

Certain Study on Eigen Vectors of Bi-Complex Matrices*

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Abstract: Theory of matrices is an integral part of algebra as well as theory of equations. There are so many applications of matrices, for this reason we thought of studying bi-complex matrices. The monograph by Price¹ contains few exercises pertaining to matrices with bi-complex entries. In this paper we discussed the method to find out eigen vectors of bi-complex matrices and also proved a result on eigen vectors of bi-complex matrices.

Keywords: Bi-complex matrices, Bi-complex polynomial, eigen vector, Fundamental Theorem of Bi-complex Algebra.

Symbols: C_0 : set of real numbers, C_1 : set of complex numbers, C_2 : set of bi-complex numbers.

1. Introduction

1.1 . Definition of Bi-Complex Matrix:

Let $A = [\xi_{mn}]_{m \times n}$ be a bi-complex matrix that is a matrix having bi-complex number entries.

$$A = \begin{bmatrix} \xi_{11} & \cdots & \xi_{1n} \\ \cdots & \cdots & \cdots \\ \xi_{1m} & \cdots & \xi_{mn} \end{bmatrix} \quad \xi_{pq} \in C_2, \quad 1 \leq p \leq m \quad \& \quad 1 \leq q \leq n .$$

Or

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$$A = \begin{bmatrix} z_{11} + i_2 w_{11} & z_{12} + i_2 w_{12} & \dots & \dots & z_{1n} + i_2 w_{1n} \\ z_{21} + i_2 w_{21} & z_{22} + i_2 w_{22} & \dots & \dots & z_{2n} + i_2 w_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ z_{m1} + i_2 w_{m1} & z_{m2} + i_2 w_{m2} & \dots & \dots & z_{mn} + i_2 w_{mn} \end{bmatrix}$$

$$A = \begin{bmatrix} z_{11} & z_{12} & \dots & \dots & z_{1n} \\ z_{21} & z_{22} & \dots & \dots & z_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ z_{m1} & z_{m2} & \dots & \dots & z_{mn} \end{bmatrix} + i_2 \begin{bmatrix} w_{11} & w_{12} & \dots & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & \dots & w_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & \dots & w_{mn} \end{bmatrix},$$

where z_{mn} & $w_{mn} \in C_1$.

Or

$$A = \begin{bmatrix} x_{11} & x_{12} & \dots & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & \dots & x_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & \dots & x_{mn} \end{bmatrix} + i_1 \begin{bmatrix} y_{11} & y_{12} & \dots & \dots & y_{1n} \\ y_{21} & y_{22} & \dots & \dots & y_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ y_{m1} & y_{m2} & \dots & \dots & y_{mn} \end{bmatrix}$$

$$+ i_2 \begin{bmatrix} u_{11} & u_{12} & \dots & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & \dots & u_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ u_{m1} & u_{m2} & \dots & \dots & u_{mn} \end{bmatrix} + i_1 i_2 \begin{bmatrix} v_{11} & v_{12} & \dots & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & \dots & v_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ v_{m1} & v_{m2} & \dots & \dots & v_{mn} \end{bmatrix},$$

where x_{mn}, y_{mn}, u_{mn} & $v_{mn} \in C_0$ and $z_{pq} = x_{pq} + i_1 y_{pq}; w_{pq} = u_{pq} + i_1 v_{pq}$.

Every bi-complex matrix $A = [\xi_{mn}]_{m \times n}$ can be expressed uniquely as (In terms of Idempotent components):

$${}^1Ae_1 + {}^2Ae_2,$$

where ${}^1A = [{}^1\xi_{mn}]$ and ${}^2A = [{}^2\xi_{mn}]$.

The idempotent representation of any bi-complex number is given by Srivastava².

