

Modulation of Heavy Nucleus Acoustic Waves in a Degenerate Relativistic Plasma

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Abstract: Basic features of an ultra-low frequency heavy nucleus acoustic (HNA) wave in a three-component (electrons, light nuclei and heavy nuclei) degenerate relativistic plasma model has been studied by taking the dynamics of the heavy nuclei. Here, the electrons and light nuclei are considered degenerate and relativistic. A nonlinear modified Schrödinger equation (mNLSE) is derived here, by employing the reductive perturbation technique. On the basis of the associated plasma parameters, the stability of the electrostatic wave mode, angular frequency, group velocity, dissipative damping coefficient, modulational instabilities, etc., are studied. The results of this paper would be useful in understanding the basic features of ultra-low frequency heavy nucleus acoustic waves in degenerate relativistic plasmas occurring in astrophysical systems, such as a white dwarf astrophysical plasma.

Keywords: Heavy nucleus acoustic wave, degenerate relativistic plasma, nonlinear Schrödinger equation, modulation instability.

1. Introduction

The nonlinear wave propagation in relativistic plasma has already been studied by many authors due to its existence in some space-plasma systems¹, laser-plasmas², Van Allen radiation belt³, Earth's magnetosphere⁴, etc. The nonlinear dynamics in degenerate quantum plasma is also drawn attention to the researchers because of its importance in some laboratory environments⁵⁻¹⁰. Ions and degenerate electrons and/or positrons are the main constituents of a degenerate plasma system¹¹⁻¹⁶. The degeneracy of electrons or positrons (lightest plasma particles) occurs at extremely low-temperature and high-density plasmas¹⁷⁻²¹. The quantum effect in many degenerate plasma systems is considered on the lightest plasma species, because the mean interparticle

distance in such a system is comparable to the de Broglie wavelength ($\lambda = h/p$, where h and p are respectively the Planck's constant and linear momentum)²². The general expression for the relativistic electron degeneracy pressure is (derived by Chandrashekar¹¹) $P_s = KN_s^\gamma$, where N_s is the particle number density of the plasma species s , where $s = i$ for ions and $s = e$ for electrons. For the ultra-relativistic case $\gamma = 4/3$ and $K = 3hc/4$ ²³⁻²⁶, and for non-relativistic case $\gamma = 5/3$ and $K = 3\pi\hbar^2/5m_e$ ²³⁻²⁶, where $\hbar$ is the reduced Planck's constant and m_e is the rest mass of an electron and c is the speed of light in a vacuum. In 1961, E. E. Salpeter²⁷ showed that the value of the relativistic parameter $1/\sqrt{1-v^2/c^2}$, where v is the particle velocity, for the lightest plasma species at ultra-high density, is extremely large, and because of this the relativistic effect cannot be ignored on the dynamics of the lightest plasma species. Because of this phenomenon, we take into consideration the degenerate relativistic plasma (DRP) for the study of nonlinear wave propagation.

Nonlinear Schrödinger equation (NLSE) is closely related to Korteweg-de Vries (KdV) equation, which is used to describe the propagation of different nonlinear waves as topological solitons²⁸. By solving the NLSE the nonlinear structures (envelope solitons) are analytically obtained. Because of compensation of the dispersion with the nonlinearity, the envelope solitons propagate in plasmas without changing the structure²⁹. The sign of the ratio of the wave dispersion, given by the dispersion coefficient (P) and nonlinearity coefficient (Q) appearing in the NLSE determines the stability of the wave amplitude. If the ratio (P/Q) is greater than zero (> 0) the envelope excitations are modulationally unstable, and it is modulationally stable when the ratio is less than zero (< 0)²⁹. Beside different plasma systems in nature, the existence of envelope solitons in laboratory plasmas has also been shown by many researchers³⁰⁻³².

Recently, S. Islam et.al.²² studied a DRP system consisting of cold electrons, cold non-degenerate inertial light nuclei and stationary heavy nuclei. In their model they considered dynamics of the cold non-degenerate inertial light nuclei. They have studied the basic features of stability of the envelope solitons. In their study, the modulation instabilities criteria are also identified.

It should be mentioned here that there are some astro-physical plasma systems where electrons, light nuclei as well as heavy nuclei exist. For example, a white dwarf (or, degenerate dwarf) is a stellar core remnant and is composed mostly of electron-degenerate matter. Other than that, white dwarf also contains light nuclei (as for example, ${}^1_1\text{H}$ or ${}^4_2\text{He}$ or ${}^{12}_6\text{C}$, etc.¹⁵⁻¹⁸) and heavy nuclei (as for example, ${}^{56}_{26}\text{Fe}$ or ${}^{85}_{37}\text{Rd}$ or ${}^{56}_{42}\text{Mo}$ etc.³³⁻³⁵).

