

A Common Fixed Point Theorem for Occasionally Weakly Compatible Mappings

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Abstract: In the present paper, we show that contractive conditions induced by implicit functions do not provide a nontrivial setting for the application of the concept of occasionally weakly compatible mapping. Further, for obtaining proper applications of occasional weak compatibility we replace contractive conditions by more general conditions that may hold for mappings satisfying contractive as well as nonexpansive and Lipschitz type conditions.

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1. Introduction and Preliminaries

In order to generalize the notion of weakly compatible mappings, Al-Thagafi and Shahzad¹ introduced the notion of occasionally weakly compatible mappings. Two self mappings f and S of a subset D of a metric space (X, d) are called occasionally weakly compatible¹ (owc in short) if $fx = Sx$ and $fSx = Sfx$ for some $x \in D$. A point x satisfying $fx = Sx$ is called a coincidence point (CP) of f and S and fx is then called a point of coincidence (POC). Thus, if f and g are owc mappings such that $fx = Sx$, $fSx = Sfx$ for some x then both fx and $fSx (=Sfx)$ are points of coincidence (POC) of f and g .

Now suppose that f and S satisfy some contractive condition induced by implicit functions^{2, 3} or any other contractive condition (for more

details on contractive conditions induced by implicit relations one can refer to papers by Ali and Imdad² or Popa³). If f and S have a common fixed point, say z , then $z = fz = Sz$, $fSz = Sfz = z$ and, therefore, f and S are owc mappings. On the other hand, if f and S are owc mappings such that $fx = Sx$ and $fSx = Sfx$ for some x then, since contractive conditions exclude the existence of two points of coincidence, we get $fx = ffx (= Sfx)$. This means that $fx = Sx$ is a unique common fixed point of f and S . Therefore, under contractive conditions the existence of a common fixed point and occasional weak compatibility are equivalent conditions and, consequently, proving the existence of fixed points by assuming owc becomes a redundant exercise. Moreover, it is now well known⁴ that in the presence of a unique fixed point or a unique point of coincidence owc reduces to weak compatibility and replacing weak compatibility by owc in the presence of a unique fixed point does not yield any generalization.

We thus see that contractive conditions do not provide a proper setting for the application of the concept of owc and a proper setting for applications of the notion of owc will be some general condition which may hold for mappings satisfying nonexpansive condition, Lipschitz type condition or a contractive condition. Moreover, owc mappings can be divided in two categories:

- (1) Mappings commuting at all the coincidence points, and
- (2) Mappings commuting on a proper subset of the set of coincidences.

In the first case the mappings are obviously pointwise R-weakly commuting⁵ or equivalently weakly compatible⁶. In the second case the mappings are noncompatible. Therefore, a proper setting for the application of owc should allow the existence of multiple fixed points or multiple points of coincidence points.

2. Main results

Define $G = \{g : R_+ \rightarrow R_+\}$ such that

- (g_1) g is nondecreasing in the 4th and 5th variables,
- (g_2) If $u, v \in R_+$ are such that $u \leq g(v, v, u, u + v, 0)$, or $u \leq g(v, u, v, u + v, 0)$ or $v \leq g(u, u, v, u + v, 0)$, or $u \leq g(v, u, v, u, u + v)$, then $u \leq hv$.

where $0 < h < 1$ is a constant,

(g_3) If $u \in R_+$ is such that $u \leq g(u, 0, 0, u, u)$ or $u \leq g(0, u, 0, u, u)$ or $u \leq g(0, 0, u, u, u)$, then $u = 0$.

The following theorem was proved by Jungck and Rhoades⁷ by using owc and contractive conditions induced by the implicit relations (see Jungck and Rhoades⁷):

Theorem 1.1⁷ Let X be a set, r a symmetric on X . Let f, g, S, T be self maps of X satisfying

$$f(X) \subset T(X), g(X) \subset S(X), \quad \text{and}$$

(1) $r(fx, gy) \leq g(r(Sx, Ty), r(fx, Sx), r(gy, Ty), r(fx, Ty), r(gy, Sx))$ for all $x, y \in X$, where $g \in G$. If $\{f, S\}$ and $\{g, T\}$ are owc, then f, g, S , and T have a unique common fixed point.

It may be observed that under the contractive condition (1) of Theorem 1.1, assumption of owc and the existence of a unique common fixed point are equivalent conditions. To see this, first suppose that the pairs $\{f, S\}$ and $\{g, T\}$ have a unique common fixed point z then, as already discussed above, $\{f, S\}$ and $\{g, T\}$ are owc.

