

On Semi-Symmetric Non-Metric Connections in a Co-Symplectic Manifold

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Abstract: The object of the present paper is to study semi -symmetric non-metric connection and semi-symmetric non-metric T-connection in a co-symplectic manifold.

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1. Introduction

In 1924, A. Friedmann and J. A. Schouten introduced the idea of semi-symmetric connection on a differentiable manifold. In 1932, H. A. Hayden studied semi-symmetric metric connection on a Riemannian manifold and it was further studied by K.Yano¹, G. Pathak and U. C. De², U. C. De and S. C. Biswas³, R. N. Singh and K. P. Pandey⁴, R. N. Singh and M. K. Pandey⁵, L. Das, U. C. De, R. N. Singh and M. K. Pandey⁶ and others. Nirmala S. Agashe and Mangala R. Chafle⁷ have studied semi-symmetric non-metric connection on a Riemannian manifold in 1992 and studied some of its properties. In this paper, we studied the properties of semi-symmetric non-metric connection and semi-symmetric non-metric T-connection in a co-symplectic manifold.

2. Preliminaries

An n -dimensional differentiable manifold M^n is an almost contact manifold, if it admits a tensor field F of type $(1, 1)$, a vector field U and 1-form u satisfying

$$(2.1) \quad (i) \quad \overline{X} + X = u(X)U, \quad (ii) \quad \overline{U} = 0,$$

where $\bar{X} = FX$. Equation (2.1) (i) and (2.1) (ii) gives

$$(2.2) \quad (i) u(\bar{X}) = 0, \quad (ii) u(U) = 1,$$

where X is any vector field.

An almost contact manifold M^n in which a Riemannian metric g of type (0, 2) satisfies

$$(2.3) \quad (i) g(\bar{X}, \bar{Y}) = g(X, Y) - u(X)u(Y), \quad (ii) g(X, U) = u(X),$$

for arbitrary vector fields X and Y , is called an almost contact metric manifold⁸. The 2-form 'F' is defined by

$$'F(X, Y) = g(\bar{X}, Y),$$

satisfies

$$(2.4) \quad (i) 'F(X, Y) = g(\bar{X}, Y) = -g(X, \bar{Y}) = -'F(Y, X),$$

$$(ii) 'F(\bar{X}, \bar{Y}) = 'F(X, Y).$$

An almost contact manifold satisfying the conditions

$$(2.5) \quad (\nabla_X F) = 0 \Leftrightarrow (\nabla_X 'F)(Y, Z) = 0,$$

is called a co-symplectic manifold⁸.

From equations (2.1) and (2.5) we can easily prove that

$$(2.6) \quad (\nabla_X u)(Y) = 0.$$

An almost contact metric manifold M^n for which

$$(2.7) \quad (\nabla_X F)(Y) = 0 \Leftrightarrow (\nabla_X F)Y + (\nabla_Y F)X = 0,$$

is satisfied, is called a nearly co-symplectic manifold⁸.

An affine connection ∇^* on (M^n, g) is semi-symmetric non-metric connection if the torsion tensor T^* of the connection ∇^* and the metric g of the manifold satisfies the following conditions⁹

$$(2.8) \quad T^*(X, Y) = u(X)Y - u(Y)X,$$

$$(2.9) \quad (\nabla_X^* g)(Y, Z) = 2\{u(Y)g(X, Z) + u(Z)g(X, Y)\},$$

for arbitrary vector fields X, Y and Z .

The relation between the semi-symmetric non-metric connection ∇^* and Riemannian connection ∇ is given by⁹

$$(2.10) \quad \nabla^*_X Y = \nabla_X Y - u(Y)X - g(X, Y)U.$$

3. Some Theorems on a Co-Symplectic Manifold with Connection ∇^*

In a co-symplectic manifold M^n with a semi-symmetric non-metric connection ∇^* given by equation (2.10), we define

$$(3.1) \quad \begin{aligned} (i) \quad & 'F(Y, Z) = g(\bar{Y}, Z), \\ (ii) \quad & 'T^*(X, Y, Z) = g(T^*(X, Y), Z), \\ (iii) \quad & H(X, Y) = -u(Y)X - g(X, Y)U, \\ (iv) \quad & 'H(X, Y, Z) = g(H(X, Y), Z). \end{aligned}$$

