

On a Class of Paracontact Metric 3-Manifold

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Abstract: The purpose of this paper is to study a paracontact metric 3-manifold whose Ricci operator commutes with the endomorphism of its tangent bundle.

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1. Introduction

Almost paracontact metric manifolds are the well known illustrations of almost para CR manifolds. In 1985, Kaneyuki and Kozai¹ defined the almost paracontact structure on pseudo-Riemannian manifold M^{2n+1} and constructed the almost paracomplex structure on $M^{2n+1} \times R$. If the paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is η -Einstein we do not have a widespread classification. This paper is organized as follows. In §2, we present some technical apparatus which is needed for further investigations. In §3, we consider a paracontact metric manifold M^3 such that $S\phi = \phi S$, and first derive that the function $\text{Tr} \ell$ is constant everywhere on M^3 . We then show that the conditions, (i) the structure is η -Einstein, (ii) Ricci operator S commutes with tensor field ϕ and (iii) ξ belongs to the k -nullity distribution of pseudo-Riemannian metric g are equivalent on M^3 . Finally, in §4, we prove that the unit torse forming vector field is concircular in this class of manifold.

2. Preliminaries

A C^∞ smooth manifold M^{2n+1} of dimension $(2n+1)$, is said to have a triplet (ϕ, ξ, η) structure, if it admits a $(1, 1)$ tensor field ϕ , a unique vector field ξ called the characteristic vector field or Reeb vector field and a 1-form η satisfying:

$$(2.1) \quad \phi^2 = I - \eta \otimes \xi \text{ and } \eta(\xi) = 1,$$

where I is the identity transformation. The endomorphism ϕ induces an almost paracomplex structure on each fibre of $D = \ker(\eta)$, the contact subbundle that is the eigen distributions $D^{\pm 1}$ corresponding to the characteristics values $\pm$ of ϕ have equal dimension n .

From the equation (2.1), it can be easily deduce that

$$(2.2) \quad \phi\xi = 0, \quad \eta\phi = 0 \text{ and } \text{rank}(\phi) = 2n.$$

This triplet structure (ϕ, ξ, η) is called an almost paracontact structure and the manifold M^{2n+1} equipped with the (ϕ, ξ, η) -structure is called an almost paracontact manifold^{1, 2} (see ³ for almost contact Riemannian case and ^{4, 5} for almost paracontact Riemannian case). If an almost paracontact manifold admits a pseudo-Riemannian metric^{6, 7}, g satisfying:

$$(2.3) \quad g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

where signature of g is necessarily $(n+1, n)$ for any vector fields X and Y . Then the quadruple- (ϕ, ξ, η, g) is called an almost paracontact metric structure and the manifold M^{2n+1} equipped with paracontact metric structure is called an almost paracontact metric manifold. With respect to g , η is metrically dual to ξ , that is

$$(2.4) \quad g(X, \xi) = \eta(X).$$

Also, equation (2.3) implies that

$$(2.5) \quad g(\phi X, Y) = -g(X, \phi Y).$$

Note: The above metric g is, of course, not unique.

Further, in addition to the above properties, if the quadruple (ϕ, ξ, η, g) satisfies:

$$d\eta(X, Y) = g(X, \phi Y),$$

for all vector fields X, Y on M^{2n+1} , then the manifold is called a paracontact metric manifold and the corresponding structure (ϕ, ξ, η, g) is called a paracontact structure with the associated metric⁷.

Now, for an almost paracontact metric manifold, there always exists a special kind of local pseudo-orthonormal basis $\{X_i, X_{i^*}, \xi\}$; where $X_{i^*} = \phi X_i$; ξ and X_i 's are space-like vector fields and X_{i^*} 's are time-like. Such a basis is called a ϕ -basis. Indicating by L and R , the Lie differentiation operator and the curvature tensor of M^{2n+1} respectively, let us define, $(1, 1)$ type tensor fields h and 1 , which are symmetric as well, by $h = \left(\frac{1}{2}\right)L_\xi \phi$, $1 = R(., \xi)\xi$. The basic properties followed by h and 1 are:

$$(2.6) \quad h\xi = 0, \quad l\xi = 0, \quad \eta oh = 0, \quad \text{Tr. } h = \text{Tr. } \phi h = 0,$$

and

$$(2.7) \quad h\phi = -\phi h \text{ (i.e., } h \text{ anti-commutes with } \phi \text{)}.$$

