

Stability and Bifurcation Analysis of Logistically Grown SIR Model with External Infection Effect of the Susceptible Class and Effect of Loss of Immunity of the Recovered Class

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(Received July 28, 2015)

Abstract: In this paper we consider an SIR model with logistic growth rate of susceptible where rate of incidence is directly affected by inhibitory factors or social or psychological factors and infection through interaction and external source also taken into consideration. In our model three equilibrium points are obtained under different conditions. One of them is trivial equilibrium point which always exists and unstable in nature in all circumstance. The other two are endemic equilibrium points exist under certain conditions and one of them becomes non-feasible under another condition. Here we examine the hopf-bifurcation depending on carrying capacity. Finally numerical simulations are done.

Keywords: Inhibition effect, Hopf bifurcation, Logistic growth rate, Lose immunity, external source of infection.

1. Introduction

The study of dynamics of the spreading of different disease is an important branch of mathematical biology. It is being studied by several authors since few years considering different spreading rate of infection such as standard incidence and saturation effects¹⁻¹¹. Birth rate of the susceptible of susceptible individual has been considered different in different models. In some model it has been considered constant^{1,2}, whereas in some model it changes logistically³⁻⁵. Our model is an extension of the models of^{3, 5, 6}.⁶ considered SIR model with logistic growth rate of the susceptible class and the rate of infection as the bilinear mass action with delay. ^{4,6}extended the model considering the rate of infection is affected by

*Presented at ICRTM 2015, University of Allahabad during July 10-12, 2015.

saturation effect. In this model we extended the model considering the external infection effect of the susceptible class.

The paper is organized as follows in first part of the paper we have formulate the model in second part stability analysis of the equilibrium points and Hopf bifurcation criterion is analyzed. Finally numerical simulation is done.

2. The Model Formulation

Here in the considered model the susceptible S-class has logistic growth rate and rate of infection is directly affected by the inhibitory factors such as different social factors and awareness. Here also the infection from the outer side i.e., external infection effect is taken into consideration. Let $S(t)$, $I(t)$, $R(t)$ and be the number of susceptible, infected and recovered individuals at time t . Then the governing differential equations proposed model is

$$(2.1) \quad \frac{ds}{dt} = rS \left(1 - \frac{S}{k} \right) - \frac{\beta SI}{1 + \alpha S} - dS - \beta_1 S + \mu R,$$

$$(2.2) \quad \frac{dI}{dt} = \frac{\beta SI}{1 + \alpha S} + \beta_1 S - (d + \gamma)I,$$

$$(2.3) \quad \frac{dR}{dt} = \gamma I - (d + \mu)R,$$

where,

r = birth rate (intrinsic growth rate) of the susceptible class.

k = Carrying capacity.

β = The transmission rate of infection.

β_1 = The external infection coefficient.

α = The parameter that measure the inhibitory factors.

d = The natural death of the population.

μ = Rate at which the recovered class losses immunity and becomes susceptible.

γ = Rate at which the infected individuals recovered.

The equation (2.1) shows that the susceptible class become infected through interaction and by external sources at the rate of β_1 become infected. So the term $+\beta_1$ enters in the second equation of the model.

3. Stability Analysis of the Model

The system admits one disease free equilibrium points which is $A_0 = (S_0, I_0, R_0) = (0, 0, 0)$ and the two endemic equilibrium points of the given system given by the following equations

$$(3.1) \quad H(S) = aS^2 + bS + c = 0,$$

where

$$\begin{aligned} a &= \frac{r}{k} \alpha (d + \gamma) (d + \mu) (1 - R_{01}), \\ b &= \left\{ \left(\frac{r}{k} + \alpha d \right) (d + \gamma) (d + \mu) + \beta_1 d \alpha (d + \mu + \gamma) + \beta r (d + \mu) \right\} (1 - R_{03}), \\ c &= d \{ \beta_1 (d + \mu + \gamma) + (d + \gamma) (d + \mu) \} (1 - R_{02}), \\ R_{01} &= \frac{\beta}{\alpha (d + \gamma)}, \\ R_{02} &= \frac{r (d + \gamma) (d + \mu)}{d \{ \beta_1 (d + \mu + \gamma) + (d + \gamma) (d + \mu) \}}, \\ R_{03} &= \frac{r \alpha (d + \gamma) (d + \mu) + \beta d (d + \mu)}{\left(\frac{r}{k} + \alpha d \right) (d + \gamma) (d + \mu) + \beta_1 d \alpha (d + \mu + \gamma) + \beta r (d + \mu)}. \end{aligned}$$

