On $\tilde{K}$-Curvature Inheritance in a Finsler Space

P. N. Pandey and Vaishali Pandey
Department of Mathematics, University of Allahabad, Allahabad
Email- pnpiaps@rediffmail.com; mathvaishali@gmail.com
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Abstract: $\tilde{K}$-Curvature inheritance and Projective $\tilde{K}$-Curvature inheritance have been discussed by S. P. Singh and J. K. Gatoto. They obtained several theorems on such transformations, especially generated by contra and concurrent vector fields. The aim of the present paper is to discuss $\tilde{K}$-Curvature inheritance and Projective $\tilde{K}$-Curvature inheritance in a Finsler Space and to generalize the theorems of S. P. Singh and J. K. Gatoto.

Keywords: $\tilde{K}$-Curvature inheritance, Projective $\tilde{K}$-Curvature inheritance, contra and concurrent vector fields.

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1. Preliminaries

Let $F_n$ be an n-dimensional Finsler space equipped with symmetric connection coefficients $\Gamma_{jk}^i(x, \xi)$. The covariant derivative of a tensor field $T_j^i$ with respect to connection coefficients $\Gamma_{jk}^i$ is given by

\begin{equation}
T'_j = \partial_k T_j^i + \left( \partial_k \Gamma_{jk}^i \right) T_j^h + T_j^h \Gamma_{jk}^h - T_j^h \Gamma_{jk}^i,
\end{equation}

where $\partial_k = \frac{\partial}{\partial x^k}$ and $\partial_k' = \frac{\partial}{\partial \xi^k}$.

The commutation formula for such covariant derivative is given by

\begin{equation}
X_{jkh}^i - X_{ikh}^j = \tilde{K}_{jkh}^i (x, \xi) X^j,
\end{equation}

where

\begin{equation}
\tilde{K}_{jkh}^i (x, \xi) = \left( \partial_k \Gamma_{jh}^i + (\partial_k \Gamma_{jh}^m) \partial_h \xi^m \right) - \left( \partial_k \Gamma_{jk}^i + (\partial_k \Gamma_{jk}^m) \partial_h \xi^m \right) + \Gamma_{mk}^i \Gamma_{jh}^m - \Gamma_{mh}^i \Gamma_{jk}^m.
\end{equation}
This tensor is called relative curvature tensor, since it depends on partial
derivatives of the field $\xi^m(x^k)$ with respect to $x^h$ [Rund]. The relative
curvature tensor $\tilde{K}^{i}_{jkh}(x,\xi)$ satisfies the following:

$$\begin{align*}
(1.4) \quad (a) \quad & \tilde{K}^{i}_{jkh} = - \tilde{K}^{i}_{jkh} \\
& (b) \quad \tilde{K}^{i}_{jkh} ; m + \tilde{K}^{i}_{jkm} ; k + \tilde{K}^{i}_{jkm} ; m ; h = 0.
\end{align*}$$

The associate tensor of the relative curvature tensor is defined as

$$\begin{align*}
(1.5) \quad & g_{jm} \tilde{K}^{m}_{ikh} = \tilde{K}^{m}_{ikh}.
\end{align*}$$

This tensor satisfies

$$\begin{align*}
(1.6) \quad & \tilde{K}^{i}_{jikh} x^i = - \tilde{K}^{i}_{jkh} x^i,
\end{align*}$$

The Lie-derivative of a tensor field $T^i_j$ with respect to the infinitesimal
transformation

$$\begin{align*}
(1.7) \quad & \bar{x}^i = x^i + \varepsilon v^i(x^i),
\end{align*}$$

where $v^i(x^i)$ is a contravariant vector field which depends upon position
coordinates only and $\varepsilon$ is an infinitesimal constant, is given by

$$\begin{align*}
(1.8) \quad & \mathcal{L} T^i_j = T^i_j ; h v^h - T^i_j v^j_h + T^i_j v^h_j,
\end{align*}$$

The Lie-derivative of the connection coefficients $\Gamma^i_{jk}(x,\xi)$ is given by

$$\begin{align*}
(1.9) \quad & \mathcal{L} \Gamma^i_{jk} = v^i_{,jk} + \tilde{K}^i_{jkh} v^h,
\end{align*}$$