Theorem 3.1 *A co-symplectic manifold M^n equipped with a semi-symmetric non-metric connection ∇^* satisfies the relations*

$$(i) \quad (\nabla^*_{\bar{X}} F)\bar{Y} = (\nabla^*_X F)Y - u(Y)\bar{X}, \\ (ii) \quad (\nabla^*_X F)(Y) = 0 \text{ if and only if } 'F(Y, X)U = u(Y)\bar{X}.$$

Proof: In view of equation (3.1)(iii) equation (2.10) becomes

$$\nabla^*_X Y = \nabla_X Y + H(X, Y).$$

From equation (2.10), we have

$$(3.2) \quad \nabla^*_X \bar{Y} = \nabla_X \bar{Y} - g(X, \bar{Y})U,$$

which gives

$$(3.3) \quad (\nabla^*_X F)Y = (\nabla_X F)Y + (\overline{\nabla_X \bar{Y}}) - g(X, \bar{Y})U - (\overline{\nabla^*_X \bar{Y}}).$$

Operating F on both sides of the equation (2.10) and using (2.1)(ii), we get

$$(3.4) \quad (\overline{\nabla^*_X \bar{Y}}) = (\overline{\nabla_X \bar{Y}}) - u(Y)\bar{X}.$$

Now using equations (2.5) and (3.4) in equation (3.3), we get

$$(3.5) \quad (\nabla^*_X F)(Y) = u(Y)\bar{X} - g(X, \bar{Y})U.$$

Barring X and Y in equation (3.5) and using (2.1)(i), we get

$$(3.6) \quad (\nabla^*_{\bar{X}} F)(\bar{Y}) = g(\bar{X}, Y)U.$$

From equations (3.5) and (3.6), we have

$$(\nabla^*_{\bar{X}} F)(\bar{Y}) = (\nabla^*_X F)Y - u(Y)\bar{X}.$$

Again from Equation (3.5), it is clear that $(\nabla^*_X F)(Y) = 0$ if and only if $u(Y)\bar{X} - g(X, \bar{Y})U = 0$, i.e. $F(Y, X)U = u(Y)\bar{X}$. This completes the proof.

Theorem 3.2 *A co-symplectic manifold M^n with semi-symmetric non-metric connection ∇^* satisfies the relation*

$$d^*F(X, Y, Z) = 2[u(X)g(Y, \bar{Z}) + u(Y)g(Z, \bar{X}) + u(Z)g(X, \bar{Y})].$$

Proof: Differentiating equation (3.1)(i) covariantly with respect to ∇^* , we have

$$(3.7) \quad (\nabla^*_X F)(Y, Z) = (\nabla^*_X g)(\bar{Y}, Z) + g((\nabla^*_X F)Y, Z).$$

Using equations (2.9) and (3.5) in above equation, we obtain

$$(3.8) \quad (\nabla^*_X F)(Y, Z) = u(Z)g(X, \bar{Y}) + u(Y)g(\bar{X}, Z).$$

Define

$$(3.9) \quad d^*F(X, Y, Z) = (\nabla^*_X F)(Y, Z) + (\nabla^*_Y F)(Z, X) + (\nabla^*_Z F)(X, Y).$$

Thus from equations (3.8) and (3.9), we have

$$d^*F(X, Y, Z) = 2[u(X)g(Y, \bar{Z}) + u(Y)g(Z, \bar{X}) + u(Z)g(X, \bar{Y})].$$

This completes the proof.

Theorem 3.3 *In a co-symplectic manifold M^n , the semi-symmetric non-metric connection given by equation (2.10) satisfies the conditions*

- (i) $(\nabla^*_X F)(Y, \bar{Z}) = 0$,
- (ii) $(\nabla^*_X F)(\bar{Y}, Z) = [u(X)u(Y) - g(X, Y)]u(Z)$,
- (iii) $(\nabla^*_{\bar{X}} F)(\bar{Y}, Z) + (\nabla^*_{\bar{Y}} F)(\bar{X}, Z) = 0$,
- (iv) $(\nabla^*_{\bar{Y}} F)(\bar{X}, Z) - (\nabla^*_{\bar{Z}} F)(X, Y) = T^*(X, Y, Z)$.

