

Stability of Fractional SEIR Epidemic Model for Ebola Outbreak*

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(Received July 12, 2015)

Abstract: Ebola outbreak became a matter of concern for the World Health Organization as a result of which data about the past and the present outbreaks were made available to the research community. The central concern of this paper is to demonstrate the Ebola epidemic model by dealing with the problem of fractional order SEIR (Susceptible-Exposed-Infected-Removed). The basic reproduction number, a threshold parameter for invasion of a disease into the susceptible population, is obtained for the proposed fractional order SEIR Ebola epidemic by the next generation matrix. The conditions to be satisfied for the stability of the disease-free equilibrium point are discussed using Routh-Hurwitz criterion. The derived theoretical results are validated through numerical simulations.

Keywords: Ebola virus, Epidemiology, Fractional order, Basic reproduction number.

1. Introduction

There have been more than twenty-five known outbreaks of Ebola virus disease in Africa since the disease was first identified in Zaire in 1976 near the Ebola River which is now in Central Africa country, Democratic Republic of the Congo and the virus name has been derived from this location since it was the source for the first outbreak¹. So far five different species of Ebola virus have been identified, namely Zaire Ebola virus [EBOV], Sudan Ebola virus [SUDV], Tai Ebola virus [TAFV], Bundibugyo Ebola virus [BDBV] and Reston Ebola virus [RESTV]². The virus spread through close contact and through contaminated medical equipment. The incubation period of the virus after entering the human body is around 2 to 21 days. The person or animal infected with Ebola virus is not contagious until explicit symptoms appear. The symptoms of Ebola include fever, headache, muscle pain, weakness, diarrhea, vomiting, abdominal pain and

*Presented at ICRTM 2015, University of Allahabad during July 10-12, 2015.

• This research work was supported by DST-SERB Project No. SB/FTP/MS-045/2013.

unexplained hemorrhage fever including bleeding and it often leads to fatal^{3,4}. Ebola haemorrhagic fever is one of the most virulent viral diseases known to human kind, causing death in 50-90 percentage of cases⁵.

Mathematical epidemiology is the building and analysis of mathematical models describing the spread and control of infectious diseases. The model developed in epidemiology provides a wider understanding of mechanisms that influence the spread of a disease and they suggest control^{6,7}. Although, numerous analysis were carried out in modeling the dynamics of epidemiological diseases most of them were restricted to integer order differential equations. In recent years, it turned out many phenomena that different fields can be described successfully by fractional-order differential equations due to its more precise descriptions for real world problem^{8,9}. Modeling of fractional-order differential equations has more advantages than classical integer order modeling, in which the effects of memory or long range interactions are neglected. Fractional order had been successfully applied in the field of biology^{10,11}, physics¹², hydrology¹³, engineering¹⁴ and finance¹⁵. The authors¹⁶ studied the asymptotic stability of disease-free equilibrium and positive fixed points for a fractional order SEIR model with vertical transmission. The author¹⁷ introduced the fractional SEIR model dependent on the population size and the death rate had been studied. The use of fractional models is increased in recent years due to the fact that fractional derivatives and integral enable the description of the memory and hereditary properties inherent in various materials and processes.

In this paper, we put forth the fractional epidemiology model for an Ebola outbreak in West Africa, and by modeling the SEIR model for Ebola we tried to find out the stability nature of the Ebola by deriving the basic reproduction number. The organization of this paper is as follows: In the following section we present the formulation of fractional SEIR Ebola model. In section 3, we derive the basic reproduction number and disease-free equilibrium points. The stability analysis of the formulated model is presented in section 4. Finally, the numerical simulations are provided in the section 5 to validate the derived theoretical predictions.

In this section, we present some definition of fractional derivatives, that is, Riemann-Liouville fractional derivative, Caputo fractional derivative. In this Caputo fractional derivative definition is most commonly used definition because of its accuracy in solutions of real world problems.

Definition 1: *The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(\bullet)$ is the Euler gamma function.

Definition 2: The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$$\begin{aligned} D^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{\alpha-n-1} f(s) ds \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f(s)^{(n)}}{(t-s)^{\alpha+n-1}} f(s) ds, \end{aligned}$$

where the function $f(t)$ have absolutely continuous derivatives up to order $(n-1)$.