1.2 . Algebraic Structure of Bi-Complex Matrices:

Let S be the set of all square matrices of order $n \times n$ define the binary composition of addition, real scalar multiplication & multiplication as follows:

If

$$A = \begin{bmatrix} \xi_{11} & \xi_{12} & \dots & \dots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \dots & \dots & \xi_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \xi_{n1} & \xi_{n2} & \dots & \dots & \xi_{nn} \end{bmatrix} \quad \& \quad B = \begin{bmatrix} \eta_{11} & \eta_{12} & \dots & \dots & \eta_{1n} \\ \eta_{21} & \eta_{22} & \dots & \dots & \eta_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \eta_{n1} & \eta_{n2} & \dots & \dots & \eta_{nn} \end{bmatrix}$$

are arbitrary member of S then

$$A + B = \begin{bmatrix} \xi_{11} + \eta_{11} & \xi_{12} + \eta_{12} & \dots & \dots & \xi_{1n} + \eta_{1n} \\ \xi_{21} + \eta_{21} & \xi_{22} + \eta_{22} & \dots & \dots & \xi_{2n} + \eta_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \xi_{n1} + \eta_{n1} & \xi_{n2} + \eta_{n2} & \dots & \dots & \xi_{nn} + \eta_{nn} \end{bmatrix},$$

$$\alpha A = \begin{bmatrix} \alpha \xi_{11} & \alpha \xi_{12} & \dots & \dots & \alpha \xi_{1n} \\ \alpha \xi_{21} & \alpha \xi_{22} & \dots & \dots & \alpha \xi_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \alpha \xi_{n1} & \alpha \xi_{n2} & \dots & \dots & \alpha \xi_{nn} \end{bmatrix}$$

and

$$AB = \begin{bmatrix} \xi_{11}\eta_{11} + \dots + \xi_{1n}\eta_{n1} & \xi_{11}\eta_{12} + \dots + \xi_{1n}\eta_{n2} & \dots & \xi_{11}\eta_{1n} + \dots + \xi_{1n}\eta_{nn} \\ \xi_{21}\eta_{11} + \dots + \xi_{2n}\eta_{n1} & \xi_{21}\eta_{12} + \dots + \xi_{2n}\eta_{n2} & \dots & \xi_{21}\eta_{1n} + \dots + \xi_{2n}\eta_{nn} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \xi_{n1}\eta_{11} + \dots + \xi_{nn}\eta_{n1} & \xi_{n1}\eta_{12} + \dots + \xi_{nn}\eta_{n2} & \dots & \xi_{n1}\eta_{1n} + \dots + \xi_{nn}\eta_{nn} \end{bmatrix}$$

with these binary compositions S is an algebra.

1.3. Some Properties of Bi-Complex Matrices:

- (1) If A is any bi-complex square matrix of order n then $\det(A)$ and determinant of transpose A are equal.
- (2) A & B are two bi-complex matrices of order n such that B is obtained from A by interchanging any two row /column of A then $|A| = -|B|$.
- (3) If any one of row /column in a square bi-complex matrix has each element in $O_2 = I_1 \cup I_2$ then matrix will be singular or non-invertible.

Proofs of these results are straight forward³

2. Bi-Complex Polynomial

The polynomial of the form $P(\xi) = \sum_{k=0}^n \alpha_k \xi^k$ where $\alpha_k, \xi_k \in C_2$ is called bi-complex polynomial in C_2 .

Zeros of bi-complex polynomial if $P(\xi_0) = 0$ for some ξ_0 , then we say that ξ_0 is a zero of this bi-complex polynomial $P(\xi_0) = 0$.