Proof: (i) Differentiating equation (3.1)(i) covariantly with respect to connection ∇^* , we have

$$\begin{aligned} (\nabla^*_X F)(Y, Z) + F(\nabla^*_X Y, Z) + F(Y, \nabla^*_X Z) \\ = (\nabla^*_X g)(\bar{Y}, Z) + g(\nabla^*_X \bar{Y}, Z) + g(\bar{Y}, \nabla^*_X Z). \end{aligned}$$

Using equations (2.5), (2.9) and (2.10) in above equation, we have

$$(3.10) \quad (\nabla^*_X F)(Y, Z) = u(Z)g(X, \bar{Y}).$$

Barring Z in above equation and using (2.2)(i), we get

$$(\nabla^*_X F)(Y, \bar{Z}) = 0.$$

(ii) Barring Y in equation (3.10) and using (2.1)(i), we get

$$(3.11) \quad (\nabla^*_X F)(\bar{Y}, Z) = [u(X)u(Y) - g(X, Y)]u(Z).$$

(iii) Again barring X in equation (3.11) and using equation (2.2)(i), we have

$$(3.12) \quad (\nabla^*_{\bar{X}} F)(\bar{Y}, Z) = u(Z)g(X, \bar{Y}).$$

Interchanging X and Y in above equation, we get

$$(3.13) \quad (\nabla^*_{\bar{Y}} F)(\bar{X}, Z) = u(Z)g(Y, \bar{X}).$$

Now from equations (3.12), (3.13) and (2.4)(i), we have

$$(\nabla^*_{\bar{X}} F)(\bar{Y}, Z) + (\nabla^*_{\bar{Y}} F)(\bar{X}, Z) = 0.$$

(iv) Barring X in equation (3.10) and using equation (2.3)(i), we have

$$(3.14) \quad (\nabla^*_{\bar{X}} F)(Y, Z) = u(Z)[g(X, Y) - u(X)u(Y)].$$

Writing two more equations by cyclic permutations of X , Y and Z from equation (3.14), we get

$$(\nabla^*_{\bar{Y}} F)(Z, X) = u(X)[g(Y, Z) - u(Y)u(Z)]$$

and

$$(\nabla^*_{\bar{Z}} F)(X, Y) = u(Y)[g(Z, X) - u(Z)u(X)].$$

$$\begin{aligned} \text{Thus } (\nabla^*_{\bar{Y}} F)(\bar{X}, Z) - (\nabla^*_{\bar{Z}} F)(X, Y) &= u(X)g(Y, Z) - u(Y)g(X, Z) \\ &= T^*(X, Y, Z). \end{aligned}$$

This completes the proof.

Theorem 3.4 *A co-symplectic manifold M^n with a semi-symmetric non-metric connection ∇^* satisfies the condition $(\nabla_X^* u)(\bar{Y}) = -F(X, Y)$.*

Proof: We have

$$u(Y) = g(Y, U).$$

Taking covariant derivative of above equation with respected to ∇^* , we have

$$(\nabla_X^* u)Y + u(\nabla_X^* Y) = (\nabla_X^* g)(Y, U) + g(\nabla_X^* Y, U) + g(Y, \nabla_X^* U),$$

which on using equations (2.9) and (2.10), reduces to

$$(4.15) \quad (\nabla_X^* u)(Y) = u(X)u(Y) + g(X, Y).$$

Now barring Y in above equation and using equation (2.2)(i), we obtain

$$(\nabla_X^* u)(\bar{Y}) = -g(\bar{X}, Y) = -F(X, Y).$$

This completes the proof.