Let $\Delta = b^2 - 4ac$, First we consider $R_{03} < 1$ or $b > 0$ then we have the following cases:

(i) When $R_{01} < 1$, $R_{02} > 1$ then the equation $H(S) = 0$ has a unique

positive root, $S^* = \frac{-b + \sqrt{\Delta}}{2a}$.

(ii) When $R_{01} > 1$, $R_{02} < 1$ then the equation $H(S) = 0$ has a unique

positive root, $S_* = \frac{-b - \sqrt{\Delta}}{2a}$.

(iii) When $R_{01} > 1$, $R_{02} = 1$ then the equation $H(S) = 0$ has a unique

positive root, $S_1^* = -\frac{b}{a}$.

- (iv) When $R_{01} = 1$, $R_{02} > 1$ then the equation $H(S) = 0$ became a linear equation with positive root $S_2^* = -\frac{c}{b}$.
- (v) When $\Delta = 0$, $R_{01} > 1$ then the equation $H(S) = 0$ gives two equal positive roots $S^* = -\frac{b}{2a}$.
- (vi) When $R_{01} > 1$, $R_{02} > 1$ and $\Delta = 0$ then the equation $H(S) = 0$ have two positive roots which are $S^* = \frac{-b + \sqrt{\Delta}}{2a}$, $S_* = \frac{-b - \sqrt{\Delta}}{2a}$.

Denoting the endemic equilibrium point by $(\bar{S}, \bar{I}, \bar{R})$ where $\bar{S}$ is root of the equation $H(S) = 0$ and $\bar{I} = \frac{(d + \mu)\bar{S}}{d(d + \mu + \gamma)}\left(r - d - \frac{r\bar{S}}{k}\right)$, exist when $\bar{S} < \left(1 - \frac{d}{r}\right)k$, $\bar{R} = \frac{\gamma}{d + \mu}\bar{I}$.

Number of positive roots of the equation (3.1) for different conditions					
R_{01}	R_{02}	Sign of Δ	Sign of a	Sign of c	Number of positive roots of equation (3.1)
< 1	> 1	> 0	+ve	-ve	One, S^*
> 1	< 1	> 0	-ve	+ve	One, S_*
> 1	$= 1$	$\Delta = b > 0$	-ve	0	One, S_1^*
$= 1$	> 1	$\Delta = b > 0$	0	-ve	One, S_2^*
> 1	-	$= 0$	-ve	-ve	One, S_*
> 1	> 1	> 0	-ve	-ve	Two, S^*, S_*

Summarizing the above about the existence of the equilibrium we have the following lemma

Lemma 1:

- (1) When $R_{01} < 1$, $R_{02} > 1$, $S^* < \left(1 - \frac{d}{r}\right)k$ then the model has a unique endemic equilibrium point $A^*(S^*, I^*, R^*)$.

- (2) When $R_{01} > 1$, $R_{02} < 1$ $S_* < \left(1 - \frac{d}{r}\right)k$, then the model has a unique endemic equilibrium point $A_*(S_*, I_*, R_*)$.
- (3) When $R_{01} > 1$, $R_{02} = 1$, $S_1^* < \left(1 - \frac{d}{r}\right)k$, then the model has a unique endemic equilibrium point $A_1^*(S_1^*, I_1^*, R_1^*)$.
- (4) When $R_{01} = 1$, $R_{02} > 1$ $S_2^* < \left(1 - \frac{d}{r}\right)k$, then the model has a unique endemic equilibrium point $A_2^*(S_2^*, I_2^*, R_2^*)$.
- (5) When $\Delta = 0$, $R_{01} > 1$ $S_* < \left(1 - \frac{d}{r}\right)k$ then the model has a unique endemic equilibrium point $A_*(S_*, I_*, R_*)$.
- (6) When $R_{01} > 1$, $R_{02} > 1$, $\Delta > 0$, $S_* < \left(1 - \frac{d}{r}\right)k$, and $S^* < \left(1 - \frac{d}{r}\right)k$ then the model have two endemic equilibrium points $A^*(S^*, I^*, R^*)$, $A_*(S_*, I_*, R_*)$.

