-\mu)} \int_{t_0}^t \frac{h^{(n)}(\tau)}{(t-\tau)^{\mu+1-n}} d\tau,$$

where $\mu > 0$ and n is a positive integer such that $n-1 < \mu \leq n$ and $\Gamma(\cdot)$ is a Gamma function.

Definition 2.2: The Laplace transform of the Caputo fractional-order derivative is

$$(2.2) \quad L\{{}_{t_0}D_t^\mu h(t); s\} = s^\mu H(s) - \sum_{k=0}^{n-1} s^{\mu-k-1} h^{(k)}(t_0), \quad n-1 < \mu \leq n.$$

When $h^{(k)}(t_0) = 0, k = 1, 2, \dots, n-1$, then

$$(2.3) \quad L\{{}_{t_0}D_t^\mu h(t); s\} = s^\mu H(s),$$

where $L\{\cdot\}$ denotes the Laplace transform and s is the variable of Laplace domain.

Theorem 2.3: The following linear autonomous system²⁰

$$(2.4) \quad D_t^\mu x = Ax \text{ and } x(0) = x_0,$$

where $0 < \mu \leq 1$, $x \in R^n$ and $A \in R^{n \times n}$ is an $n \times n$ matrix, is asymptotically stable if and only if $|\arg(\lambda)| > \frac{\mu\pi}{2}$ is satisfied for all eigenvalues (λ) of the matrix A . In this case, each component of the states decays towards 0 like $t^{-\mu}$. Also, this system is stable if and only if $|\arg(\lambda)| \geq \frac{\mu\pi}{2}$ is satisfied for all eigenvalues (λ) of the matrix A with those critical eigenvalues satisfying $|\arg(\lambda)| = \frac{\mu\pi}{2}$ having geometric multiplicity of one.

Theorem 2.4: For the nonlinear fractional-order system²¹

$$(2.5) \quad D_t^\mu x = g(x),$$

where $0 < \mu \leq 1$, $x \in R^n$ and g is a smooth nonlinear function, the equilibrium x_0 is locally asymptotically stable if $|\arg(\lambda)| > \frac{\mu\pi}{2}$ is satisfied for all eigenvalues (λ) of the Jacobian matrix $J = \frac{\partial g}{\partial x}$ evaluated at the equilibrium x_0 , where x_0 satisfies $g(x_0) = 0$.

In this paper, we consider the fractional-order Hindmarsh-Rose(HR) neuron model with time delay and the description as follows

$$(2.6) \quad \begin{cases} D^\mu u = au^2 - bu^3 + v - w(t - \tau) + I, \\ D^\mu v = c - du^2 - v, \\ D^\mu w = r(\beta(u + \delta) - w), \end{cases}$$

where μ is the fractional-order ($0 < \mu < 1$), u represents the membrane potential, v is associated with the fast current of Na^+ or K^+ ions and w is associated with the slow current of, for example Ca^+ ions. The parameters $a, b, c, d, r, \beta, \delta, I$ are real constants. $\tau > 0$ is time delay and I represents the external current in biological neurons.

Let (u^*, v^*, w^*) be the equilibrium point of the system (2.6). From (2.6), we can obtain the linearizing equations as follows

$$(2.7) \quad \begin{cases} D^\mu \tilde{u} = 2au^* \tilde{u} - 3bu^{*2} \tilde{u} + \tilde{v} - \tilde{w}(t - \tau), \\ D^\mu \tilde{v} = -2du^* \tilde{u} - \tilde{v}, \\ D^\mu \tilde{w} = r(\beta \tilde{u} - \tilde{w}), \end{cases}$$

with the following linear transform, $\tilde{u} = u - u^*$, $\tilde{v} = v - v^*$ and $\tilde{w} = w - w^*$. Equation (2.7) can be rewritten as in the following matrix form

$$(2.8) \quad D^\mu U = JU(t) + HU(t - \tau),$$

where $U = (\tilde{u}, \tilde{v}, \tilde{w})^T$ and J is the Jacobian matrix of system (2.6) at the equilibrium point (u^*, v^*, w^*) , such as