Thus, influenced by S. Islam et.al.²², in this study, the nonlinear propagation of a low frequency acoustic wave is considered and the plasma is assumed to consist of electrons, light nuclei and heavy nuclei relevant to the above mentioned astrophysical plasma system. We consider the plasma where the inertia is provided by the mass of the heavier nuclei species and the restoring force is provided by the degenerate pressure of the relativistic lighter nuclei and the electrons. In this plasma model, the dynamics of the heavy nuclei is considered. As a result, compared with the ion-acoustic waves in usual electron-ion plasma, the frequency of the perturbed electrostatic wave becomes very low, which is here termed as ultra-low frequency heavy nucleus acoustic (HNA) wave³⁵. We, therefore, study the nonlinear dynamics of the ultra-low frequency HNA wave in a degenerate relativistic plasma system with degenerate electrons, degenerate light nuclei and inertial heavy nuclei. For this investigation a modified nonlinear Schrödinger equation (mNLSE) is derived. This manuscript is organized as follows: The governing equations of our plasma model and the derivation of a mNLSE are given in Section II. Modulation instability of the ultra-low frequency HNA wave and solution of the mNLSE are described in Section III. Results are discussed in Section IV. Finally, Sec. IV concludes the paper.

2. Mathematical Model and Derivation of Modified Nonlinear Schrödinger Equation

In this study we consider the dynamics of ultra-low frequency HNA wave in a relativistic plasma consisting of electrons, light nuclei and heavy nuclei. The electrons and the light nuclei are assumed to be degenerate and relativistic. The inertia is provided by the mobile heavy ions and degenerate electron and light ions are inertialess. The equilibrium state of such a plasma system under consideration is defined as: $Z_l N_{l0} + Z_h N_{h0} = N_{e0}$, where Z_l and Z_h are respectively the charge state of the light and heavy nuclei, and N_{so} is the equilibrium number density of the

plasma species s ($s = e$ for electron, $s = l$ for light ion and $s = h$ for heavy ion). For simplicity of presentation, we consider a one-dimensional geometry. The nonlinear dynamics of the ultra-low frequency DRP system in such a geometry is described by the following set of equations:

$$(2.1) \quad \frac{\partial N_h}{\partial T} + \frac{\partial}{\partial X}(N_h V_h) = 0,$$

$$(2.2) \quad \frac{\partial V_h}{\partial T} + V_h \frac{\partial V_h}{\partial X} = -\frac{\partial \Phi}{\partial X} + \eta \frac{\partial^2 V_h}{\partial X^2},$$

and

$$(2.3) \quad \frac{\partial^2 \Phi}{\partial X^2} = (1 + \alpha)N_e - \alpha N_l - N_h,$$

where N_s is the particle number density of the species s and is normalized by N_{s0} , the velocity of the heavy nuclei, v_h is normalized by $C_s = (Z_h m_e c^2 / m_h)^{1/2}$ (m_s is the mass of the plasma species s), the electrostatic potential ϕ is normalized by $m_e c^2 / e$ (e is the magnitude of charge of an electron), the space variable X is normalized by $\lambda_0 = (m_e c^2 / 4\pi Z_h N_{h0} e^2)^{1/2}$, the time variable T is normalized by $\omega_{ph}^{-1} = (4\pi Z_h^2 N_{h0} e^2 / m_h)^{1/2}$, the kinematic viscosity of heavy ions is $\eta = \mu_h C_s / m_h N_{h0} \lambda_0^2$ (μ_h is the dynamic viscosity) and $\alpha = Z_l N_{l0} / Z_h N_{h0}$.

For the propagation of the ultra-low frequency electrostatic perturbation mode (the HNA wave), we consider inertialess electrons and light nuclei. Thus, the number densities of the light nuclei and the electrons in the degenerate relativistic plasma system are expressed as³⁶

$$(2.4) \quad N_l = \left(1 - \frac{\gamma_l - 1}{\gamma_l K_1} \Phi \right)^{1/(\gamma_l - 1)},$$

and

$$(2.5) \quad N_e = \left(1 + \frac{\gamma_e - 1}{\gamma_e K_2} \Phi \right)^{1/(\gamma_e - 1)},$$

where γ_l and γ_e are respectively the relativistic parameters of light nuclei and electrons, $K_1 = K_l N_{l0}^{\gamma_l - 1} / Z_l m_e c^2$ and $K_2 = K_e N_{e0}^{\gamma_e - 1} / m_e c^2$. After the expansions, Eqs. (2.4) and (2.5) are equivalent to the followings:

$$(2.6) \quad N_l = 1 - f_{l1}\Phi + f_{l2}\Phi^2 - f_{l3}\Phi^3 + \dots,$$

and

$$(2.7) \quad N_e = 1 + f_{e1}\Phi + f_{e2}\Phi^2 + f_{e3}\Phi^3 + \dots,$$

where for ultra-relativistic case $f_{l1} = -3/4K_1$, $f_{l2} = 3/16K_1^2$, $f_{l3} = -1/64K_1^3$, $f_{e1} = 3/4K_2$, $f_{e2} = 3/16K_2^2$, $f_{e3} = 1/64K_2^3$ and for non-relativistic case $f_{l1} = -3/4K_1$, $f_{l2} = 3/50K_1^2$, $f_{l3} = 1/250K_1^3$, $f_{e1} = -3/5K_2$, $f_{e2} = 3/50K_2^2$, $f_{e3} = -1/250K_2^3$.

After using the expansions in Eqs.(2.6) and (2.7) and by suppressing the subscripts on N_h and V_h , the set of equations, Eqs.(2.1)-(2.3), take the following form:

$$(2.8) \quad \frac{\partial N}{\partial T} + \frac{\partial}{\partial X}(NV) = 0,$$

$$(2.9) \quad \frac{\partial V}{\partial T} + V \frac{\partial V}{\partial X} = -\frac{\partial \Phi}{\partial X} + \eta \frac{\partial^2 V}{\partial X^2},$$

and

$$(2.10) \quad \frac{\partial^2 \Phi}{\partial X^2} = 1 + \mu_1\Phi + \mu_2\Phi^2 + \mu_3\Phi^3 - N,$$

where $\mu_1 = (1 + \alpha)f_{e1} + \alpha f_{l1}$, $\mu_2 = (1 + \alpha)f_{e2} - \alpha f_{l2}$ and $\mu_3 = (1 + \alpha)f_{e3} + \alpha f_{l3}$. The coefficient of viscosity is very small in some experimental cases and hence we consider $\eta \ll \eta^0 \epsilon^{3/2}$, where the perturbation parameter ϵ is a very small quantity ($0 < \epsilon < 1$). To derive the nonlinear Schrödinger equation for the ultra-low frequency HNA wave in a degenerate quantum plasma, we consider the following stretched coordinates:

$$(2.11) \quad \xi = \epsilon(X - v_0 T),$$

and

$$(2.12) \quad \tau = \varepsilon^2 T,$$

where ε is a very small quantity as has been mentioned earlier and the constant v_0 is the group velocity of the HNA wave and will be determined from the compatibility condition of the reductive perturbation technique. We expand the normalized variables N , V and ϕ as follows:

$$(2.13) \quad N = 1 + \sum_n \varepsilon^n \sum_l N_l^{(n)}(\xi, \tau) e^{il(kX - \omega T)},$$

$$(2.14) \quad V = \sum_n \varepsilon^n \sum_l V_l^{(n)}(\xi, \tau) e^{il(kX - \omega T)},$$

$$(2.15) \quad \Phi = 1 + \sum_n \varepsilon^n \sum_l \Phi_l^{(n)}(\xi, \tau) e^{il(kX - \omega T)},$$

where ω is the angular frequency and k is the wave number of the carrier ultra-low frequency HNA wave. The l -th harmonic of the n -th order slowly varying field variables N , V and ϕ are respectively $N_l^{(n)}(\xi, \tau)$, $V_l^{(n)}(\xi, \tau)$ and $\Phi_l^{(n)}(\xi, \tau)$. After using the stretched coordinates (Eqs.(2.11) and (2.12)) and expanding the variables from Eqs.(2.13)-(2.15), we get two coordinate systems namely (X, T) and (ξ, T) . In the first coordinate system (X, T) , the field quantities of the carrier varies rapidly with k and ω . On the other hand there is a slowly modulated wave packet in the second coordinate system (ξ, T) . After substituting all the above expansions, Eqs.(2.11)-(2.15), in Eqs. (2.8)-(2.10) we get a new set of equations, which gives the l -th harmonic of the n -th order reduced equations. Here, we don't list these reduced equations because of their very large length. From the reduced equations by taking the first harmonic of the first order quantities, i.e., $n=1$ and $l=1$, we get the linear dispersion relation of the ultra-low frequency HNA wave:

$$(2.16) \quad \omega^2 = \frac{k^2}{k^2 + \mu_1}.$$

Similarly, by taking the first harmonic of the second order quantities, i.e., $n=2$ and $l=1$, from the resulting compatibility condition, we obtain the group velocity

$$(2.17) \quad v_0 = \frac{\omega}{k}(1 - \omega^2).$$

We omit the detailed derivation of the mNLSE and just mention in short the important steps. The second order part of the second harmonic $\Phi_2^{(2)}$ is obtained in terms of $\Phi_1^{(1)}\Phi_1^{(1)}$ from the reduced equations when $n=2$ and $l=2$. Similarly, when $n=3$ and $l=0$, $\Phi_0^{(2)}$ is obtained in terms of $|\Phi_1^{(1)}|^2$ from the reduced equations. Finally, substituting the expressions for $\Phi_2^{(2)}$ and $\Phi_0^{(2)}$ into the reduced equations with $n=3$ and $l=1$, we get finally the following mNLSE:

$$(2.18) \quad i \frac{\partial a}{\partial \tau} + P \frac{\partial^2 a}{\partial \xi^2} + Q|a|^2 a = -iYa,$$

where $a = \Phi_1^{(1)}$ and

$$(2.19) \quad P = \frac{\omega^3}{2k^2} \left(v_0 \frac{a_1}{\omega^2} + 2v_0 \frac{ka_2}{\omega^3} - \frac{a_2}{\omega^2} - 1 \right),$$

$$(2.20) \quad Q = \frac{\omega^3}{2k^2} \left(2(a_4 + a_7) - \frac{k^2}{\omega^2}(a_5 + a_8) - 2\frac{k^3}{\omega^3}(a_6 + a_9) \right)$$

and the damping coefficient Y is due to the kinematic viscosity of the heavy nuclei which is given by

$$(2.21) \quad Y = \frac{k^2}{2}\eta.$$

Equation (2.18) is the required mNLSE for the evolution of the ultra-low frequency HNA wave in a degenerate relativistic plasma. It is here noted that in the absence of collision (i.e. when right hand side of Eq. (2.18) is zero), the solution of the mNLSE is called a bright soliton if $P/Q > 0$ and a dark soliton when $P/Q < 0$ ^{32, 38}. On the other hand, there is no stationary envelop solitonic solution in presence of dissipation ($Y \neq 0$) in the system.

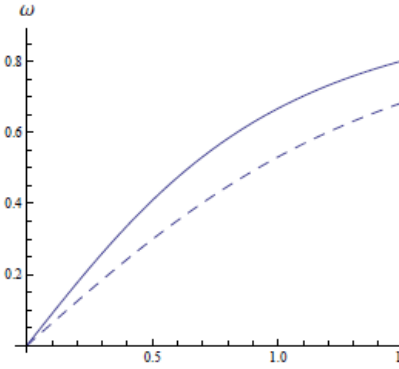


Figure 1. The variation of the angular frequency ω with respect to the wave number of the carrier electrostatic wave k . Here, $Z_h = 26$, $Z_l = 1$ (solid curve) and $Z_l = 6$ (dotted curve); $N_{e0} = 10^{31} \text{ cm}^{-3}$, $N_{l0} = 0.005N_{e0}$.

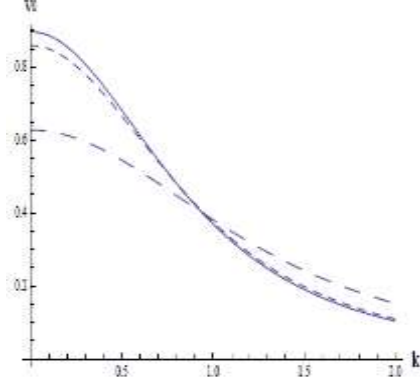


Figure 2. The variation of the group velocity v_0 versus wave number of the carrier electrostatic wave k . Here, $Z_l = 1$ (solid curve), $Z_l = 2$ (dotted curve) and $Z_l = 6$ (long-dashed curve); $N_{e0} = 10^{31} \text{ cm}^{-3}$, $N_{l0} = 0.005N_{e0}$.