Also, $hX = \lambda X \Rightarrow h\phi X = -\lambda\phi X$, i.e., if λ is an eigen value of h with the corresponding eigen vector X , then $-\lambda$ is also an eigen value of h corresponding to the eigen vector ϕX . If S denotes the Ricci operator and ∇ denotes the Levi-Civita connection of the metric g , then using the above properties of h and 1 , we can easily calculate the following formulas for a paracontact metric manifold M^{2n+1} :

$$(2.8) \quad \nabla_X \xi = -\phi X + \phi hX, \quad \nabla_\xi \xi = 0,$$

$$(2.9) \quad \nabla_\xi \phi = 0,$$

$$(2.10) \quad \text{Tr} l = g(S\xi, \xi) = \text{Tr} h^2 - 2n,$$

$$(2.11) \quad \phi l \phi + l = 2(h^2 - \phi^2),$$

$$(2.12) \quad \nabla_{\xi} h = -\phi - \phi l + h^2 \phi.$$

If the Reeb vector field ξ is Killing, i.e., equivalently $h=0$, then the paracontact metric manifold M^{2n+1} is called a K -paracontact manifold⁷. On a 3-dimensional pseudoRiemannian manifold, since the conformal curvature tensor vanishes identically, therefore the curvature tensor R takes the form⁸

$$(2.13) \quad R(X, Y)Z = g(Y, Z)SX - g(X, Z)SY + g(SY, Z)X \\ - g(SX, Z)Y - \frac{r}{2}\{g(Y, Z)X - g(X, Z)Y\},$$

where r is the scalar curvature of the manifold and the Ricci operator S is defined by

$$(2.14) \quad g(SX, Y) = \text{Ric}(X, Y),$$

where Ric is the Ricci tensor.

3. k -Nullity Distribution

In contact geometry, the notion of k -nullity distribution is introduced by Tanno⁹. The k -nullity distribution of a Riemannian manifold (M, g) , for a real number k , is a distribution

$$(3.1) \quad N(k): p \rightarrow N_p(k) \\ = \left[Z \in \Gamma(T_p M) : R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\} \right],$$

for any $X, Y \in \Gamma(T_p M)$, where R and $\Gamma(T_p M)$ respectively denotes the curvature tensor and the tangent vector space of M^{2n+1} at any point $p \in M$. If the characteristic vector field ξ of a paracontact metric manifold belongs to the k -nullity distribution then the following relation holds⁹

$$(3.2) \quad R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y).$$

Definition 3.1: A paracontact metric manifold is said to be η -Einstein⁶, if the Ricci operator S can be written in the following form:

$$(3.3) \quad S = aI + b\eta \otimes \xi,$$

where a and b are some functions.

Proposition 3.1: Let $M^3(\phi, \xi, \eta, g)$ be a paracontact metric manifold with $S\phi = \phi S$, then we have

$$\begin{aligned} (i) \quad & \phi l = l\phi. \\ (ii) \quad & l = h^2 - \phi^2. \\ (iii) \quad & \xi \text{Tr} l = 0. \end{aligned}$$

Proof: From Eqs. (2.1), (2.2), (2.10), (2.13) and $S\phi = \phi S$, we find

$$(3.4) \quad \phi l = l\phi.$$

By the virtue of Eqs. (2.11), (2.12) and (3.4), we obtain that

$$(3.5) \quad l = h^2 - \phi^2$$

and

$$\begin{aligned} \nabla_{\xi} h &= -\phi + h^2\phi - \phi l = -\phi + h^2\phi - \phi(h^2 - \phi^2) \\ &= -\phi + h^2\phi - \phi h h + \phi^3 = h^2\phi + h\phi h \end{aligned}$$

that is,

$$(3.6) \quad \nabla_{\xi} h = 0.$$

Differentiating Eq. (3.5) with respect to ξ and using Eqs. (2.9) and (3.6), we find that

$$(3.7) \quad \nabla_{\xi} l = 0 \Rightarrow \xi \text{Tr} l = 0.$$

This ends the proof.

Theorem 3.1: Let $M^3(\phi, \xi, \eta, g)$ be a paracontact metric manifold with $S\phi = \phi S$. Then $\text{Tr} l = \text{constant}$ everywhere on M^3 .