2. Mathematical Formulation

We formulate here the system modeling to describe the spread of the disease in human population. In this model the population. In this model the population is supposed to be divided into different classes based on their disease status. At time t , $S(t)$ is the number of people in the susceptible compartment where the people are capable of becoming infected, $E(t)$ is the number of people in the exposed compartment where the people are incubating the infection, $I(t)$ is the number of people in the infected compartment where the people are infected or infectious with the virus and $R(t)$ is the number of people in the removed compartment where the people are consider to be recovered or died. The total population at time t represented by $N(t)$ is considered as a constant and is the sum of the population in the compartments $S(t)$, $E(t)$, $I(t)$ and $R(t)$.

The governing equation describing the evolution and dynamics of SEIR model can be described by a set of differential equations as follows

$$(2.1) \quad \begin{cases} \frac{dS}{dt} = \frac{-\beta S(I+qE)}{N}, \\ \frac{dE}{dt} = \frac{\beta S(I+qE)}{N} - \delta E, \\ \frac{dI}{dt} = \delta E - \gamma I, \\ \frac{dR}{dt} = \gamma I, \end{cases}$$

In the above equation, we assume that $N(t)=S(t)+E(t)+I(t)+R(t)$ is constant and denotes the total number of population in the system, where there is new birth and deaths of people are not considered. The parameter β is transmission rate of disease from susceptible to exposed, q is a weight factor added to the model since it is known that a susceptible individual has higher chance of getting infected from an infectious individual than from a latent individual. Similarly, the parameters $\frac{1}{\delta}$ and $\frac{1}{\gamma}$ are the average durations of incubation and infectiousness respectively.

Fractional order calculus is more suitable than the integer order ones in modeling the epidemiological systems. Therefore, we extend the derivatives to an arbitrary order. Then, the model becomes

$$(2.2) \quad \begin{cases} D^\alpha S(t) = \frac{-\beta S(I+qE)}{N}, \\ D^\alpha E(t) = \frac{\beta S(I+qE)}{N} - \delta E, \\ D^\alpha I(t) = \delta E - \gamma I, \\ D^\alpha R(t) = \gamma I, \end{cases}$$

where $\alpha \in (0, 1]$ is the derivative order and the parameter β, γ, δ, q have the same meaning as in (2.1).

3. Analysis of SEIR Model

Basic Reproduction Number: The basic reproduction number, R_0 is defined as the expected number of secondary infections produced by an individual in a completely susceptible population. If $R_0 < 1$, the transmission chains are not self-sustaining and are unable to generate a major epidemic. On the other hand, if $R_0 > 1$, then the number of infected individuals will increase with each generation and the disease will spread. The basic reproduction number is a threshold parameter for invasion of a disease into the susceptible population. But, R_0 may not be good in measure of disease transmission in case of disease that begun to spread and condition favoring. In many diseases, the final size of the epidemics is measured using the R_0 . It could also be useful for guiding the numbers of antiviral and vaccines that would be needed to achieve control to the disease¹⁸.

There are many methods available to find the R_0 value for a epidemic model such as the survival function, rearranging the largest eigen value of

the Jacobian matrix, the next-generation method, calculations from the intrinsic growth rate, existence of the endemic equilibrium, the average age of infection, the final size equation and so on. Here, in this paper we used *Next-generation method* to calculate the R_0 value¹⁹. In the SEIR model for Ebola, the population was divided into four compartments: susceptible $S(t)$, exposed $E(t)$, infectious $I(t)$ and recovered or death $R(t)$ as in (2.2).