$$J = \begin{bmatrix} 2au^* - 3bu^{*2} & 1 & 0 \\ -2du^* & -1 & 0 \\ r\beta & 0 & -r \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, taking Laplace transform on both sides of equation (2.8) one can obtain

$$(2.9) \quad \begin{cases} s^\mu U_1(s) - s^{\mu-1} \delta_1(0) = 2au^* U_1(s) - 3bu^{*2} U_1(s) + U_2(s) \\ \quad - e^{-s\tau} \left(U_3(s) + \int_{\tau}^0 e^{-st} \delta_3(t) dt \right), \\ s^\mu U_2(s) - s^{\mu-1} \delta_2(0) = U_2(s) - 2du^* U_1(s), \\ s^\mu U_3(s) - s^{\mu-1} \delta_3(0) = r\beta U_1(s) - rU_3(s), \end{cases}$$

where $(U_1(s), U_2(s), U_3(s))$ and $(\delta_1(0), \delta_2(0), \delta_3(0))$ are represents the Laplace transform and initial conditions of $(\tilde{u}(t), \tilde{v}(t), \tilde{w}(t))$. Equation (2.9) can be rewritten as in the vector form as follows

$$(2.10) \quad \Delta(s) \begin{pmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{pmatrix} = \begin{pmatrix} k_1(s) \\ k_2(s) \\ k_3(s) \end{pmatrix},$$

where

$$(2.11) \quad \Delta(s) = \begin{pmatrix} s^\mu - 2au^* + 3bu^{*2} & -1 & e^{-s\tau} \\ 2du^* & s^\mu + 1 & 0 \\ -r\beta & 0 & s^\mu + r \end{pmatrix}$$

and

$$(2.12) \quad \begin{cases} k_1(s) = s^{\mu-1} \delta_1(0) - e^{-s\tau} \int_{\tau}^0 e^{-st} \delta_3(t) dt, \\ k_2(s) = s^{\mu-1} \delta_2(0), \\ k_3(s) = s^{\mu-1} \delta_3(0). \end{cases}$$

Thus, $\Delta(s)$ is the characteristic matrix of system (2.6) for simplicity and $k_1(s), k_2(s), k_3(s)$ is the remaining nonlinear part of system (2.6). Also, $\det(\Delta(s))$ is a characteristic polynomial of (2.6) and the distribution of $\det(\Delta(s))$'s eigenvalues are entirely used to studied the stability of system (2.6)

3. Results

In this section, we establish some stability theorems to ensure the stability and hopf bifurcation of considered HR neuron model with time delays.

Theorem 3.1: *If all the roots of the characteristic equation $\det(\Delta(s))=0$ have negative real parts, then the zero solution of system (2.6) is Lyapunov globally asymptotically stable.*

Proof: Multiplying by s on both sides of the equation (2.10), one can obtain

$$(3.1) \quad \Delta(s) \begin{pmatrix} sU_1(s) \\ sU_2(s) \\ sU_3(s) \end{pmatrix} = \begin{pmatrix} sk_1(s) \\ sk_2(s) \\ sk_3(s) \end{pmatrix}.$$

If all the roots of the transcendental equation $\det(\Delta(s))=0$ lie in open left half complex plane, i.e., $\operatorname{Re}(s) < 0$, then we consider (3.1) in $\operatorname{Re}(s) \geq 0$. In this restricted area, (3.1) has a unique solution $(sU_1(s), sU_2(s), sU_3(s))$. Thus, we have

$$\lim_{s \rightarrow 0, \operatorname{Re}(s) \geq 0} sU_i(s) = 0, \quad i = 1, 2, 3.$$

From the assumption of all roots of the characteristic equation $\det(\Delta(s))=0$ and the final-valued theorem of Laplace transform²³, we get

$$\lim_{t \rightarrow \infty} \tilde{u}(t) = \lim_{s \rightarrow 0, \operatorname{Re}(s) \geq 0} sU_1(s) = 0,$$

$$\lim_{t \rightarrow \infty} \tilde{v}(t) = \lim_{s \rightarrow 0, \operatorname{Re}(s) \geq 0} sU_2(s) = 0,$$

$$\lim_{t \rightarrow \infty} \tilde{w}(t) = \lim_{s \rightarrow 0, \operatorname{Re}(s) \geq 0} sU_3(s) = 0.$$

Hence, we conclude that the considered fractional-order HR neuron model with time delay system (2.6) is Lyapunov globally asymptotically stable.