where

$$S^* = \frac{-b + \sqrt{\Delta}}{2a}, I^* = \frac{(d + \mu)S^*}{d(d + \mu + \gamma)} \left(r - d - \frac{rS^*}{k} \right), R^* = \frac{\gamma}{(d + \mu)} I^*,$$

$$S_* = \frac{-b - \sqrt{\Delta}}{2a}, I_* = \frac{(d + \mu)S_*}{d(d + \mu + \gamma)} \left(r - d - \frac{rS_*}{k} \right), R_* = \frac{\gamma}{(d + \mu)} I_*,$$

$$S_*^* = -\frac{b}{2a}, I_*^* = \frac{(d + \mu)S_*^*}{d(d + \mu + \gamma)} \left(r - d - \frac{rS_*^*}{k} \right), R_*^* = \frac{\gamma}{(d + \mu)} I_*^*,$$

$$S_1^* = -\frac{b}{a}, I_1^* = \frac{(d + \mu)S_1^*}{d(d + \mu + \gamma)} \left(r - d - \frac{rS_1^*}{k} \right), R_1^* = \frac{\gamma}{(d + \mu)} I_1^*,$$

$$S_2^* = -\frac{c}{b}, I_2^* = \frac{(d + \mu)S_2^*}{d(d + \mu + \gamma)} \left(r - d - \frac{rS_2^*}{k} \right), R_2^* = \frac{\gamma}{(d + \mu)} I_2^*.$$

Since practically $A_0(S_0, I_0, R_0)$ is not important because in this case all the individual population goes to extinction and so stability analysis about this

point is not taken into consideration. It can be shown that this point is an unstable equilibrium point.

Theorem 1: The endemic equilibrium point $A^*(S^*, I^*, R^*)$ will be asymptotically stable if the following conditions are satisfied

$$R_{01} < 1, R_{02} > 1, \frac{k}{2} \left\{ 1 - \frac{d + \beta_1}{r} \right\} < S^* < k \left(1 - \frac{d}{r} \right), r > d + \beta_1,$$

$$\text{where, } S^* = \frac{-b + \sqrt{\Delta}}{2a}, I^* = \frac{(d + \mu)S^*}{d(d + \mu + \gamma)} \left(r - d - \frac{rS^*}{k} \right), R^* = \frac{\gamma}{(d + \mu)} I^*.$$

Proof: The characteristic equation about the point $A^*(S^*, I^*, R^*)$ is

$$(3.2) \quad \begin{vmatrix} r \left(1 - \frac{2S^*}{k} \right) - \frac{\beta I^*}{(1 + \alpha S^*)^2} - d - \beta_1 - \lambda & -\frac{\beta S^*}{1 + \alpha S^*} & \mu \\ \frac{\beta I^*}{(1 + \alpha S^*)^2} + \beta_1 & \frac{\beta I^*}{(1 + \alpha S^*)} - d - \gamma - \lambda & 0 \\ 0 & \gamma & -(d + \mu + \lambda) \end{vmatrix} = 0,$$

which gives

$$(3.3) \quad \lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3 = 0,$$

where

$$c_1 = 2d + \mu + \beta_1 - r + \frac{2rS^*}{k} + \frac{\beta I^*}{(1 + \alpha S^*)^2} + \beta_1 \frac{S^*}{I^*},$$

$$c_2 = \frac{\beta I^* (2d + \mu + \gamma)}{(1 + \alpha S^*)^2} + \left(d + \mu + \frac{\beta_1 S^*}{I^*} \right) \left(d + \beta_1 - r + \frac{2rS^*}{k} \right) + \frac{\beta_1 S^*}{I^*} \left(d + \mu + \frac{\beta I^*}{(1 + \alpha S^*)} \right),$$