The expressions of the parameters $a_1 - a_9$ used in Eqs.(2.19) and (2.20), are given below:

$$(2.22) \quad a_1 = \frac{k}{\omega} \left(v_0 \frac{k}{\omega} - 1 \right),$$

$$(2.23) \quad a_2 = v_0 \frac{k}{\omega} - 1,$$

$$(2.24) \quad a_3 = \frac{1}{\omega^2} (ka_2 + \omega_1),$$

$$(2.25) \quad a_4 = \frac{1}{3k^2} \left(\mu_2 - \frac{3k^4}{2\omega^4} \right),$$

$$(2.26) \quad a_5 = \frac{k^2}{\omega^2} a_4 + \frac{3k^4}{2\omega^4},$$

$$(2.27) \quad a_6 = \frac{k}{\omega} \left(a_4 + \frac{k^2}{2\omega^2} \right),$$

$$(2.28) \quad a_7 = \frac{2\mu_2 - \frac{1}{v_0^2} \frac{k^2}{\omega^2} \left(1 + 2v_0 \frac{k}{\omega} \right)}{-\mu_2 + \frac{1}{v_0^2}},$$

$$(2.29) \quad a_8 = 2\mu_2 - \mu_1 a_7,$$

$$(2.30) \quad a_9 = \frac{1}{v_0} \left(a_7 + \frac{k^2}{\omega^2} \right).$$

3. Modulation Instability of Hna Wave and Solution of The Modified Nonlinear Schrödinger Equation

We analyze the modulational instability of the dissipative quantum ultra-low frequency plasma, which is described by the mNLSE (Eq. (2.18)), only for $P/Q > 0$. The amplitude of the wave can be expressed as³⁷

$$(3.1) \quad a(\xi, \tau) = \left[a_0 + \delta a(\xi, \tau) \right] e^{-i \int_0^\tau \Delta(\tau) d\tau - Y\tau},$$

where a_0 is the amplitude of carrier wave and Δ is a possible nonlinear frequency shift. By taking

$$(3.2) \quad \delta a(\xi, \tau) = (U + iW) e^{i \left(K\xi - \int_0^\tau \Omega(\tau) d\tau \right)},$$

where U and V are the real and imaginary parts of $\delta a(\xi, \tau)$ respectively, the dispersion relation for the modulation of the ultra-low frequency HNA wave is obtained. The nonlinear relation is given as follows:

$$(3.3) \quad \Omega^2 = (PK^2)^2 \left(1 - \frac{K_c^2(\tau)}{K^2} \right),$$

where $K_c^2(\tau) = (2Q|a_0|^2 / P) \exp(-2Y\tau)$. The imaginary part of the square root of Ω gives the local instability growth rate, which is stated as follows:

$$(3.4) \quad \Gamma \equiv \text{Im}[\Omega(\tau)] = |PK^2| \left| \left(\frac{K_c^2(\tau)}{K^2} - 1 \right) \right|.$$

The instability growth will cease when

$$(3.5) \quad \tau \geq \tau_{\max} = \frac{1}{2Y} \ln \left(\frac{2Q|a_0|^2}{PK^2} \right)$$

and for this reason in presence of collision, $\tau_{\max}$ is the modulation instability period, which is absent in the plasma without collision.

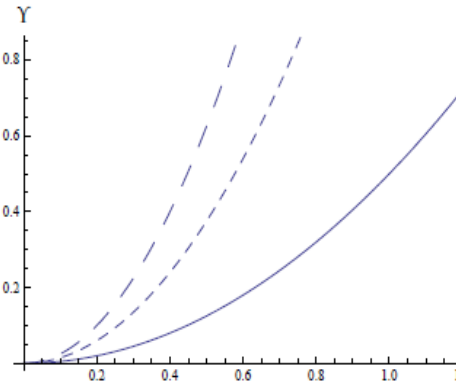


Figure 3. The variation of dissipative damping coefficient Y versus wave number of the carrier electrostatic wave k for three values of kinematic viscosity of the heavy nuclei such as $\eta=1$ (solid curve), $\eta=3$ (dotted curve) and $\eta=5$ (long-dashed curve).

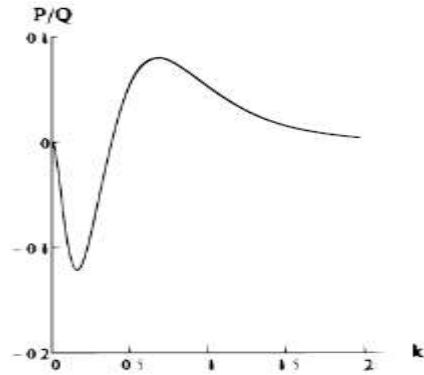


Figure 4. The variation of the ratio P/Q with respect to the wave number of the carrier electrostatic wave k . Here, $N_{e0} = 10^{31} \text{ cm}^{-3}$, $Z_h = 42$, $N_{i0} = 0.005N_{e0}$, and $Z_l = 2$.