4. Nijenhuis Tensor

The Nijenhuis tensor N^* in terms of semi-symmetric non-metric connection ∇^* is given by⁸

$$(4.1) \quad N^*(X, Y) = (\nabla_{\bar{X}}^* F)Y - (\nabla_{\bar{Y}}^* F)X - \overline{(\nabla_X^* F)Y} + \overline{(\nabla_Y^* F)X}.$$

Theorem 4.1 *A co-symplectic manifold M^n equipped with semi-symmetric non-metric connection ∇^* is integrable.*

Proof: Barring X in equation (3.5) and using equations (2.1)(i) and (2.3)(i), we get

$$(4.2) \quad (\nabla_{\bar{X}}^* F)(Y) = 2u(X)u(Y)U - u(Y)X - g(X, Y)U.$$

Interchanging X and Y in above equation, we get

$$(4.3) \quad (\nabla_{\bar{Y}}^* F)(X) = 2u(X)u(Y)U - u(X)Y - g(X, Y)U.$$

Operating F on both sides of equation (3.5) and using equation (2.1), we get

$$(4.4) \quad \overline{(\nabla_X^* F)(Y)} = -u(Y)X + u(X)u(Y)U.$$

Interchanging X and Y in above equation, we obtain

$$(4.5) \quad \overline{(\nabla_y^* F)(X)} = -u(X)Y + u(Y)u(X)U.$$

Using equations (4.2), (4.3), (4.4) and (4.5) in equation (4.1), we get

$$N^*(X, Y) = 0,$$

which shows that M^n is integrable.

This completes the proof.

5. Contravariant Almost Analytic Vector Fields on a Co-Symplectic Manifolds

A vector field V is said to be contravariant almost analytic, if the Lie-derivative of F with respect to V vanishes identically, i.e.

$$(5.1) \quad (L_V F)X = 0,$$

for all X . The equation (5.1) is equivalent to the equation

$$[V, \bar{X}] = \overline{[V, X]}.$$

In a co-symplectic manifold, the above equation becomes

$$\nabla_{\bar{X}} V - \overline{\nabla_X V} = 0 \text{ if and only if } \overline{\nabla_X V} + \nabla_X V = 0.$$

Theorem 5.1 *If a vector field V is a contravariant almost analytic with respect to Riemannian connection ∇ on M^n equipped with semi-symmetric non-metric connection ∇^* , then*

$$\nabla_{\bar{X}}^* V - \overline{\nabla_X^* V} = -g(\bar{X}, V)U = -F(X, V)U.$$

Proof: From equation (2.10), we have

$$(5.2) \quad \nabla_X^* V = \nabla_X V - u(V)X - g(X, V)U,$$

for any vector field V . Barring X in above equation, we get

$$(5.3) \quad \nabla_{\bar{X}}^* V = \nabla_{\bar{X}} V - u(V)\bar{X} - g(\bar{X}, V)U.$$

Operating F on both side of equation (5.2), we have

$$(5.4) \quad \overline{\nabla^*_X V} = \overline{\nabla_X V} - u(V) \overline{X}.$$

Subtracting equation (5.4) from equation (5.3), we get

$$(5.5) \quad \nabla^*_{\overline{X}} V - \overline{\nabla^*_X V} = \nabla_{\overline{X}} V - \overline{\nabla_X V} - g(\overline{X}, V)U,$$

since V is contravariant almost analytic with respect to connection ∇ , we have

$$\nabla_{\overline{X}} V - \overline{\nabla_X V} = 0 \Rightarrow \nabla^*_{\overline{X}} V - \overline{\nabla^*_X V} = -g(\overline{X}, V)U = -F(X, V)U.$$

This completes the proof.