The commutation formulae for Lie-differentiation and covariant
differentiation are given by

$$\begin{align*}
(1.10) \quad & \mathcal{L}(T^i_j) ; k - \mathcal{L}(T^i_j) ; k = T^i_j \mathcal{L} \Gamma^i_{h,k} - T^i_j \mathcal{L} \Gamma^h_{j,k},
\end{align*}$$

and

$$\begin{align*}
(1.11) \quad & (\mathcal{L} \Gamma^i_{jk}) ; h - (\mathcal{L} \Gamma^i_{jk}) ; k = \mathcal{L} \tilde{K}^i_{jkh},
\end{align*}$$

2. $\tilde{K}$-Curvature inheritance

An infinitesimal transformation

$$\begin{align*}
(2.1) \quad & \bar{x}^i = x^i + \varepsilon v^i(x^i),
\end{align*}$$

is called a $\tilde{K}$-Curvature inheritance if the Lie-derivative of the relative
curvature tensor $\tilde{K}^i_{jkh}$ is proportional to itself, i.e.
(2.2) \[ \mathcal{E} \tilde{K}_{jkh} = \alpha(x) \tilde{K}_{jkh} , \]

where \( \alpha(x) \) is a non-zero scalar field depending on \( x^i \). If we use the terminology of P. N. Pandey, \( \tilde{K} \)-Curvature inheritance may be called as \( \tilde{K} \)-Lie-recurrence. Obviously, in the above definition, the relative curvature tensor is assumed to be non-zero.

The infinitesimal transformation (2.1) is called a \( \tilde{K} \)-Curvature collineation if

\[ \mathcal{E} \tilde{\Gamma}_{jkh}^i = 0 . \]

Thus, we see that a \( \tilde{K} \)-Curvature inheritance cannot be a curvature collineation. In other words, we may say that the sets of \( \tilde{K} \)-Curvature inheritances and \( \tilde{K} \)-Curvature collineations are disjoint.

The necessary and sufficient condition for the infinitesimal transformation (2.1) to be an affine motion is given by

\[ \mathcal{E} \Gamma_{jkh} = 0 . \]

In view of (1.11), (2.4) gives (2.3). Thus an affine motion is a \( \tilde{K} \)-Curvature collineation. Hence an affine motion cannot be a \( \tilde{K} \)-Curvature inheritance. J. K. Gatoto and S. P. Singh studied a \( \tilde{K} \)-Curvature inheritance which is also an affine motion. Obviously, there exists no such \( \tilde{K} \)-Curvature inheritance, and therefore Theorem (1.1) and Lemma (1.1) of J. K. Gatoto and S. P. Singh are meaningless. Since every homothetic transformation is an affine motion, a homothetic transformation cannot be a \( \tilde{K} \)-Curvature inheritance. Thus, Theorem (1.3) of Gatoto and Singh is misleading.

Let us consider a recurrent space characterised by

\[ \tilde{K}_{jkh}^i = \lambda_m \tilde{K}_{jkh} , \]

where \( \lambda_h \) is a non-zero covariant vector field and \( \tilde{K}_{jkm} \neq 0 \). The vector field \( \lambda_h \) is called the recurrence vector. If it admits the \( \tilde{K} \)-Curvature inheritance (2.1), we have (2.2).

Differentiating (2.2) covariantly with respect to \( x^m \), we have

\[ (\mathcal{E} \tilde{K}_{jkh}^i)_{;m} = (\alpha_{;m} + \alpha \lambda_m) \tilde{K}_{jkh} , \]

while operating \( \mathcal{E} \) on both sides of (2.5), we get
From (2.6) and (2.7), we find
\[ (\varepsilon \tilde{K}^i_{jkh;m})_m = (\varepsilon \lambda^i_m + \alpha \lambda^i_m) \tilde{K}^i_{jkh}. \]

From (2.8), we conclude

**Theorem 2.1:** In a recurrent space, \( \tilde{K} \)-Curvature inheritance and covariant differentiation for the connection \( \Gamma^i_{jk}(x,\xi) \) commute if and only if
\[ \varepsilon \lambda^i_m = \alpha \lambda^i_m. \]

In view of (1.2), we have
\[ \lambda^i_m = \lambda^i_m - \varepsilon \lambda^i_m. \]

From (2.5), we have
\[ \lambda^i_m = \lambda^i_m - \varepsilon \lambda^i_m. \]