Theorem 3.2: *If all the roots of the characteristic equation $\Delta(s) = \det(s^\mu I - J - He^{-s\tau}) = 0$ have negative real parts, then the zero solution of system (2.6) is Lyapunov asymptotically stable.*

Proof: From (2.9) we can rewrite as follows

$$(3.2) \quad \Delta(s)U(s) = k(s),$$

where $U(s) = (U_1(s), U_2(s), U_3(s))^T$ is represents the Laplace transform of $u(t) = (\tilde{u}(t), \tilde{v}(t), \tilde{w}(t))^T$ with $U(s) = L(u(t))$. Then $\Delta(s)$ and $k(s)$ are same as in (2.11) and (2.12).

Now, we can obtain the characteristic equation of (2.6) as follows

$$(3.3) \quad \Delta(s) = \det(s^\mu I - J - He^{-s\tau}) = 0.$$

Followed by the paper ²², we conclude that all the roots of equation (3.3) have negative real parts then the system (2.6) is Lyapunov asymptotically stable.

4. Numerical Simulation

In this section, we provide Numerical simulations, which are used to verify our theoretical results. The fractional-order HR neuron model with time delays (2.6) have chosen the following parameter values as $a = 3.0$, $b = 1.0$, $c = 1.0$, $d = 5.0$, $r = 0.006$, $\beta = 4.0$, $\delta = 1.6$ and time delay $\tau = 1.0$. In this paper, we consider the parameters I and μ are variable. Given different values of the parameters I and μ that shows different behavior of considered HR system (2.6). In Figure 1-3 shows that the time trajectory and phase portrait between $u(t)$ and $w(t)$ in which the parameter $I = 1.3$ and the fractional order $\mu = 0.9, 0.7$ and 0.5 respectively. For the different values of fractional-order $\mu = 0.9, 0.7, 0.5$, it exhibits the spiking/bursting behaviors for HR neuron model (2.6). The time response and State trajectory between $u(t)$ and $v(t)$ with the parameter $I = 3.0, 4.5$ and the fractional order $\mu = 0.95$ are depicts in Figure 4-5. The increasing the values of external direct current $I = 3.0, 4.5$, the periodic bursting behavior and regular bursting behavior of HR neuron model (2.6) are shown in Figure 4-5.

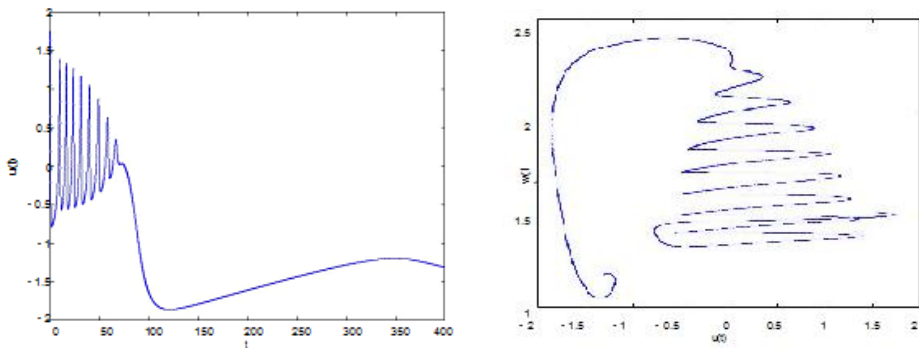


Figure 1: Time response and state trajectory of fractional-order HR neuron model when $I = 1.3$ and $\mu = 0.9$.