Considering the infected compartments E and I give

$$F = \begin{pmatrix} \frac{\beta S(I+qE)}{N} \\ 0 \end{pmatrix}, \quad V = \begin{pmatrix} \delta E \\ -\delta E + \gamma I \end{pmatrix},$$

Hence

$$F = \begin{pmatrix} \beta q & \beta \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \delta & 0 \\ -\delta & \gamma \end{pmatrix},$$

and the next generation matrix is

$$(3.1) \quad K = FV^{-1} = \begin{pmatrix} \frac{\beta q}{\delta} + \frac{\beta}{\gamma} & \frac{\beta}{\gamma} \\ 0 & 0 \end{pmatrix}.$$

In the next generation method, R_0 is given by $R_0 = \rho(FV^{-1})$, such that there are no other eigen values of K with modulus greater than the R_0 and we have from (3.1)

$$R_0 = \frac{\beta q}{\delta} + \frac{\beta}{\gamma},$$

which can be written as

$$(3.2) \quad R_0 = \frac{\beta}{\gamma} \left(1 + \frac{q\gamma}{\delta} \right).$$

We evaluate the equilibrium points of (2) by letting,

$$(3.3) \quad D^\alpha S(t)=0, D^\alpha E(t)=0, D^\alpha I(t)=0, D^\alpha R(t)=0, \alpha \in (0,1].$$

The disease-free equilibrium point obtained by solving (2.2) using (3.3) is

$$(S^*, E^*, I^*) = \left(\frac{N}{R_0}, 0, 0 \right) = E_1.$$

4. Stability Analysis

4.1. Trivial equilibrium and its Stability:

The stability analysis of the system can be done based upon the standard linearization technique and using Jacobian matrix

$$(4.1) \quad J = \begin{pmatrix} \frac{-\beta q E^* - \beta I^*}{N} & \frac{-\beta S^* q}{N} & \frac{-\beta S^*}{N} \\ \frac{\beta q E^* + \beta I^*}{N} & \frac{\beta S^* q}{N} - \delta & \frac{\beta S^*}{N} \\ 0 & \delta & -\gamma \end{pmatrix}$$

Using (4.1), the characteristic equation at the trivial equilibrium point $E_0 = (0, 0, 0)$ reduces to

$$(4.2) \quad \lambda^\alpha (-\lambda^\alpha - \delta)(-\lambda^\alpha - \gamma)\alpha = 0.$$

Obviously, equation (4.2) has a positive root $\lambda^\alpha = \delta, \gamma$ ($0 < \alpha \leq 1$). Then the trivial equilibrium point E_0 of the system (2.2) is always unstable (Saddle point).

4.2. Disease free equilibrium and its stability:

In this section, we investigate the stability of (2.2) at the disease free equilibrium point $(S^*, E^*, I^*) = \left(\frac{N}{R_0}, 0, 0\right) = E_1$. The corresponding Jacobian matrix at E_1 , point is in the form of the reproduction number R_0 then (4.1) can be written as

$$(4.3) \quad J = \begin{pmatrix} 0 & \frac{-\beta q}{R_0} & \frac{-\beta}{R_0} \\ 0 & \frac{\beta q}{R_0} - \delta & \frac{\beta}{R_0} \\ 0 & \delta & -\gamma \end{pmatrix}$$

This as the characteristic equation

$$(4.4) \quad -\lambda^{3\alpha} + \lambda^{2\alpha}(\alpha q - \delta - \gamma) + \lambda^\alpha(\alpha q \gamma - \delta \gamma - \delta \alpha) = 0,$$

where for our simplicity we have consider $\frac{\beta}{R_0} = \alpha$. From (4.4), we have one root $\lambda^\alpha = 0$, another two roots are obtained from the quadratic equation,

$$(4.5) \quad -\lambda^{2\alpha} + \lambda^\alpha(\alpha q - \delta - \gamma) + (\alpha q \gamma - \delta \gamma - \delta \alpha) = 0.$$

Thus, we have:

Theorem 1: *The disease-free equilibrium E_1 is locally asymptotically stable if $R_0 < 1$ and is unstable $R_0 > 1$.*

Proof: The disease – free equilibrium is locally asymptotically stable if all the eigenvalues, λ_i^α ($i=1, 2, 3$) of the Jacobian matrix satisfy the condition.

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}.$$

The eigenvalues of Jacobian matrix are $\lambda_i = 0$ and the other two roots determined by the equation (4.5), where $\alpha = \frac{\beta}{R_0}$ and $R_0 = \frac{\beta}{\gamma} \left(1 + \frac{q\gamma}{\delta}\right)$. Hence, the disease-free equilibrium E_1 is locally asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$.