6. Curvature Tensor of Semi-Symmetric Non-Metric Connection ∇^*

Let R^* be the curvature tensor of co-symplectic manifold M^n with respect to semi-symmetric non-metric connection given by equation (2.10), then we have

$$(6.1) \quad R^*(X, Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X, Y]} Z.$$

From equations (2.6), (2.10) and (6.1), we have

$$(6.2) \quad R^*(X, Y, Z) = R(X, Y, Z) + g(Y, Z)X - g(X, Z)Y + u(Y)u(Z)X - u(X)u(Z)Y + [u(X)g(Y, Z) - u(Y)g(X, Z)]U.$$

From above equation, we have

$$(6.3) \quad \begin{aligned} 'R^*(X, Y, Z, W) = & 'R(X, Y, Z, W) + g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ & + u(Y)u(Z)g(X, W) - u(X)u(Z)g(Y, W) + u(X)u(W)g(Y, Z) \\ & - u(Y)u(W)g(X, Z), \end{aligned}$$

where $'R^*(X, Y, Z, W) = g(R^*(X, Y, Z), W)$ and $'R(X, Y, Z, W) = g(R(X, Y, Z), W)$.

Putting $X = W = e_i$ in equation (6.3) and taking summation over i , $1 \leq i \leq n$, we get

$$(6.4) \quad \text{Ric}^*(Y, Z) = \text{Ric}(Y, Z) + ng(Y, Z) + (n-2)u(Y)u(Z).$$

From above equation, we have

$$(6.5) \quad r^* = r + (n-1)(n+2).$$

Projective curvature tensor P and Conformal curvature tensor C of the manifold M^n are given by⁸

$$(6.6) \quad P(X, Y, Z) = R(X, Y, Z) - \frac{1}{(n-1)} [\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y]$$

and

$$(6.7) \quad C(X, Y, Z) = R(X, Y, Z) - \frac{1}{(n-2)} [\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y].$$

Divergence of the curvature tensor of a Riemannian manifold M^n is given by⁸

$$(6.8) \quad (\text{div}R)(X, Y, Z) = (\nabla_X \text{Ric})(Y, Z) - (\nabla_Y \text{Ric})(X, Z).$$

A manifold M^n is called R-harmonic¹⁰ if

$$(6.9) \quad (\text{div}R)(X, Y, Z) = 0.$$

Divergence of the conformal curvature tensor C is given by¹⁰

$$(6.10) \quad (\text{div}C)(X, Y, Z) = \frac{(n-3)}{(n-2)} [\nabla_X \text{Ric}(Y, Z) - (\nabla_Y \text{Ric})(X, Z)] - \frac{1}{2(n-1)} [g(Y, Z)\text{dr}(X) - g(X, Z)\text{dr}(Y)].$$

A Riemannian manifold M^n is said to be conformally conservative¹⁰, if

$$(6.11) \quad (\text{div}C)(X, Y, Z) = 0.$$

Theorem 6.1 *In a co-symplectic manifold M^n , Bianchi's first identity with respect to semi-symmetric non-metric connection ∇^* is given by*

$$R^*(X, Y, Z) + R^*(Y, Z, X) + R^*(Z, X, Y) = 0.$$

Proof: Writing two more equations by cyclic permutations of X, Y and Z from the equation (6.2) and adding them to equation (6.2), we get the desired result.

Theorem 6.2 *A co-symplectic manifold M^n , with semi-symmetric non-metric connection ∇^* , satisfies the relations*

- (i) $'R^*(X, Y, Z, W) + 'R^*(Y, X, Z, W) = 0,$
- (ii) $'R^*(X, Y, Z, W) + 'R^*(X, Y, W, Z) = 0,$
- (iii) $'R^*(X, Y, Z, W) = 'R^*(Z, W, X, Y).$

Proof:

- (i) Interchanging X and Y in equation (6.3) and adding it to equation (6.3), we obtain required result.
- (ii) Interchanging Z and W in equation (6.3) and adding it to equation (6.3), we have the required result.
- (iii) Interchanging the pair of slots in equation (6.3) and using the fact that $'R(X, Y, Z, W) = 'R(Z, W, X, Y)$, we get the required result.