Fundamental Theorem of bi-complex algebra:

Theorem 2.1: *A bi-complex polynomial of degree n with non-singular leading coefficient has exactly n^2 roots counted according to their multiplicities.*

Theorem 2.2: *If no two roots of $P(\xi_0) = 0$ are equal then $P(\xi_0)$ can be factored into linear factors in $n!$ essentially different ways¹.*

3. Characteristic Value Problem

Given a square bi-complex matrix A of order n , the problem is how to determine the scalar λ and non-zero vector X which simultaneously satisfy the equation.

$$(3.1) \quad AX = \lambda X$$

This is known as characteristic value problem.

$$\text{Let } A = [\xi_{mn}]_{m \times n} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

Since $\lambda X = \lambda IX$, equation (3.1) can be written as

$$\begin{bmatrix} \xi_{11} & \dots & \dots & \xi_{1n} \\ \xi_{21} & \dots & \dots & \xi_{2n} \\ \dots & \dots & \dots & \dots \\ \xi_{n1} & \dots & \dots & \xi_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix},$$

or

$$\begin{bmatrix} \xi_{11} - \lambda & \dots & \dots & \xi_{1n} \\ \xi_{21} & \xi_{22} - \lambda & \dots & \xi_{2n} \\ \dots & \dots & \dots & \dots \\ \xi_{n1} & \dots & \dots & \xi_{nn} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix},$$

or

$$(3.1) \quad (A - \lambda I)X = 0.$$

This is homogenous system of linear equations whose coefficient matrix is $(A - \lambda I)$. Since a non-zero vector X is required, it is necessary for this coefficient matrix to have determinant equal to zero i.e., $|A - \lambda I| = 0$.

4. Definitions

Characteristic equations:

The equation $|A - \lambda I| = 0$ i.e., $|{}^1A - {}^1\lambda I|e_1 + |{}^2A - {}^2\lambda I|e_2 = 0$, is called characteristic equation of A .

Characteristic matrix:

The matrix $(A - \lambda I) = ({}^1A - {}^1\lambda I)e_1 + ({}^2A - {}^2\lambda I)e_2$, is called characteristic matrix of A .

Characteristic polynomial:

The expansion of determinant $|A - \lambda I|$ yields a polynomial in λ , $P(\lambda)$ which is called characteristic polynomial of matrix A .

Eigen values of bi-complex matrix A :

By fundamental theorem of bi-complex algebra we know that a bi-complex polynomial of degree n has n^2 roots in C_2 so by idempotent combination of roots of $|{}^1A - \lambda I| = 0$ & $|{}^2A - \lambda I| = 0$, we get the roots of bi-complex matrix A and these n^2 roots are called eigen values of bi-complex matrix A .

If no two roots of $P(\lambda) = 0$ are equal then $P(\lambda)$ can be factored into linear factors in $\mathbb{C}_2$ essentially different ways¹.

3.1. Eigen Vectors of a Bi-Complex Matrix A :

Definition: Let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \quad \& \quad A = \begin{bmatrix} \xi_{11} & \dots & \dots & \xi_{1n} \\ \xi_{21} & \dots & \dots & \xi_{2n} \\ \dots & \dots & \dots & \dots \\ \xi_{n1} & \dots & \dots & \xi_{nn} \end{bmatrix}.$$

If λ is an eigen value of A , then $\exists$ a non-zero vector X s.t. $AX = \lambda X$ or $[A - \lambda I]X = 0$ & this equation has non trivial solution for X , every such solution for X (nontrivial) is called eigen vector of A .

3.2. Now we proceed to find eigen vectors of bi-complex matrix A :

Let

$$A = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix},$$

where

$$\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22} \in C_2.$$

In terms of idempotent $A = {}^1Ae_1 + {}^2Ae_2$,

$$\text{i.e.} \quad \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} = \begin{bmatrix} {}^1\xi_{11} & {}^1\xi_{12} \\ {}^1\xi_{21} & {}^1\xi_{22} \end{bmatrix} e_1 + \begin{bmatrix} {}^2\xi_{11} & {}^2\xi_{12} \\ {}^2\xi_{21} & {}^2\xi_{22} \end{bmatrix} e_2,$$

where ${}^1A$ & ${}^2A$ & are complex idempotent matrices of bi-complex matrix A having eigen values ${}^1\lambda_1, {}^1\lambda_2$ & ${}^2\lambda_1, {}^2\lambda_2$ (Dixit⁴) eigen vector corresponding to roots ${}^1\lambda_1$ is obtained by solving

$$(3.3) \quad ({}^1A - {}^1\lambda_1 I) X_1 = 0$$

This is a homogenous system of linear equation; whose coefficient matrix is $({}^1A - {}^1\lambda_1 I)$ and since non zero vector X_1 is required it is necessary for this coefficient matrix to have determinant equal to zero i.e. $|{}^1A - {}^1\lambda_1 I| = 0$ and we obtained non trivial solution for X_1 .