Proof: By virtue of Eqs. (2.1), (2.2), (2.10) and $S\phi = \phi S$, we have

$$(3.8) \quad S\xi = \text{Tr}l(\xi).$$

From equation (2.13), using the definition of l and Eq. (3.8), we have for any X

$$\begin{aligned} lX = R(X, \xi)\xi &= g(\xi, \xi)SX - g(X, \xi)S\xi + g(S\xi, \xi)X \\ &\quad - g(SX, \xi)\xi - \left(\frac{r}{2}\right)(g(\xi, \xi)X - g(X, \xi)\xi), \end{aligned}$$

which gives

$$(3.9) \quad lX = SX + (\text{Tr}l - r/2)X + \eta(X)(r/2 - 2\text{Tr}l)\xi.$$

Since, $\eta(SX)\xi = \eta(X)(\text{Tr}l)\xi$. As a matter of fact if at a point $p \in M^3$ there exists $X \in \Gamma(T_p M^3)$ such that $lX = 0$ for X, ξ , then $l = 0$ at p . So, let us assume that $l \neq 0$ on a neighbourhood of a point p . From Eqs. (2.5) and (3.4), we get

$$g(\phi X, lX) = 0.$$

Therefore, lX is parallel to X for any X orthogonal to ξ . Now putting $lX = fX$ for any X orthogonal to ξ , where f is any smooth function. Taking a local pseudo-orthonormal frame $\{e_1, e_2, \xi\}$, where $-g(e_1, e_1) = g(e_2, e_2) = g(\xi, \xi) = 1$. Then, by definition

$$(3.10) \quad \text{Tr}l = \sum_{i=1}^3 g(e_i, e_i)g(le_i, e_i) = 2f.$$

Thus for any X , we can write

$$(3.11) \quad lX = f\phi^2 X.$$

In light of Proposition (3.1) and Eq. (3.10), we have

$$(3.12) \quad \xi f = 0.$$

With the help of Eqs. (2.1), (3.9), (3.10) and (3.11), we find

$$(3.13) \quad SX = aX + b\eta(X)\xi,$$

Where

$$(3.14) \quad a = \frac{r}{2} - f, \quad b = 3f - \frac{r}{2}.$$

From the second identity of Bianchi, we get

$$(3.15) \quad (\nabla_X R)(Y, \xi, Z) + (\nabla_Y R)(\xi, X, Z) + (\nabla_\xi R)(X, Y, Z) = 0.$$

Employing Eq. (3.13) in Eq. (2.13), we have

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)(aX + b\eta(X)\xi) - g(X, Z)(aY + b\eta(Y)\xi) \\ &\quad + g(aY + b\eta(Y)\xi, Z)X - g(aX + b\eta(X)\xi, Z)Y \\ &\quad - (r/2)(g(Y, Z)X - g(X, Z)Y) \\ &= ag(Y, Z)X + b\eta(X)g(Y, Z)\xi - ag(X, Z)Y - b\eta(Y)g(X, Z)\xi \\ &\quad + ag(Y, Z)X + b\eta(Y)g(\xi, Z)X - ag(X, Z)Y - b\eta(X)g(\xi, Z)Y \\ &\quad - (r/2)(g(Y, Z)X - g(X, Z)Y) \\ &= \{(2a - r/2)g(Y, Z)X + b\eta(Y)\eta(Z)X\} \\ &\quad - \{(2a - r/2)g(X, Z)Y + b\eta(X)\eta(Z)Y\} \\ &\quad + b\eta(X)g(Y, Z)\xi - b\eta(Y)g(X, Z)\xi \end{aligned}$$

that is,

$$(3.16) \quad R(X, Y)Z = \{(\gamma g(Y, Z) + b\eta(Y)\eta(Z))X - (\gamma g(X, Z) + b\eta(X)\eta(Z))Y\} \\ + b\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\}\xi,$$

where $\gamma = 2a - r/2$. From Eq. (3.16) for $Z = \xi$, we get

$$\begin{aligned} R(X, Y)\xi &= \{\gamma\eta(Y) + b\eta(Y)\}X - \{\gamma\eta(X) + b\eta(X)\}Y \\ &\quad + b\{\eta(X)\eta(Y) - \eta(Y)\eta(X)\}\xi \\ &= (\gamma + b)(\eta(Y)X - \eta(X)Y). \end{aligned}$$