Theorem 6.3 *In a co-symplectic manifold M^n the Ricci tensor of semi-symmetric non-metric connection ∇^* is symmetric.*

Proof: Interchanging Y and Z in equation (6.4), we get

$$\begin{aligned} \text{Ric}^*(Z, Y) &= \text{Ric}(Z, Y) + \text{ng}(Z, Y) + (n-2)u(Y)u(Z) \\ &= \text{Ric}^*(Y, Z). \end{aligned}$$

Theorem 6.4 *If the Ricci tensor of the semi-symmetric non-metric connection ∇^* in a co-symplectic manifold M^n vanishes, then the manifold is quasi-Einstein manifold.*

Proof: If $\text{Ric}^*(Y, Z) = 0$ then from equation (6.4), we have

$$\begin{aligned} \text{Ric}(Y, Z) &= -\text{ng}(Y, Z) - (n-2)u(Y)u(Z) \\ &= a g(Y, Z) + b u(Y)u(Z), \end{aligned}$$

where $a = -n$ and $b = -(n-2)$,

which shows that manifold is quasi-Einstein manifold.

Theorem 6.5 *If the Ricci tensor of the semi-symmetric non-metric connection ∇^* in a co-symplectic manifold M^n vanishes, then the manifold is R-harmonic and conformally conservative.*

Proof: If $\text{Ric}^*(Y, Z) = 0$ then from equation (6.4), we have

$$\text{Ric}(Y, Z) = -ng(Y, Z) - (n-2)u(Y)u(Z),$$

from which, we get

$$(6.12) \quad (\nabla_X \text{Ric})(Y, Z) = -n(\nabla_X g)(Y, Z) - (n-2)(\nabla_X u)(Y)u(Z) - (n-2)u(Y)(\nabla_X u)(Z).$$

Using equation (2.6) in above equation, we get

$$(6.13) \quad (\nabla_X \text{Ric})(Y, Z) = 0.$$

Also from equation (6.5) and the condition $\text{Ric}^*(Y, Z) = 0$, we get

$$(6.14) \quad dr(X) = 0.$$

Using equations (6.13) and (6.14) in equations (6.8) and (6.10), we get the required result.

7. Projective Curvature Tensor of the Semi-Symmetric Non-Metric Connection ∇^*

Projective curvature tensor of co-symplectic manifold with respect to semi-symmetric non-metric connection ∇^* is given by

$$(7.1) \quad P^*(X, Y, Z) = R^*(X, Y, Z) - \frac{1}{(n-1)}[\text{Ric}^*(Y, Z)X - \text{Ric}^*(X, Z)Y].$$

Using equations (6.2) and (6.4) in above equation, we get

$$(7.2) \quad P^*(X, Y, Z) = P(X, Y, Z) - \frac{1}{(n-1)}[g(Y, Z)X - g(X, Z)Y - u(Y)u(Z)X + u(X)u(Z)Y] + [u(X)g(Y, Z) - u(Y)g(X, Z)]U,$$

where $P(X, Y, Z)$ is the projective curvature tensor of the co-symplectic manifold M^n with respect to the Levi-Civita connection ∇ .

Theorem 7.1 *In a co-symplectic manifold M^n with semi-symmetric non-metric connection ∇^* , we have*

$$P^*(X, Y, Z) + P^*(Y, Z, X) + P^*(Z, X, Y) = P(X, Y, Z) + P(Y, Z, X) + P(Z, X, Y).$$

Proof: Writing two more equations by cyclic permutations of X, Y, Z from equation (7.2) and adding them to equation (7.2), we get the required result.

Theorem 7.2 *In a co-symplectic manifold M^n , if the curvature tensor of the semi-symmetric non-metric connection is equal to the Weyl-projective curvature tensor of the Levi-Civita connection, then the manifold is quasi-Einstein manifold.*

Proof: Let $R^*(X, Y, Z) = P(X, Y, Z)$, which by using equations (6.2) and (6.6), gives

$$\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y = (n-1)[\{g(X, Z)Y - g(Y, Z)X + u(X)u(Z)Y - u(Y)u(Z)X\} + \{u(Y)g(X, Z) - u(X)g(Y, Z)\}U],$$

which gives

$$\begin{aligned} \text{Ric}(Y, Z)g(X, W) - \text{Ric}(X, Z)g(Y, W) &= (n-1)[\{g(X, Z)g(Y, W) \\ &- g(Y, Z)g(X, W) + u(X)u(Z)g(Y, W) - u(Y)u(Z)g(X, W)\} + \{u(Y)g(X, Z) \\ &- u(X)g(Y, Z)\}u(W)]. \end{aligned}$$

Now putting $X = W = e_i$ in above equation and taking summation over i , $1 \leq i \leq n$, we get

$$\text{Ric}(Y, Z) = -ng(Y, Z) - (n-2)u(Y)u(Z),$$

which shows that M^n is a quasi-Einstein manifold.