$$c_3 = d(d + \mu + \gamma) \left\{ \frac{\beta I^*}{(1 + \alpha S^*)^2} + \beta_1 \right\} + (d + \mu) \frac{\beta_1 S^*}{I^*} \left(d - r + \frac{2rS^*}{k} \right),$$

and

$$\begin{aligned}
c_1 c_2 - c_3 = & \frac{\beta^2 I^{*2} (2d + \mu + \gamma)}{(1 + \alpha S^*)^4} + \frac{\beta I^*}{(1 + \alpha S^*)^2} \left[\left(-r + \frac{2rS^*}{k} \right) \left(3d + 2\mu + \gamma + \frac{\beta_1 S^*}{I^*} \right) (d + \mu) \right. \\
& \left. (3d + \mu + 2\beta_1) + \frac{2\beta_1 S^*}{I^*} (2d + \mu + \beta_1) \right. \\
& \left. + \left(d + \gamma - \frac{\beta_1 S^*}{I^*} \right) \left(d + 2\beta_1 + \frac{\beta_1 S^*}{I^*} \right) \right] \\
& + \left(-r + \frac{2rS^*}{k} \right)^2 \left(d + \mu + \frac{\beta_1 S^*}{I^*} \right) + \left(-r + \frac{2rS^*}{k} \right) \left\{ 2(d + \beta_1) \left(d + \mu + \frac{\beta_1 S^*}{I^*} \right) + (d + \mu)^2 \right\} \\
& + 2(d + \mu) \frac{\beta_1 S^*}{I^*} + \beta_1 \left(d + \gamma - \frac{\beta_1 S^*}{I^*} \right) \left\{ (d + \mu)^2 + (d + \mu) \frac{\beta_1 S^*}{I^*} + \beta_1 \left(d + \gamma - \frac{\beta_1 S^*}{I^*} \right) \right\} \\
& + (d + \mu + \frac{\beta_1 S^*}{I^*}) (d + \beta_1)^2 + \left(d + \beta_1 + \frac{\beta_1 S^*}{I^*} \right) \left\{ (d + \mu)^2 + (d + \mu) \frac{\beta_1 S^*}{I^*} + \beta_1 \left(d + \gamma - \frac{\beta_1 S^*}{I^*} \right) \right\} \\
& + (d + \beta_1) (d + \mu) \frac{\beta_1 S^*}{I^*} + \beta \mu \gamma,
\end{aligned}$$

c_1 and c_2 are positive under the conditions stated in the theorem, c_3 is always positive and $c_1 c_2 - c_3 > 0$. Therefore Routh Hurwitz criterion is satisfied and the Eigen values must have negative real part. Hence the solutions in the neighborhood of endemic equilibrium point $A^*(S^*, I^*, R^*)$ will be stable in nature. Hence the theorem is proved.

Theorem 2: The endemic equilibrium point $A_*(S_*, I_*, R_*)$ will be asymptotically stable if the following conditions are satisfied

$$R_{01} > 1, R_{02} < 1, \frac{k}{2} \left\{ 1 - \frac{d + \beta_1}{r} \right\} < S_* < k \left(1 - \frac{d}{r} \right), \quad r > d + \beta_1,$$

$$\text{where } S_* = \frac{-b - \sqrt{\Delta}}{2a}, I_* = \frac{(d + \mu) S_*}{d(d + \mu + \gamma)} \left(r - d - \frac{r S_*}{k} \right), R_* = \frac{\gamma}{(d + \mu)} I_*.$$