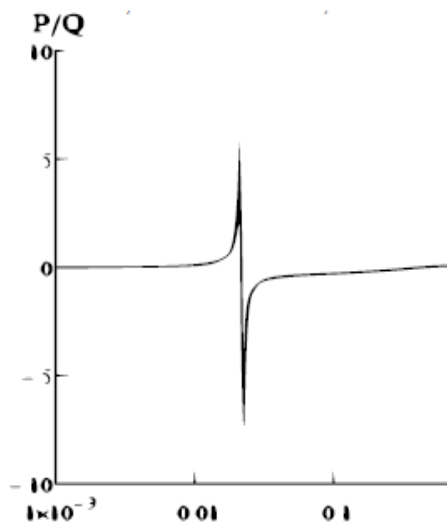


Figure 5. The variation of the ratio P/Q with respect to the wave number of the carrier electrostatic wave k . Here, $N_{e0} = 10^{31} \text{ cm}^{-3}$, $Z_h = 42$, $N_{i0} = 0.005N_{e0}$ and $Z_l = 2$.

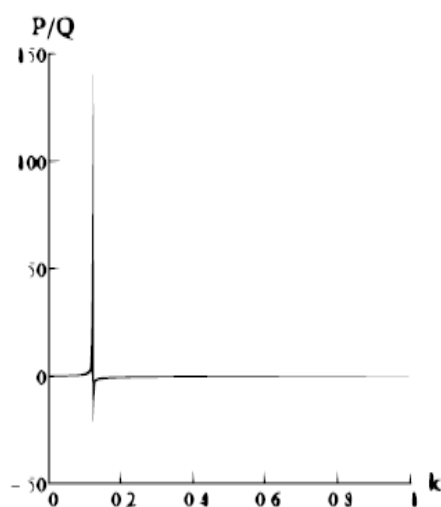


Figure 6. The variation of the ratio P/Q with respect to the wave number of the carrier electrostatic wave k . Here, $N_{e0} = 10^{31} \text{ cm}^{-3}$, $Z_h = 42$, $N_{i0} = 0.01N_{e0}$ and $Z_l = 2$.

Now we are interested to find the solution of the mNLSE, Eq. (18), within the instability region (where $P/Q > 0$). By substitution of $\Phi(\xi, \tau) = a(\xi, \tau) \exp(-Y\tau)$ into the mNLSE (Eq. (2.18)) we get³⁹

$$(3.6) \quad i \frac{\partial \Phi}{\partial \tau} + P \frac{\partial^2 \Phi}{\partial \xi^2} + e^{-2Y\tau} Q |\Phi|^2 \Phi = 0.$$

Here in the Taylor series expansion of $\exp(-2Y\tau)$, we consider only the first order and constant terms because of the very small value of the damping parameter coefficient Y . Substituting this approximate Taylor series expansion in Eq. (3.6) the following equation is obtained.

$$(3.7) \quad i \frac{\partial \Phi}{\partial \tau} + P \frac{\partial^2 \Phi}{\partial \xi^2} + Q \sigma_D |\Phi|^2 \Phi = 0,$$

where $\sigma_D = \exp(-2Y\tau) \approx 1 - 2Y\tau$. Now by substituting $\zeta(\xi, \tau) = \sigma_D(\tau)\xi$, $\chi = \sigma_D(\tau)\tau$ and $\hat{\Phi} = \Phi \exp(iY\sigma_D(\tau)\xi^2/2P)$ in Eq. (3.7), the following standard NLSE is obtained:

$$(3.8) \quad i \frac{\partial \hat{\Phi}}{\partial \chi} + P \frac{\partial^2 \hat{\Phi}}{\partial \zeta^2} + Q |\hat{\Phi}|^2 \hat{\Phi} = 0.$$

The standard NLSE, Eq. (3.8), has a rational solution, which is localized both in the directions of ζ and χ (i.e. in the direction of ξ and τ), and is localized in nonzero background. Therefore, the ultimate rogue wave solution for the dissipative standard nonlinear Schrödinger equation, Eq. (1.8), is obtained as³⁹

$$(3.9) \quad a(\xi, \tau) = \sqrt{\frac{P}{Q}} \left[\frac{4 + 16iP\sigma_D(\tau)\tau}{1 + 16\{P\sigma_D(\tau)\tau\}^2 + 4\{\sigma_D(\tau)\xi\}^2} - 1 \right] \\ \times \exp \left[-i \left\{ \frac{Y\sigma_D(\tau)\xi^2}{2P} \right\} \right] \sqrt{\sigma_D(\tau)}.$$