Switching the values of b and γ in the above equation, we have

$$(3.17) \quad R(X, Y)\xi = f(\eta(Y)X - \eta(X)Y).$$

Using Eq. (3.17), we obtain

$$(3.18) \quad R(Y, \xi)\xi = f(Y - \eta(Y)\xi),$$

$$(3.19) \quad R(\nabla_X Y, \xi)\xi = f(\nabla_X Y - \eta(\nabla_X Y)\xi),$$

$$(3.20) \quad R(Y, \nabla_X \xi)\xi = f(\eta(\nabla_X \xi)Y - \eta(Y)\nabla_X \xi),$$

$$(3.21) \quad \begin{aligned} R(Y, \xi)\nabla_X \xi = & \left\{ \gamma\eta(\nabla_X \xi) + b\eta(\nabla_X \xi) \right\} Y \\ & - \left\{ \gamma g(Y, \nabla_X \xi) + b\eta(Y)\eta(\nabla_X \xi) \right\} \xi \\ & + b \left\{ \eta(Y)\eta(\nabla_X \xi) - g(Y, \nabla_X \xi) \right\} \xi. \end{aligned}$$

From Eq. (3.18), we also have

$$(3.22) \quad \begin{aligned} (\nabla_X R)(Y, \xi, \xi) + R(\nabla_X Y, \xi)\xi + R(Y, \nabla_X \xi)\xi + R(Y, \xi)\nabla_X \xi \\ = (Xf)(Y - \eta(Y)\xi) + f \left\{ \nabla_X Y - ((\nabla_X \eta)Y)\xi - \eta(\nabla_X Y)\xi - \eta(Y)\nabla_X \xi \right\}. \end{aligned}$$

Hence for X, Y orthogonal to ξ , we get from Eqs. (3.19)-(3.22) that

$$(\nabla_X R)(Y, \xi, \xi) + f(\nabla_X Y) = (Xf)Y + f(\nabla_X Y),$$

this implies that

$$(3.23) \quad (\nabla_X R)(Y, \xi, \xi) = (Xf)Y.$$

Applying $Z = \xi$ in Eq. (3.15), we get

$$(3.24) \quad (\nabla_X R)(Y, \xi, \xi) + (\nabla_Y R)(\xi, X, \xi) + (\nabla_\xi R)(X, Y, \xi) = 0.$$

From Eq. (3.17), we can write

$$\begin{aligned}
 (3.25) \quad & (\nabla_{\xi} R)(X, Y, \xi) + R(\nabla_{\xi} X, Y)\xi + R(X, \nabla_{\xi} Y)\xi + R(X, Y)\nabla_{\xi} \xi \\
 & = (\xi f)\{\eta(Y)X - \eta(X)Y\} + f\left\{\left((\nabla_{\xi} \eta)Y\right)X - \left((\nabla_{\xi} \eta)X\right)Y\right\} \\
 & + f\left\{\eta(\nabla_{\xi} Y)X + \eta(Y)\nabla_{\xi} X - \eta(\nabla_{\xi} X)Y - \eta(X)\nabla_{\xi} Y\right\}.
 \end{aligned}$$

Using Eqs. (2.8) and (3.12) in Eq. (3.25), we get

$$\begin{aligned}
 & (\nabla_{\xi} R)(X, Y, \xi) + R(\nabla_{\xi} X, Y, \xi) + R(X, \nabla_{\xi} Y, \xi) \\
 & = f\left\{\eta(Y)\nabla_{\xi} X - \eta(\nabla_{\xi} X)Y + \eta(\nabla_{\xi} Y)X - \eta(X)\nabla_{\xi} Y\right\} \\
 & + f\left\{\left((\nabla_{\xi} \eta)Y\right)X - \left((\nabla_{\xi} \eta)X\right)Y\right\} \\
 & = R(\nabla_{\xi} X, Y, \xi) + R(X, \nabla_{\xi} Y, \xi) \\
 & + f\left(g(Y, \nabla_{\xi} \xi)X - g(X, \nabla_{\xi} \xi)Y\right).
 \end{aligned}$$

that is

$$(3.26) \quad (\nabla_{\xi} R)(X, Y, \xi) = 0.$$

From Eqs. (3.24) and (3.26), we find that $(\nabla_X R)(Y, \xi, \xi) = (\nabla_Y R)(X, \xi, \xi)$ and by the use of (3.23) this implies, $(Xf)Y = (Yf)X$, for X and Y orthogonal to ξ . Therefore, $Xf = 0$, for X orthogonal to ξ , but from (3.12) we have $\xi f = 0$, so the function f is constant everywhere on M^3 . Therefore by virtue of Eq. (3.10), we have $Trl = \text{constant}$ everywhere on M^3 . This completes the proof of the theorem.

Theorem 3.2: If $M^3(\phi, \xi, \eta, g)$ be a paracontact metric manifold, then the following conditions are equivalent:

- (i) M^3 is η -Einstein.
- (ii) $S\phi = \phi S$.
- (iii) ξ belongs to the k -nullity distribution.