Proof: The characteristic equation about the point $A_*(S_*, I_*, R_*)$ is

$$(3.4) \quad \begin{vmatrix} r\left(1 - \frac{2S_*}{k}\right) - \frac{\beta I_*}{(1 + \alpha S_*)^2} - d - \beta_1 - \lambda & -\frac{\beta S_*}{1 + \alpha S_*} & \mu \\ \frac{\beta I_*}{(1 + \alpha S_*)^2} + \beta_1 & \frac{\beta I_*}{(1 + \alpha S_*)} - d - \gamma - \lambda & 0 \\ 0 & \gamma & -(d + \mu + \lambda) \end{vmatrix} = 0,$$

which gives

$$(3.5) \quad \lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3 = 0,$$

where

$$\begin{aligned} c_1 &= 2d + \mu + \beta_1 - r + \frac{2rS_*}{k} + \frac{\beta I_*}{(1 + \alpha S_*)^2} + \beta_1 \frac{S_*}{I_*}, \\ c_2 &= \frac{\beta I_* (2d + \mu + \gamma)}{(1 + \alpha S_*)^2} + \left(d + \mu + \frac{\beta_1 S_*}{I_*}\right) \left(d + \beta_1 - r + \frac{2rS_*}{k}\right) \\ &\quad + \frac{\beta_1 S_*}{I_*} \left(d + \mu + \frac{\beta I_*}{(1 + \alpha S_*)}\right), \\ c_3 &= d(d + \mu + \gamma) \left\{ \frac{\beta I_*}{(1 + \alpha S_*)^2} + \beta_1 \right\} + (d + \mu) \frac{\beta_1 S_*}{I_*} \left(d - r + \frac{2rS_*}{k}\right) \end{aligned}$$

and

$$c_1 c_2 - c_3 = \frac{\beta^2 I_*^2 (2d + \mu + \gamma)}{(1 + \alpha S_*)^4} + \frac{\beta I_*}{(1 + \alpha S_*)^2} + \frac{\left[\left(-r + \frac{2rS_*}{k}\right) \left(3d + 2\mu + \gamma + \frac{\beta_1 S_*}{I_*}\right) \right.}{\left. (d + \mu)(3d + \mu + 2\beta_1) \right.} \\ \left. + \frac{2\beta_1 S_*}{I_*} (2d + \mu + \beta_1) \right. \\ \left. + \left(d + \gamma - \frac{\beta_1 S_*}{I_*}\right) \left(d + 2\beta_1 + \frac{\beta_1 S_*}{I_*}\right) \right]$$

$$\begin{aligned}
& + \left(-r + \frac{2rS_*}{k} \right)^2 \left(d + \mu + \frac{\beta_1 S_*}{I_*} \right) + \left(-r + \frac{2rS_*}{k} \right) \left\{ + (d + \mu)^2 + 2(d + \mu) \frac{\beta_1 S_*}{I_*} \right. \\
& \quad \left. + \beta_1 \left(d + \gamma - \frac{\beta_1 S_*}{I_*} \right) \right\} \\
& + \left(d + \mu + \frac{\beta_1 S_*}{I_*} \right) (d + \beta_1)^2 + \left(d + \beta_1 + \frac{\beta_1 S_*}{I_*} \right) \left\{ (d + \mu)^2 + (d + \mu) \frac{\beta_1 S_*}{I_*} \right. \\
& \quad \left. + \beta_1 \left(d + \gamma - \frac{\beta_1 S_*}{I_*} \right) \right\} \\
& + (d + \beta_1)(d + \mu) \frac{\beta_1 S_*}{I_*} + \beta \mu \gamma.
\end{aligned}$$

c_1 and c_2 are positive under the conditions stated in the theorem, c_3 is always positive and $c_1 c_2 - c_3 > 0$. Therefore Routh Hurwitz criterion is satisfied and the eigen values must have negative real part. Hence the solutions in the neighborhood of endemic equilibrium point $A_*(S_*, I_*, R_*)$ will be stable in nature. Hence the theorem is proved.

It is obvious from the statement of c_1 , c_2 and c_3 that sign of them can be controlled by changing the value of carrying capacity (k).

4. Hopf Bifurcation Around Positive Equilibrium

Since the expression of c_1, c_2, c_3 and $c_1 c_2 - c_3$ depends on carrying capacity k . The sign of c_1, c_2, c_3 and $c_1 c_2 - c_3$ can be controlled by changing the values of k . A Hopf bifurcation of the system is expected for some range of k where $c_1 c_2 - c_3 = 0$.