4. Results

In the following section, we study some linear and nonlinear aspects of the ultra-low frequency HNA wave in the degenerate relativistic plasma system. We here study stability of the modulated wave. To have some numerical appreciation of the results of this investigation, we have numerically analyzed angular frequency, group velocity, for different relevant plasma parameters and also analyzed the solution, Eq. (3.9), of the mNLSE of the HNA wave. Here, we consider the following typical parameters relevant to some astrophysical context, such as, for a white dwarf³⁶: $N_{e0} = 10^{31} \text{ cm}^{-3}$, N_{i0}/N_{e0} varies between 0.005–0.01, $\gamma_i = \gamma_e = 4/3$, $h = 1.054 \times 10^{-27} \text{ erg.s}$, $Z_h = 26$ or 42 and $Z_l = 1$ or 2 or 6, $a_0 = 0.1$. The variation of the angular frequency ω with respect to the wave number k of the carrier electrostatic ultralow frequency HNA wave is shown in Figure 1. Here we consider that charge state of heavy ion is $Z_h = 26$ (for Fe) and that of light ions are $Z_l = 1$ (for H) and $Z_l = 6$ (for C), which are shown in solid curve and dotted curve respectively with the equilibrium electron number

density $N_{e0} = 10^{31} \text{ cm}^{-3}$ and the light nuclei number density $N_{l0} = 0.005N_{e0}$. Figure 1 shows that if the wave number of the carrier wave is increased then the angular frequency also increases toward the saturation value which is the characteristic of ion-acoustic wave. Also large value of charge state of light nuclei gives lower value of the angular frequency.

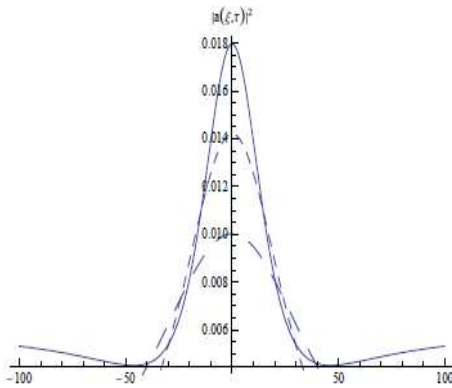


Figure 7. The variation of the modulus-square of the amplitude of the nonlinear electrostatic wave $|a(\xi, \tau)|^2$ as a function of the position coordinate ξ for three values of kinematic viscosity of the heavy nuclei such as $\eta=1$ (solid curve), $\eta=3$ (dotted curve) and $\eta=5$ (long-dashed curve); $N_{e0} = 10^{31} \text{ cm}^{-3}$, $N_{l0} = 0.005N_{e0}$, $Z_l = 1$ and $Z_l = 6$.

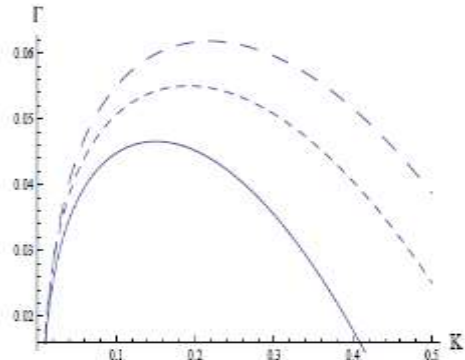


Figure 8. The variation of the instability growth rate Γ with respect to the modulation wave numbers K for three values of carrier electrostatic wave number $k=0.7$ (solid curve), $k=0.9$ (dotted curve) and $k=1.1$ (long-dashed curve); $N_{e0} = 10^{31} \text{ cm}^{-3}$, $N_{l0} = 0.005N_{e0}$, $Z_l = 1$, $Z_l = 26$ and $a_0 = 0.1$.

Figure 2 is a graph of group velocity v_0 versus wave number k of the carrier electrostatic wave for different values of charge state of light ion Z_l . The solid curve, the dotted curve and the long-dash curve are for $Z_l = 1$, $Z_l = 2$ and $Z_l = 6$ respectively with the equilibrium electron number density $N_{e0} = 10^{31} \text{ cm}^{-3}$, and the light nuclei number density $N_{l0} = 0.005N_{e0}$. From the figure it is seen that the group velocity v_0 decreases with the wave number of the carrier k . Also we observe that when the wave number is below 0.95 (approximately), the group velocity is higher for low charge state of the light nuclei and after that the group velocity is lower for low charge state of