So from (3.3)

$$\begin{bmatrix} {}^1\xi_{11} - {}^1\lambda_1 & {}^1\xi_{12} \\ {}^1\xi_{21} & {}^1\xi_{22} - {}^1\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $|{}^1A - {}^1\lambda_1 I| = 0$, so we obtain system of one linear equation i.e.

$$({}^1\xi_{11} - {}^1\lambda_1) x_1 + {}^1\xi_{12} x_2 = 0$$

Take $x_2 = k_2$, where k_2 arbitrary then $x_1 = \frac{-{}^1\xi_{12} k_2}{({}^1\xi_{11} - {}^1\lambda_1)}$, so

$${}^1X_1 = \begin{bmatrix} \frac{-{}^1\xi_{12} k_2}{({}^1\xi_{11} - {}^1\lambda_1)} \\ k_2 \end{bmatrix}.$$

Similarly, we obtain ${}^1X_2$ corresponding to root ${}^1\lambda_2$ and ${}^2X_1$ & ${}^2X_2$ corresponding to root ${}^2\lambda_1$ & ${}^2\lambda_2$ respectively.

So the eigen vectors of bi-complex matrix A can be obtained by linear combination of these eigen vectors of complex matrices ${}^1A$ & ${}^2A$ i.e.

suppose Y_1, Y_2, Y_3 & Y_4 are representing eigen vectors of bi-complex matrix A then

$$Y_1 = {}^1X_1e_1 + {}^2X_1e_2,$$

$$Y_2 = {}^1X_1e_1 + {}^2X_2e_2,$$

$$Y_3 = {}^1X_2e_1 + {}^2X_1e_2,$$

$$Y_4 = {}^1X_2e_1 + {}^2X_2e_2.$$

3.3. Generalized Result for $n \times n$ Bi-Complex Matrix:

Theorem 3.3: *If μ_1 & μ_2 are two distinct eigen values of an n -square matrix A , and Y_1 and Y_2 are non-singular eigen vectors associated with these eigen values respectively then Y_1 and Y_2 are linearly independent.*

Proof: Since Y_1 & Y_2 are eigen vectors corresponding to the eigen values μ_1 and μ_2 of the matrix A , we have

$$(3.4) \quad AY_1 = \mu_1 Y_1 \text{ \& } AY_2 = \mu_2 Y_2.$$

Let x & y be scalars such that

$$(3.5) \quad xY_1 + yY_2 = 0.$$

Pre-multiplying by A , we obtain

$$x(AY_1) + y(AY_2) = 0,$$

by (3.4), we have

$$x(\mu_1 Y_1) + y(\mu_2 Y_2) = 0,$$

from (3.5) $yY_2 = -xY_1$, so

$$x\mu_1 Y_1 + \mu_2 (-xY_1) = 0,$$

$$\text{i.e.} \quad x(\mu_1 - \mu_2)Y_1 = 0,$$

but $\mu_1 \neq \mu_2$, by hypothesis and since Y_1 is eigen vector so $Y_1 \neq 0$ hence $x = 0$ similarly we have $y = 0$ so from (3.12) it follows that Y_1 & Y_2 are linearly independent.

5. Conclusion

In this paper we have obtained eigen vectors of bi-complex matrices corresponding to distinct and non zero eigen values. Furthermore, we have proved a generalized result on eigen vectors of matrices in C_2 .

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