Theorem 3: *The System undergoes a Hopf bifurcation for $R_{01} < 1$, $R_{02} > 1$, $\Delta > 0$, when the carrying capacity k passes through the critical value k_c with the restriction $\frac{2S^*}{k} < 1 - \frac{d}{r}$ and when $\left(\frac{dQ_1}{dk} \right)_{k=k_c}$ does not changes sign.*

Proof: Hopf bifurcation will occur if $c_1(k)c_2(k) - c_3(k) = 0$ with $c_i(k) > 0$ and $\frac{d}{dk}(\text{real } \lambda) \neq 0$ at $k = k_c$.

For $c_1(k)c_2(k) - c_3(k) = 0$ with $c_i(k) > 0$; then the characteristic equation $\lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3 = 0$ becomes $(\lambda + c_1)(\lambda^2 + c_2) = 0$ having roots $-c_1, \pm i\sqrt{c_2}$. So there are purely imaginary Eigen values and one is strictly negative real Eigen value. We assume that for k is in the neighborhood of $k = k_c$ the roots have the form $\lambda_1 = Q_1(k) + iQ_2(k)$, $\lambda_2 = Q_1(k) - iQ_2(k)$ and $\lambda_3 = -Q_3(k)$ where $Q_i(k)$, $i=1, 2, 3$ are real. In view of the above roots, the corresponding characteristic equation will be

$$(4.1) \quad \lambda^3 + (Q_3 - 2Q_1)\lambda^2 + (Q_1^2 + Q_2^2 - 2Q_1Q_2)\lambda + Q_3(Q_1^2 + Q_2^2) = 0.$$

Comparing we get

$$c_1 = Q_3 - 2Q_1, c_2 = Q_1^2 + Q_2^2 - 2Q_1Q_2, c_3 = Q_3(Q_1^2 + Q_2^2).$$

Since $Q_1(k) = 0$ at $k = k_c$ then from the above we get

$$(4.2) \quad (c_1 + 2Q_1)c_2 = c_3 - 2Q_1(c_1 + 2Q_1)^2.$$

Differentiating both sides of (4.2) w.r.t. k , we obtain

$$(4.3) \quad (c_1 + 2Q_1)\frac{dc_2}{dk} + c_2\left(\frac{dc_1}{dk} + 2\frac{dQ_1}{dk}\right) = \frac{dc_3}{dk} - \left\{2\frac{dQ_1}{dk}(c_1 + 2Q_1)^2 + 2Q_1\frac{d}{dk}(c_1 + 2Q_1)^2\right\}.$$

Using the condition at $k = k_c$, $Q_1(k) = 0$ we get

$$\left(\frac{dQ_1}{dk}\right)_{k=k_c} = -\left\{\frac{\frac{d}{dk}(c_1c_2 - c_3)}{2c_1^2 + c_2}\right\}_{k=k_c}.$$