light nuclei. The variation of dissipative damping coefficient Y versus wave number k of the carrier electrostatic wave is shown in Figure 3. The curves are for three values of kinematic viscosity of the heavy nuclei such as $\eta=1$ (solid curve), $\eta=3$ (dotted curve) and $\eta=5$ (long-dashed curve). Figure 3 shows that damping coefficient is increasing if we increase the wave number of the carrier. There is also a noticeable effect of kinematic viscosity of the heavy nuclei on the damping coefficient. This figure shows that damping coefficient increases drastically with the kinematic viscosity. Figures 4-6 are graphs of the ratio, P/Q , of the dispersion coefficient P and the nonlinear coefficient Q of the mNLSE, Eq. (1.8), versus the wave number k of the carrier electrostatic HNA wave. We determine for what value of the carrier wave number the electrostatic HNA wave mode is modulationally stable and for what values of k it is unstable. In the curves in Figures 4-6, we consider $N_{e0}=10^{31} \text{ cm}^{-3}$, $Z_h=42$ and $Z_l=2$. In Figure 4, $N_{l0}=0.005N_{e0}$. Here the wave mode is modulationally stable when the wave number, k , of the carrier wave ranges from 0 to 0.379 and after that value of k , the wave amplitude is unstable. In Figure 5, $N_{l0}=0.008N_{e0}$. Here the wave mode is modulationally unstable when the wave number, k , of the carrier wave ranges from 0 to 0.021 and after that value of k , the wave amplitude is stable. Finally, in Figure 6, $N_{l0}=0.01N_{e0}$. We observe that the wave mode is modulationally unstable when the wave number, k , of the carrier wave ranges from 0 to 0.121, then from 0.121 to 0.4 it is stable, and again after that value of k , the wave amplitude is unstable.

Figure 7 shows the variation of the modulus-square of the amplitude of the nonlinear electrostatic wave $|a(\xi, \tau)|^2$ as a function of the position coordinate ξ for three different values of the kinematic viscosity of the heavy nuclei η . The solid curve is for $\eta=1$ whereas the dotted curve and the long-dashed curve are for $\eta=3$ and $\eta=5$ respectively. The other parameters are the following: $N_{e0}=10^{31} \text{ cm}^{-3}$, $N_{l0}=0.005N_{e0}$, $Z_l=1$, and $Z_h=26$. We observe from the figure that the modulated amplitude drastically decreases with the increment of the kinematic viscosity η .

Figure 8 shows the variation of the growth rate of the modulation stability of the ultra-low frequency HNA wave with respect to the modulation wave number K taking carrier wave number k as a parameter. It is observed from the figure that the growth rate increases with the modulation wave

number up to a certain value of K and then it decreases. It is further observed that the peak value of the growth rate increases with the carrier wave number k . The solid curve is for $k=0.7$ where as the dotted curve and the long-dashed curve are for $k=0.9$ and $k=1.1$ respectively. The other parameters are the following: $N_{e0}=10^{31}\text{cm}^{-3}$, $N_{i0}=0.005N_{e0}$, $Z_i=1$, and $Z_i=26$. It is also found that this peak shifts to the right in the K -axis as the increase of the carrier wave number k .

5. Discussions

In this research work, we have reported the basic features of the linear wave dispersion, as well as the modulation stability of the ultra-low frequency HNA wave in a degenerate relativistic plasma. By considering the dynamics of the heavy nuclei we have derived the mNLSE by employing the reductive perturbation technique. For this, we have taken the electrons and the light nuclei are inertialess degenerate relativistic particles and the heavy nuclei have been taken as inertial and nonrelativistic. Depending on the range of the carrier wave number, the amplitude of the modulated wave becomes both stable as well as unstable. For comparatively lower values of the unperturbed number density of the lighter nuclei the critical wave number is comparatively large. However, as the unperturbed number density of the lighter nuclei is increased, the critical wave number of the modulated ultra-low frequency HNA wave decreases. Furthermore, for very large number density of the lighter nuclei, the critical wave number even changes twice depending on the carrier wave number. For lower values of the carrier wave number, the wave is modulationally unstable. The modulation growth rate has also been studied and it is found that for larger carrier wave number, the growth rate increases and the peak value of the growth rate shifts towards the larger values of the modulated wave number. The relevance of the current research in some astrophysical context is indicated, such as the white dwarf plasma where both light nuclei as well as heavy nuclei exist in addition to the electrons. It should be mentioned here that in the current research, we have not included the pressure and the self-gravitational effect of the heavy nuclei dynamics on the light nuclei and the electrons, which however, may have great importance in some astrophysical systems. This is beyond the scope of the present paper. The current research should be important and valuable in understanding the nonlinear behavior of localized modulated waves in some astrophysical compact objects, for example, white dwarf and neutron stars, which are thought to contain

relativistically degenerate electrons and light nuclei and in addition to the heavy nuclei.

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