Proposition 1: Consider the following three dimensional fractional order systems,

$$D^\alpha u = f(u),$$

where $\alpha \in (0, 1]$ and let u^* is an equilibrium point of the above system, then its characteristic polynomial is given as

$$P(\lambda) = -\lambda^{3\alpha} + \lambda^{2\alpha} P_1 + \lambda^\alpha P_2 = 0,$$

where $P_1(\alpha q - \delta - \gamma)$, $P_2(\alpha q \gamma - \delta \gamma - \delta \alpha) = 0$ and $P_3 = 0$.

Its discriminant is defined as,

$$K(P) = (P_1 P_2)^2 - 4(P_2)^2.$$

According to Routh- Hurwitz conditions for fractional order of Ebola system

- (i) If $K(P) > 0$, then the necessary and sufficient condition for the equilibrium point E_1 to be locally asymptotically stable is $P_1 > 0, P_1 P_2 > P_3$, which implies $P_1 P_2 > 0$ as $P_3 = 0$.
- (ii) If $K(P) < 0, P_1 > 0, P_2 > 0$ and $\alpha \in [0, 1)$, then E_1 is locally asymptotically stable.
- (iii) If $K(P) < 0, P_1 > 0, P_2 < 0$ and $\alpha > 2/3$, then E_1 is unstable.

5. Numerical Simulation

Since, most of the fractional order differential equations do not have exact analytic solutions, approximation and numerical methods can be used to show the result. In this section, we provide numerical simulations to validate the stability results obtained in the previous section. We have calculated the numerical experiments for the system with different sets of parameter value and also we have selected different arbitrary value for α .

We consider the parametric value and by using that parametric value we calculate the basic reproduction number R_0 to see the effect of the disease in the society. The parameter values are

$$(5.1) \quad q=0.035, \beta=0.7704, N=118805278, \delta=\frac{1}{12}, \gamma=\frac{1}{7}.$$

Moreover, the initial conditions are fixed as

$$S(0)=N\frac{m}{100}, E(0)=0, I(0)=15, R(0)=0,$$

where m is fixed as 75, that is, the susceptible population is a percentage of the total population N . For the parametric values given in (5.1), the R_0 value we obtained is 0.1123. When the value of $\alpha=1$ for the system and for the parametric value (5.1), the population size of susceptible compartment slightly decreases from as the time increases the susceptible case absolutely decreases. This indicates that the rate of the susceptible cases is either reduced or which is moved to the exposed compartment or to the infected compartment. But, the exposed and infected compartment also decline. This implies that there is no transmission in between the compartments but the population of susceptible, exposed and infected fall off in their respective compartment itself. The removed size has been elevated, which is presented

in Figure (1). This symbolize that numerically, when $R_0 < 1$, the disease died out, but here by our calculation we obtain R_0 value 0.1123 which is less than one. In this case, the disease spreading is reduced and the result showed that the susceptible, exposed and infected populations were trimmed while the recovered population increased.

We consider the same parametric values and by changing the values of α as $\alpha = 0.95$ and $\alpha = 0.85$ again we obtain in Figure (2) and Figure (3), clearly the same product that the disease died out but at different points. We adopt the parametric values so that R_0 is greater than one. The values to be

$$(5.2) \quad q=0.065, \beta=0.7704, N=118805278, \delta=\frac{1}{12}, \gamma=\frac{1}{7}.$$

Here, we have modified the q value that is we have increased the q value from 0.035 to 0.065. Where q is said to be the contact rate. The main factor for the disease spreading is the direct contact with the infected persons as here we have raised the direct contact rate which also equally increases the disease spreading rate in the humanity. Hence, by using these parameter values (5.2) we have $R_0 = 5.9937$. The Figure (3), highlights the variation in the susceptible compartments due to the increase in the contact rate and likewise it explains the change when $R_0 > 1$, for different values α . As in previous case we consider values of α as 1, 0.95 and 0.85.