Using the values of c_1, c_2 and c_3 we get

$$\left(\frac{dQ_1}{dk} \right)_{k=k_c} = -\frac{1}{2c_1^2 + c_2} \left[\begin{aligned} & \left\{ L + \frac{\beta c_2}{(1+\alpha S^*)^2} \right\} D \left(r - d - \frac{2rS^*}{k} \right) S^{*'} + \left\{ L + \frac{\beta c_2}{(1+\alpha S^*)^2} \right\} \frac{DrS^*}{k^2} \\ & + \frac{DrS^{*2}}{k^2} \frac{\beta c_1}{(1+\alpha S^*)^2} + \frac{\beta c_1}{(1+\alpha S^*)^2} DS^* \left(r - d - \frac{2rS^*}{k} \right) \left(-r + \frac{2rS^*}{k} \right) \\ & - \frac{4\alpha\beta^2 I^{*2} S^{*'}}{(1+\alpha S^*)^5} (2d + \mu + \gamma) - \frac{2\alpha\beta I^* S^{*'}}{(1+\alpha S^*)^3} c_1 \left(-r + \frac{2rS^*}{k} \right) - \frac{2\alpha\beta I^* S^{*'}}{(1+\alpha S^*)^3} c_2 \\ & + I^{*-2} \left[\begin{aligned} & c_3 DS^* \left(-r + \frac{2rS^*}{k} \right)^2 + c_3 c_5 \left(-r + \frac{2rS^*}{k} \right) - \frac{rS^{*3}}{k^2} c_3 \left(-r + \frac{2rS^*}{k} \right) \\ & + \beta_1 DS^{*'} \left(-r + \frac{2rS^*}{k} \right)^3 + \beta_1 c_5 \left(-r + \frac{2rS^*}{k} \right)^2 - \beta_1 \frac{rS^{*3}}{k^2} \left(-r + \frac{2rS^*}{k} \right)^2 \\ & + DT_1 S^{*'} \left(-r + \frac{2rS^*}{k} \right) + T_1 c_5 - T_1 \frac{rS^{*3}}{k^2} + c_4 DS^* (d + \mu - \beta_1) \\ & \left(-r + \frac{2rS^*}{k} \right) + c_4 c_5 (d + \mu - \beta_1) - \frac{rS^{*3}}{k^2} c_4 (d + \mu - \beta_1) \end{aligned} \right] \\ & + k^{-2} \left[c_6 k S^{*'} + D_1 k S^* \left(-r + \frac{2rS^*}{k} \right) - c_6 S^* - D_1 S^* \left(-r + \frac{2rS^*}{k} \right) \right] \\ & + \frac{\beta\beta_1 I^*}{(1+\alpha S^*)^2} \left(5d + 2\mu + \gamma - d\beta_1 - \beta_1^2 \frac{S^*}{I^*} - \beta_1 \frac{S^*}{I^*} \right) \end{aligned} \right]$$

where

$$D = \frac{d + \mu}{d(d + \mu + \gamma)}, L = \frac{(2d + \mu + \gamma)2\beta^2 I^*}{(1 + \alpha S^*)^4}, c_1 = 3d + 2\mu + \gamma + \beta_1 \frac{S^*}{I^*}, S^{*'} = \frac{dS^*}{dk},$$

$$c_2 = (d + \mu)^2 + 2(d + \beta_1)(d + \mu) + 2\beta_1 \frac{S^*}{I^*} (2d + \mu + \beta_1)$$

$$+ \left(d + \gamma - \beta_1 \frac{S^*}{I^*} \right) \left(d + 2\beta_1 + \beta_1 \frac{S^*}{I^*} \right),$$

$$c_3 = \frac{\beta\beta_1 I^*}{(1 + \alpha S^*)^2} + 2\beta_1 (\mu + 2d), c_4 = \beta_1 \left(d + \beta_1 + \beta_1 \frac{S^*}{I^*} \right), c_5 = (dD + I^*) S^{*'},$$

$$c_6 = \frac{2rD\beta I^*}{(1+\alpha S^*)^2} + 2rN, D_1 = 4r \left(d + \mu + \beta_1 \frac{S^*}{I^*} \right),$$

$$N = 2(d + \beta_1) \left(d + \mu + \beta_1 \frac{S^*}{I^*} \right) + (d + \mu)^2 + 2(d + \mu) \beta_1 \frac{S^*}{I^*} \\ + \beta_1 \left(d + \gamma - \beta_1 \frac{S^*}{I^*} \right).$$

Hence, when $k < k_c$ then the solution will be stable and for $k > k_c$ the solution will be unstable in nature in the neighborhood A^* . Thus Hopf bifurcation occurs when the carrying capacity crosses the critical value $k = k_c$. Hence the result is proved.

5. Numerical Simulation

The numerical simulation is done considering several values of the parameters. We consider $r=14, \beta=1, d=2, \gamma=0.001, \mu=0.01, \alpha=0.011, \beta_1=0.01$ and for different values of k we found the graphical presentation of the system of equations (2.1)-(2.3). For $k=100$ the endemic equilibrium point is $(2.0442, 11.9670, 0.0060)$ and figure-1(a), 1(b), and 1(c) is the corresponding graphical presentation. It is clear from the figure that the system is stable and the endemic equilibrium point is an attracting fixed point.