7. Conformal Curvature Tensor of the Semi-Symmetric Non-Metric Connection ∇^*

Conformal curvature tensor C^* of the co-symplectic manifold relative to semi-symmetric non-metric connection ∇^* is given by

$$\begin{aligned} (8.1) \quad C^*(X, Y, Z) &= R^*(X, Y, Z) - \frac{1}{(n-2)} [\text{Ric}^*(Y, Z)X - \text{Ric}^*(X, Z)Y + \\ &g(Y, Z)Q^*X - g(X, Z)Q^*Y] + \frac{r^*}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y]. \end{aligned}$$

Using equations (6.2), (6.4) and (6.5) in above equation, we get

$$\begin{aligned} (8.2) \quad C^*(X, Y, Z) &= C(X, Y, Z) - \frac{(n+2)}{(n-2)} [g(Y, Z)X - g(X, Z)Y] \\ &+ [u(X)g(Y, Z) - u(Y)g(X, Z)]U. \end{aligned}$$

Theorem 8.1 *A co-symplectic manifold M^n with semi-symmetric non-metric connection ∇^* satisfies the condition*

$$C^*(X, Y, Z) + C^*(Y, Z, X) + C^*(Z, X, Y) = C(X, Y, Z) + C(Y, Z, X) + C(Z, X, Y).$$

Proof: Writing two more equations by the cyclic permutations of X, Y and Z from equation (8.2) and adding them to equation (8.2), we get the required result.

Theorem 8.2 *In a co-symplectic manifold M^n , if the curvature tensor of the semi-symmetric non-metric connection is equal to the conformal curvature tensor of the Levi-Civita connection, then M^n is quasi-Einstein manifold.*

Proof: Let $R^*(X, Y, Z) = C(X, Y, Z)$, then

$$(8.3) \quad 'R^*(X, Y, Z, W) = 'C(X, Y, Z, W),$$

where $'C(X, Y, Z, W) = g(C(X, Y, Z), W)$.

Using equations (6.3) and (6.7) in equation (8.3), we get

$$(8.4) \quad g(Y, Z)g(X, W) - g(X, Z)g(Y, W) + [u(Y)g(X, W) - u(X)g(Y, W)]u(Z) \\ + [u(X)g(Y, Z) - u(Y)g(X, Z)]u(W) = -\frac{1}{(n-2)} [\text{Ric}(Y, Z)g(X, W) \\ - \text{Ric}(X, Z)g(Y, W) + \text{Ric}(X, W)g(Y, Z) - \text{Ric}(Y, W)g(X, Z)] \\ + \frac{r}{(n-1)(n-2)} [u(Y)g(X, W) - u(X)g(Y, W)]u(Z).$$

Now, putting $X = W = e_i$ in above equation and taking summation over i , $1 \leq i \leq n$, we get

$$\text{Ric}(Y, Z) = -ng(Y, Z) - (n-2)u(Y)u(Z),$$

which shows that M^n is quasi-Einstein manifold.

8. Semi-Symmetric Non-Metric T-Connection

If in addition to equations (2.8) and (2.9)

$$(9.1) \quad \nabla_X^* U = 0 \text{ or } (\nabla_X^* u)(Y) = 0,$$

hold for arbitrary vector fields X and Y, then the connection ∇^* is said to be semi-symmetric non-metric T-connection.