Differentiating (2.10) covariantly and using (2.5), we get
\[ (A^m_{l;p} + \lambda^m_p A^m_{l}) \tilde{K}^i_{jkh} = 2\lambda^i_p (\tilde{K}^r_{jkh} \tilde{K}^i_{rml} - \tilde{K}^i_{rkh} \tilde{K}^r_{jml} - \tilde{K}^i_{jrk} \tilde{K}^r_{kml}), \]
which in view of (2.10), gives
\[ A^m_{l;p} = \lambda^m_p A^m_{l}. \]

From (1.4 b) and (2.5), we have
\[ \lambda^i_m \tilde{K}^i_{jkh} + \lambda^i_h \tilde{K}^i_{jmk} + \lambda^i_k \tilde{K}^i_{jhm} = 0. \]

Multiplying (2.10) by \( \lambda^i_p \) and taking skew-symmetric part with respect to the indices \( l, m \) and \( p \), we have
\[ \lambda^i_p A^m_{l} + \lambda^m_p A^i_{l} + \lambda^i_j A^i_{m} = 0, \]
due to (2.12).

Thus, we have

**Theorem 2.2:** The tensor \( A^m_{l} (= \lambda^m_l - \lambda^i_l m) \) is recurrent in a \( \tilde{K} \)-recurrent Finsler space and satisfies the identity (2.13).

Operating (2.10) by the operator of Lie-differentiation and using (2.2), we get
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(2.14) \[ \mathcal{L} A_{lm} = \alpha A_{lm}. \]

This leads to

**Theorem 2.3:** The tensor $A_{lm}$ is Lie-recurrent with respect to a $\tilde{K}$-Curvature inheritance.

J. K. Gatoto and S. P. Singh\(^2\) assumed $\mathcal{L} A_{lm} = -\alpha A_{lm}$ in proving Theorem (1.4), which cannot be true in view of the above theorem. Theorem (1.5) of Gatoto and Singh\(^2\) states that a general recurrent Finsler space does not admit a $\tilde{K}$-Curvature inheritance if it becomes an affine motion. In fact, there is no Finsler space admitting $\tilde{K}$-Curvature inheritance which is an affine motion.

### 3. Projective $\tilde{K}$-Curvature inheritance

A $\tilde{K}$-Curvature inheritance is called a Projective $\tilde{K}$-Curvature inheritance if it is also a Projective motion\(^1\).

The necessary and sufficient condition for an infinitesimal transformation to be a Projective motion is given by

\[ \mathcal{L} \Gamma^i_{jk} = \delta^i_j p_h + \delta^i_k p_j. \]

Using condition (3.1) in equation (1.11), we have

\[ \delta^i_j p_{h;k} + \delta^i_h p_{j;k} - \delta^i_j p_{k;h} - \delta^i_k p_{j;h} = \alpha(x) \tilde{K}^i_{jhh}, \]

\[ \delta^i_j (p_{h;k} - p_{k;h}) + \delta^i_h p_{j;k} - \delta^i_k p_{j;h} = \alpha(x) \tilde{K}^i_{jkh}, \]

\[ 2 \delta^i_j p_{h;k} + 2 \delta^i_h p_{j;k} = \alpha(x) \tilde{K}^i_{jhk}, \]

where each square bracket denotes the skew-symmetric part with respect to the indices enclosed in it.

Therefore we may state

**Theorem 3.1:** In a Finsler space $F_n$ admitting a Projective $\tilde{K}$-Curvature inheritance, the relative curvature tensor $\tilde{K}^i_{jhh}$ can be expressed in terms of derivatives of $p_j$ in the form (3.2).

Gatoto and Singh proved three theorems assuming a Projective $\tilde{K}$-Curvature inheritance as a motion, homothetic transformation and an affine motion. These theorems are misleading because a Projective $\tilde{K}$-Curvature
inheritance cannot be a motion, homothetic transformation or an affine motion.

4. Special $\tilde{\mathbf{K}}$-Curvature inheritance

In this section, we study the $\tilde{\mathbf{K}}$-Curvature inheritance generated by a contra vector field and a concurrent vector field. These vector fields are respectively characterised by

(4.1) $v_{i; j} = 0$

and

(4.2) $v_{i; j} = \rho \delta_{j}^{i}$,

where $\rho$ is a constant.

These types of inheritances were discussed by Gatoto and Singh\(^1\), P. N. Pandey showed that the above types of vector fields generate an affine motion. Therefore the question of $\tilde{\mathbf{K}}$-Curvature inheritance or Projective $\tilde{\mathbf{K}}$-Curvature inheritance in a Finsler space does not arise.

References