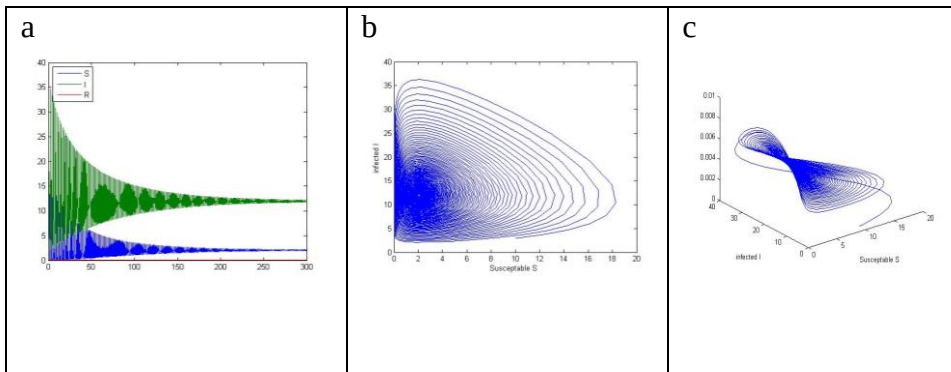


Figure1: Graphical presentation for $k=100, r=14, \beta=1, d=2, \gamma=0.001, \mu=0.01, \alpha=0.011, \beta_1=0.01$

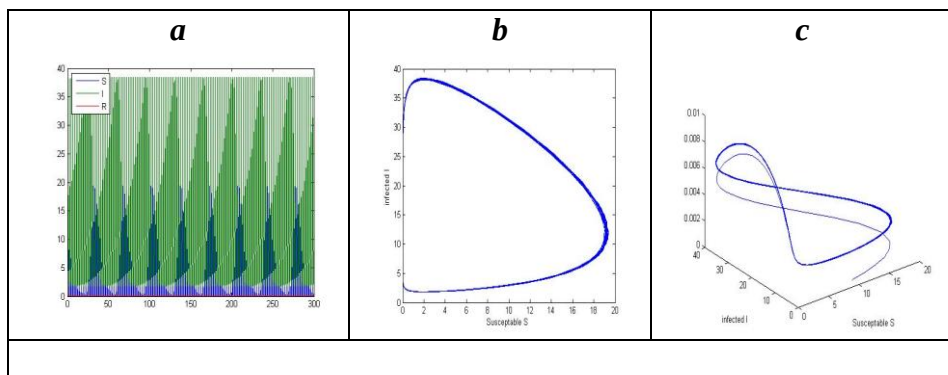


Figure 2: Graphical presentation for $k = 126.98$, $r = 14$, $\beta = 1$, $d = 2$, $\gamma = 0.001$, $\mu = 0.01$, $\alpha = 0.011$, $\beta_1 = 0.01$

Now if we increase the value of k to the critical value $k = k_c = 126.98$ and all other parameter remain unchanged then the endemic equilibrium point is $(2.0443, 12.0292, 0.0060)$ and the corresponding graphical presentation is shown in the figure-2(a), (b) and (c). From the figure-2 it is clear that periodic solution occurs at $k = k_c$. and when we increase the value of k then nature of the solution changes and it becomes chaotic. Thus Hopf bifurcation occurs at $k = k_c$. On the other hand if we set the effect of external infection coefficient $\beta_1 = 0$ then the for $k = 126.98$ the solution pattern becomes chaotic in nature but Hopf bifurcation occurs at $k = k_c = 124.99$. That is introduction of the term β_1 increases the stability range.

6. Conclusions

In this paper considered the SIR model with logistic growth rate of susceptible and effect of loss of immunity of the recovered class and the inhibitory effect on the incidence and infection of susceptible through interaction and by external sources is also taken into consideration. Here three equilibrium points obtained. One of them is disease free and two are endemic equilibrium points. But the number of endemic equilibrium point becomes some time one depending on the parameters. The solution in the neighborhood of disease free equilibrium point $A_0(0,0,0)$ is always unstable in nature. The solution in the neighborhood of endemic equilibrium point of trajectories under goes a Hopf-bifurcation depending on the values of

carrying capacity k . The range of stability zone increases with the introduction of the external infection term.

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