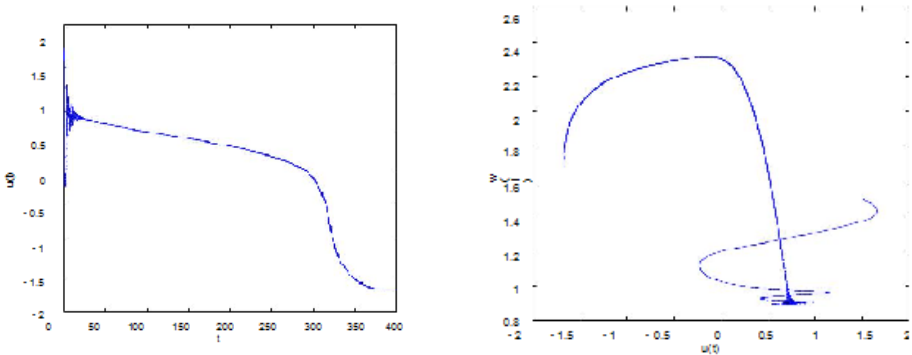


Figure 2: Time response and state trajectory of fractional-order HR neuron model with the parameters when $I = 1.3$ and $\mu = 0.7$.

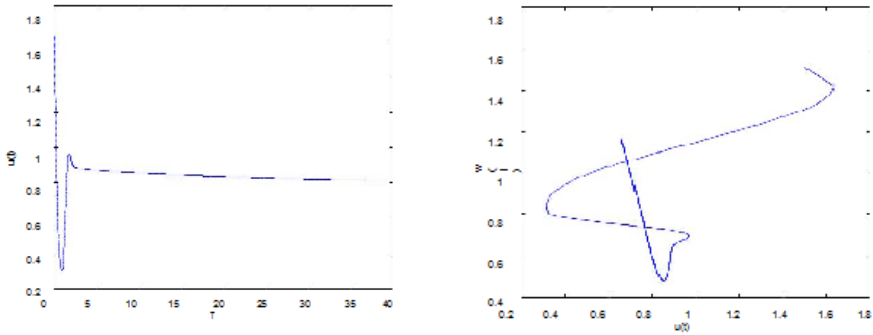


Figure 3: Time response and state trajectory of fractional-order HR neuron model with the parameters when $I = 1.3$ and $\mu = 0.5$.

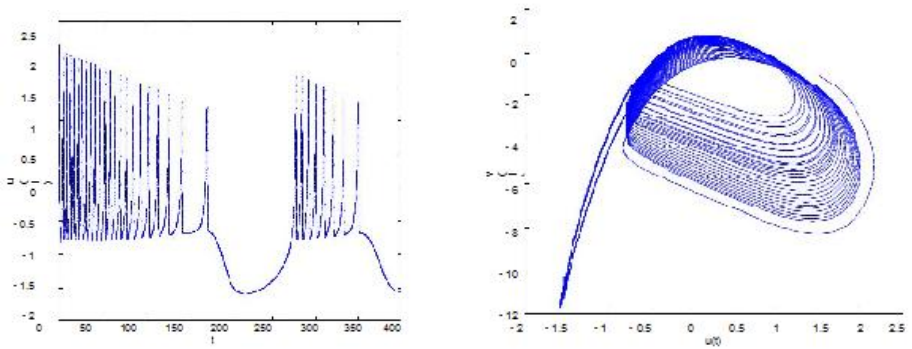


Figure 4: Time response and state trajectory of fractional-order HR neuron model with the parameters when $I = 3.0$ and $\mu = 0.95$.

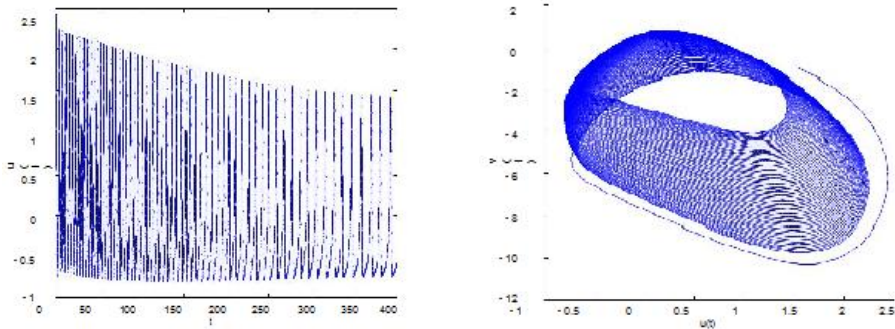


Figure 5: Time response and state trajectory of fractional-order HR neuron model with the parameters when $I = 4.5$ and $\mu = 0.95$.