6. Conclusion

In this paper, we have introduced the SEIR model in the fractional order for the Ebola. We found the value of R_0 to explain the behavior of the disease and we also studied the stable at the trivial and disease-free equilibrium points and showed the characteristic with the help of Jacobian method. We have seen the nature of the basic reproduction number, it is unstable when $R_0 > 1$ and stable when $R_0 < 1$. The numerical simulations showed the efficiency of result which we have derived analytically. It is clear that the spread of infectious disease for two reasons, first, the disease was a lethal, tenacious and another was a fear and misunderstanding that fuelled high-risk behaviors. Also, we have shown in numerical, if the contact is high the disease spreading is also high, this clearly shows the disease is spread through the direct contact between the populations.

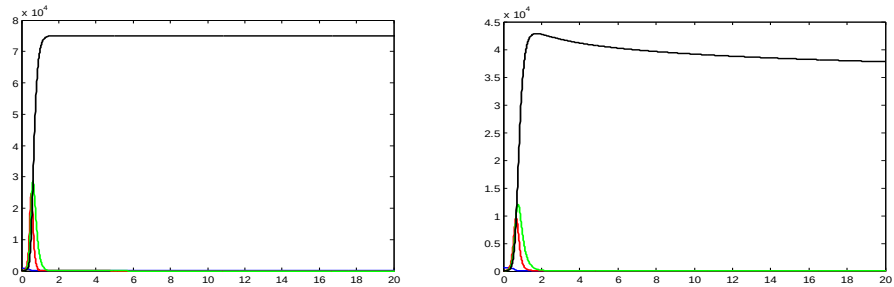


Figure 1: The susceptible, exposed, infected and removed population with the value $\alpha = 1$ and $R_0 < 1$, The (Right) Figure is susceptible, exposed, infected and removed population with the value $\alpha = 0.95$ and $R_0 < 1$.

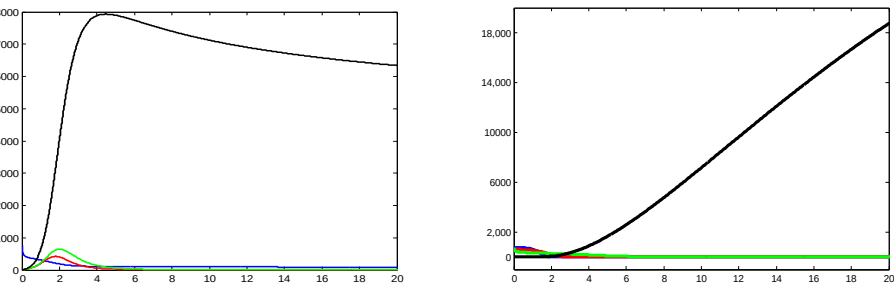


Figure 2: The susceptible, exposed, infected and removed population with the value $\alpha = 0.85$ and $R_0 < 1$. The (Right) Figure is susceptible, exposed, infected and removed population with the value $\alpha = 1$, $q=0.065$ and $R_0 > 1$.

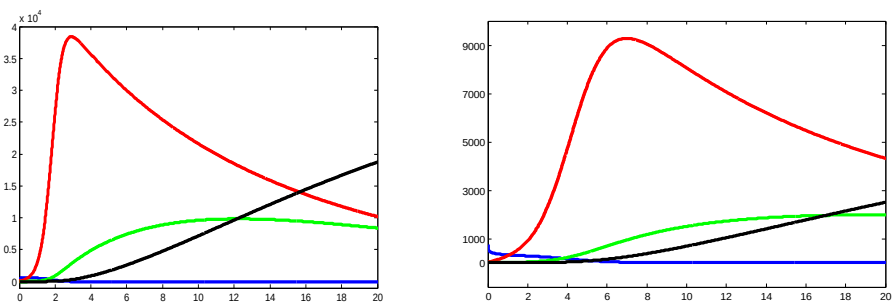


Figure 3: The susceptible, exposed, infected and removed population with the value $\alpha = 0.95$, $q=0.065$ and $R_0 > 1$. The (Right) Figure is susceptible, exposed, infected and removed population with the value $\alpha = 0.95$, $q=0.065$ and $R_0 > 1$.

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