Proof: (i) $\Rightarrow$ (ii).

Let M^3 be η -Einstein that is, $S = aI + b\eta \otimes \xi$. Therefore

$$S\phi = a\phi + b(\eta \circ \phi) \otimes \xi = a\phi.$$

Also, $\phi S = a\phi + b\eta \otimes \phi\xi = a\phi$. Hence $S\phi = \phi S$.

(ii) $\Rightarrow$ (iii). Let $S\phi = \phi S$. Then, we have from Eqs. (3.10) and (3.17) that

$$R(X, Y)\xi = \frac{Trl}{2}(\eta(Y)X - \eta(X)Y).$$

By the theorem 3.1 we have $Trl = \text{constant}$. Take $Trl = k/2$ then

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y),$$

this implies that ξ belongs to the k -nullity distribution.

(iii) $\Rightarrow$ (i). Let ξ belongs to the k -nullity distribution. Then,

$$(3.27) \quad R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y),$$

where k is a constant. Contracting above expression with respect to X , we have

$$Ric(Y, \xi) = k(3\eta(Y) - \eta(Y)) = 2k\eta(Y),$$

that is, $S\xi = 2k\xi$ and so from Eq. (2.13), we find

$$(3.28) \quad R(X, Y)\xi = \eta(Y)SX - \eta(X)SY + (2k - r/2)(\eta(Y)X - \eta(X)Y).$$

Comparing Eqs. (3.27) and (3.28), we get

$$\eta(Y)\{SX + (k - r/2)X\} = \eta(X)\{SY + (k - r/2)Y\}.$$

Taking Y orthogonal to ξ and $X = \xi$, we have

$$SY = (r/2 - k)Y,$$

and so for any Z ,

$$SZ = (r/2 - k)Z + (3k - r/2)\eta(Z)\xi.$$

This implies that $S = aI + b\eta \otimes \xi$, where $a = r/2 - k$ and $b = 3k - r/2$.

Therefore M^3 is η -Einstein. This completes the proof of the theorem.

Example 3.1:¹⁰ Let us consider a Lie group $\mathbb{E}$ of dimension three which is real and connected, and consider its Lie algebra σ with the basis $\{\psi_j\}_{j=1}^3$ of left invariant vector fields. Then σ is determined by:

$$(3.29) \quad [\psi_1, \psi_2] = \delta\psi_3, [\psi_1, \psi_3] = \theta\psi_2, [\psi_2, \psi_3] = \theta\psi_1, \quad \delta \neq 0.$$

Now, we define (ϕ, ξ, η, g) -structure on $\mathbb{E}$ by:

$$(3.30) \quad \begin{cases} \phi\psi_1 = \psi_2, \phi\psi_2 = \psi_1, \\ \phi\psi_3 = 0, \xi = \psi_3, \eta(\psi_3) = 1, \\ g(\psi_1, \psi_1) = -g(\psi_2, \psi_2) = g(\psi_3, \psi_3) = 1, \\ g(\psi_i, \psi_j) = 0, \forall i \neq j \in \{1, 2, 3\}. \end{cases}$$

Then $(\mathbb{E}, \phi, \xi, \eta, g)$ is a paracontact metric 3-manifolds with $\delta = -2$. For the Levi-Civita connection ∇ with respect to this left invariant metric g , we have

$$(3.31) \quad \begin{cases} \nabla_{\psi_1}\psi_1 = 0, \nabla_{\psi_1}\psi_2 = -\psi_3, \nabla_{\psi_1}\psi_3 = -\psi_2 \\ \nabla_{\psi_2}\psi_1 = \psi_3, \nabla_{\psi_2}\psi_2 = 0, \nabla_{\psi_2}\psi_3 = -\psi_1, \\ \nabla_{\psi_3}\psi_1 = -(\theta+1)\psi_2, \nabla_{\psi_3}\psi_2 = -(\theta+1)\psi_1, \\ \nabla_{\psi_3}\psi_3 = 0. \end{cases}$$

Here the condition $S\phi = \phi S \Leftrightarrow \theta = 0$. Further, it can be easily seen that

$$(3.32) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

and

$$(3.33) \quad S(X, Y) = 2(g(X, Y) - 2\eta(X)\eta(Y)).$$

4. Torse-Forming Vector Fields

Definition 4.1: A vector field U on a pseudo-Riemannian manifold M defined by $g(X, U) = u(X)$ for any $X \in \Gamma(TM)$ is said to be torse-forming vector field¹¹ (see also¹²⁻¹⁴) if it satisfies

$$(4.1) \quad (\nabla_X u)(Y) = sg(X, Y) + \alpha(X)u(Y),$$

where s and α are called the conformal scalar and the generating form of U , respectively.