5. Conclusion

In this paper, we dealt with the problem of stability analysis of fractional-order Hindmarsh-Rose (HR) neuron model with time-delays. First, we presented the fractional-order HR neuron model with time delays. By using the stability theory of linear fractional-order systems, some new sufficient conditions to ensure the stability results of fractional-order HR model with time-delays were derived. Finally, numerical simulations have been given to show the effectiveness of our main results.

References

1. Podlubny, *Fractional differential equations*, Academic Press, New York 1999.
2. O. Heaviside, *Electromagnetic theory*, Chelsea, New York 1971.
3. E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, *Molecular and Quantum Acoustics*, **23**(2002) 397–404.
4. A. El-Sayed, A. El-Mesiry, and H. El-Saka, On the fractional-order logistic equation, *Applied Mathematics Letters*, **20**(2007) 817–823.
5. N. M. Grahovac, and M. M. Zigic, Modelling of the hamstring muscle group by use of fractional derivatives, *Computers & Mathematics with Applications*, **59**(2010) 1695–1700.
6. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
7. I. Grigorenko and E. Grigorenko, Chaotic dynamics of the fractional Lorenz system, *Physical review letters*, **91**(2003) 034101.
8. S. T. Mohammad and H. Mohammad, A necessary condition for double scroll attractor existence in fractional-order systems, *Physics Letters A*, **367**(2007) 102–113.

9. D. G. Varsha and B. Sachin, Chaos in fractional ordered Liu system, *Computers and Mathematics with Applications*, **59**(2010) 1117–1127.
10. J. L. Hindmarsh and R.M.Rose, A model of the nerve impulse using two first-order differential equations, *Nature*, **296**(1982) 162–164.
11. M. Perc, Stochastic resonance on excitable small-world networks via a pacemaker, *Physical Review E*, **76**(2007) 066203.
12. Y. Q. Che, J. Wang, S. G. Cui, B. Deng, X. L. Wei, W. L. Chan and K. M. Tsang, Chaos synchronization of coupled neurons via adaptive sliding mode control, *Nonlinear Analysis; Real World Applications*, **12**(2011) 3199–3206.
13. A. Shilnikov and M. Kolomiets, Methods of the qualitative theory for the Hindmarsh-Rose model: A case study, *International Journal of Bifurcation and chaos*, **18**(2008) 2141–2168.
14. A. L. Hodgkin and A. F. Gyxket, A quantitative description of membrane current and its application to conduction and excitation in nerve, *The Journal of physiology*, **117**(1952) 500–544.
15. R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane, *Biophysical journal*, **1**(1961) 445–466.
16. M. Shi and Z. Wang, Abundant bursting patterns of a fractional-order Morris-Lecar neu-ron model, *Communications in Nonlinear Science and Numerical Simulation*, **19**(2014) 1956–1969.
17. X. Shi and Z. Wang, Adaptive synchronization of time delay Hindmarsh-Rose neuron system via self-feedback, *Nonlinear Dynamics*, **69**(2012) 2147–2153.
18. D. Jun, Z. Guang-jun, X. Yong, Y. Hong and W. Jue, Dynamic behavior analysis of fractional-order Hindmarsh-Rose neuronal model, *Cognitive Neurodynamics*, **8**(2014) 167–175.
19. X. Yong, K. YanMei, L. Yong and W. Ying, Firing properties and synchronization rate in fractional-order Hindmarsh-Rose model neurons, *Science China: Technological Sciences*, **57**(2014) 914–922.
20. D. Matignon, Stability Results for Fractional Differential Equations with Applications to Control Processing, In *Proceeding IMACS-SMC 1996, Lille, France*, (1996) 963–968.
21. E. Ahmed, A. El-sayed and H. El-Saka, Equilibrium Points, Stability and Numerical Solutions of Fractional Order Predator-Prey and Rabies Models, *Journal of Mathematical Analysis and Applications*, **352**(2007) 542–553.
22. W. Deng, C. Li and J. Lu, Stability analysis of linear fractional differential system with multiple time delays, *Nonlinear Dynamics*, **48**(2007) 409–416.
23. E. J. Muth, *Transform Methods with Applications to Engineering and Operations Research*, Prentice-Hall, New Jersey 1977.