Also from equations (2.10) and (9.1), we have

$$(9.2) \quad \nabla_X^* U + X - u(X)U = 0 \Leftrightarrow g(X, Y) + u(X)u(Y) = 0.$$

From equation (6.2), we have

$$R^*(X, Y, Z) = R(X, Y, Z) + [u(Y)u(Z) + g(Y, Z)](X) - [u(X)u(Z) + g(X, Z)](Y) + [g(Y, Z)u(X) - g(X, Z)u(Y)]U.$$

In view of equation (3.14), above equation reduces to

$$(9.3) \quad R^*(X, Y, Z) = R(X, Y, Z) + (\nabla_Y^* u)(Z)(X) - (\nabla_X^* u)(Z)(Y) + [g(Y, Z)u(X) - g(X, Z)u(Y)]U.$$

From equations (9.1) and (9.3), we have

$$(9.4) \quad R^*(X, Y, Z) = R(X, Y, Z) + [g(Y, Z)u(X) - g(X, Z)u(Y)]U.$$

Contracting above equation with respect to X, we get

$$(9.5) \quad \text{Ric}^*(Y, Z) = \text{Ric}(Y, Z) + g(Y, Z) - u(Y)u(Z),$$

from which, we have

$$(9.6) \quad r^* = r + (n-1).$$

Equations (9.4), (9.5) and (9.6) denote the curvature tensor, Ricci tensor and scalar curvature respectively with respect to semi-symmetric non-metric T-connection.

Theorem 9.1 *In a co-symplectic manifold M^n , if the curvature tensor of the semi-symmetric non-metric T-connection is equal to the Weyl-projective curvature tensor of the Levi-Civita connection, then M^n is quasi-Einstein manifold.*

Proof: Let $R^*(X, Y, Z) = P(X, Y, Z)$.

Using equations (9.4) and (6.6) in above equation, we get

$$\begin{aligned} R(X, Y, Z) + [g(Y, Z)u(X) - g(X, Z)u(Y)]U \\ = R(X, Y, Z) - \frac{1}{(n-1)}[\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y], \end{aligned}$$

which gives

$$\begin{aligned} \text{Ric}(Y, Z)g(X, W) - \text{Ric}(X, Z)g(Y, W) \\ = -(n-1)[g(Y, Z)u(X) - g(X, Z)u(Y)]u(W). \end{aligned}$$

Putting $X = W = e_i$ in above equation and taking summation over i , $1 \leq i \leq n$, we get

$\text{Ric}(Y, Z) = -g(Y, Z) + u(Y)u(Z)$,
which shows that M^n is quasi-Einstein manifold.

Theorem 9.2 *In a co-symplectic manifold M^n , if the curvature tensor of the semi-symmetric non-metric T-connection is equal to the conformal curvature tensor of the Levi-Civita connection, then M^n is quasi-Einstein manifold.*

Proof: Let $R^*(X, Y, Z) = C(X, Y, Z)$, then

$$(9.7) \quad 'R^*(X, Y, Z, W) = 'C(X, Y, Z, W),$$

where $'C(X, Y, Z, W) = g(C(X, Y, Z), W)$.

Using equations (6.7) and (9.4) in above equation, we get

$$\begin{aligned} [u(X)g(Y, Z) - u(Y)g(X, Z)] u(W) = & -\frac{1}{(n-2)} [\text{Ric}(Y, Z)g(X, W) \\ & - \text{Ric}(X, Z)g(Y, W) + \text{Ric}(X, W)g(Y, Z) - \text{Ric}(Y, W)g(X, Z)] \\ & + \frac{r}{(n-1)(n-2)} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned}$$

Now, putting $X = W = e_i$ in above equation and taking summation over i , $1 \leq i \leq n$, we get

$$\text{Ric}(Y, Z) = -g(Y, Z) + u(Y)u(Z),$$

which shows that M^n is quasi-Einstein manifold.

Theorem 9.3 *In a co-symplectic manifold M^n , the Ricci tensor of the semi-symmetric non-metric T-connection is symmetric.*

Proof: Interchanging Y and Z in equation (9.5), we get

$$(9.8) \quad \text{Ric}^*(Z, Y) = \text{Ric}(Y, Z) + g(Y, Z) - u(Z)u(Y).$$

Subtracting equation (9.8) from equation (9.5) and using the fact that

$\text{Ric}(Y, Z) = \text{Ric}(Z, Y)$, we get the desired result.

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