A torse forming vector field U is called

- (i) recurrent if $s = 0$,
- (ii) parallel if $s = 0$ and $\alpha = 0$,
- (iii) concurrent if $s = 1$ and $\alpha = 0$,
- (iv) concircular if the generating form α is a gradient,
- (v) convergent if it is concircular and $s = \text{const. exp}(\alpha)$.

Theorem 4.1: Let $M^3(\phi, \xi, \eta, g)$ be a paracontact metric manifold with $S\phi = \phi S$. Then the unit torse forming vector field in M^3 is concircular.

Proof: For a unit torse forming vector field $\hat{U}$ corresponding to U , if we define $g(X, \hat{U}) = v(X)$, then

$$(4.2) \quad v(X) = u(X) / \sqrt{|u(U)|}.$$

From Eqs. (4.1) and (4.2), we have

$$(4.3) \quad (\nabla_X v)(Y) = \mu g(X, Y) + \alpha(X)v(Y),$$

where $\mu = s / \sqrt{|u(U)|}$. Using $Y = \hat{U}$ and $v(\hat{U}) = 1$, Eq. (4.3) gives

$$(4.4) \quad \alpha(X) = -\mu v(X),$$

and hence Eq. (4.3) can be expressed in the following form

$$(4.5) \quad (\nabla_X v)(Y) = \mu \{g(X, Y) - v(X)v(Y)\},$$

which shows that v is closed. Now differentiating Eq. (4.5) covariantly and using the Ricci identity, we obtain

$$(4.6) \quad \begin{aligned} v(R(X, Y)Z) &= (Y\mu)\{g(X, Z) - v(X)v(Z)\} \\ &\quad - (X\mu)\{g(Y, Z) - v(Y)v(Z)\} + \mu^2\{g(X, Z)v(Y) - g(Y, Z)v(X)\}. \end{aligned}$$

By the use of Eqs. (2.4), (3.13), (3.14), (3.17) and (4.2), we have from Eq. (4.6)

$$(4.7) \quad \{\eta(X) - \eta(\hat{U})v(X)\}\{f + (\hat{U}\mu) + \mu^2\} = 0,$$

which gives

$$\begin{cases} (I) \quad \eta(X) - \eta(\hat{U})v(X) = 0, \\ (II) \quad f + (\hat{U}\mu) + \mu^2 = 0. \end{cases}$$

If Eq. (I) holds, then putting $X = \xi$ in Eq. (I), we have $\eta(\hat{U}) = \pm 1$. This implies that

$$(4.8) \quad \eta(X) = \pm v(X).$$

From Eqs. (2.8), (4.5) and (4.8), we have $\mu = \pm c$ (constant). Therefore $\alpha(X) = \mp cv(X)$. Hence the vector field $\hat{U}$ is concircular. If Eq. (II) holds, then $\{\eta(X) - \eta(\hat{U})v(X)\} \neq 0$. Then from Eq. (4.6), we have

$$(4.9) \quad -g(SX, \hat{U}) = (X\mu) + (\hat{U}\mu)v(X) + 2\mu^2v(X).$$

Putting $X = \xi$ and using Eq. (3.8) in Eq. (4.9), we have

$$(4.10) \quad \xi\mu = -(\mu^2 + f)\eta(\hat{U}).$$

In view of Eq. (4.10) and $v(\xi) = \eta(\hat{U})$, Eq. (4.6) yields for $Y = Z = \xi$,

$$(4.11) \quad X\mu = -(\mu^2 + f)v(X).$$

From Eqs. (4.4) and (4.11), we get

$$(a) \quad Y\alpha(X) = (\mu^2 + f)v(X)v(Y) - \mu(Yv(X)).$$

We can also obtain

$$(b) \quad X\alpha(Y) = (\mu^2 + f)v(Y)v(X) - \mu(Xv(Y))$$

and

$$(4.12) \quad \alpha([X, Y]) = -\mu v([X, Y]).$$

By the consequences of Eqs. (a), (b) and (4.12), we have

$$d\alpha(X, Y) = -\mu((dv)(X, Y)).$$

This implies that $\hat{U}$ is concircular. This completes the proof of the theorem.

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