On the other hand, if $\{f, S\}$ and $\{g, T\}$ are owc mappings then there exists u and v in X such that $fu=Su, fSu=Sfu(=ffu=SSu)$ and $gv=Tv, gTv=Tgv(=ggv=TTv)$. Condition (1) now straight away implies that

$$\begin{aligned} r(fu, gv) &\leq g(r(Su, Tv), r(fu, Su), r(gv, Tv), r(fu, Tv), r(gv, Su)) \\ &= g(r(fu, gv), 0, 0, r(fu, gv), r(gv, fu)), \end{aligned}$$

In view of (g_3) this implies, $r(fu, gv)=0$, hence $fu=gv$. Again using condition (1) we get

$$\begin{aligned} r(fu, gTv) &\leq g(r(Su, TTv), r(fu, Su), r(gTv, TTv), r(fu, TTv), r(gTv, Su)) \\ &= g(r(fu, gTv), 0, 0, r(fu, gTv), r(gTv, fu)), \end{aligned}$$

and

$$\begin{aligned} r(fSu, gv) &\leq g(r(SSu, Tv), r(fSu, SSu), r(gv, Tv), r(fSu, Tv), r(gv, SSu)) \\ &= g(r(fSu, gv), 0, 0, r(fSu, gv), r(gv, fSu)), \end{aligned}$$

which implies $fu = gTv$, $fSu = gv$ and $fu = Tv = ffu = Sfu = TTv = gTv$ is a common fixed Point of f and g . Uniqueness of the fixed point follows from condition (1).

We thus see that under the contractive condition (1) of Theorem 1.1⁷, assumption of owc and the existence of a unique common fixed point are equivalent conditions and replacing weak compatibility by owc does not yield any generalization. This shows that in Theorem 1.1⁷ contractive conditions induced by implicit relations do not provide a nontrivial setting for the application of owc.

For providing nontrivial applications of owc contractive conditions should be replaced by more general conditions that may hold for mappings satisfying contractive as well as nonexpansive and Lipschitz type conditions. We adopt this approach in the next theorem (Theorem 1.2) and this approach can also be used for improving results of Jungck and Rhoades⁷ and all other results for owc mappings that have been obtained under contractive conditions.

Theorem 1.2: Let (X, d) be a symmetric space with symmetric r and let f and S be occasionally weakly compatible self-mappings of X satisfying (i) $r(Sx, S^2x) \neq \max\{r(Sx, fSx), r(S^2x, fSx)\}$, whenever $Sx \neq S^2x$. Then f and S have a common fixed point.

Proof: Since f and S are owc, there exists a point u in X such that $fu = Su$ and $fSu = Sfu$. This in turn yields $ffu = fSu = Sfu = SSu$. If $fu \neq f^2u$ then using (i) we get $r(Su, S^2u) \neq \max\{r(Su, fSu), r(S^2u, fSu)\} = d(Su, S^2u)$, a contradiction. Hence $Su = fSu = SSu$ and Su is a common fixed point of f and S .

We now give an example to illustrate Theorem 1.2.

Example 1.1: Let $X = [0, 1]$ and d be the usual symmetric metric on X . Define $f, g: X \rightarrow X$ as follows

$$fx = \frac{\left(\sqrt{5 - 4(2x - 1)^2} - 1 \right)}{4},$$

$$Sx = \frac{(1 - x)}{3}.$$

Then f and S satisfy all the conditions of the above theorem and have two coincidence points $x = 1$, $x = 1/4$ and a common fixed point $x = 1/4$. It

may be verified in this example that f and S are owc maps. f and S are owc since they commute at one of their coincidence points $x = 1/4$. It is also easy to verify that f and S satisfy the condition

$$r(Sx, S^2x) < \max\{r(Sx, fSx), r(S^2x, fSx)\}.$$

References

1. M. A. Al-Thagafi and N. Shahzad, Generalized I-nonexpansive selfmaps and Invariant Approximations, *Acta. Math. Sin.*, **24** (2008) 867–876.
2. J. Ali and M. Imdad, An Implicit Function Implies Several Contraction Conditions, *Sarajevo J. Math.*, **4** (2008) 269–285.
3. V. Popa, Some Fixed Point Theorems for Compatible Mappings Satisfying an Implicit Relation, *Demonstratio Math.*, **32(1)** (1999) 157-163.
4. D. Doric, Z. Kadelburg and S. Radenovic, A Note on Occasionally Weakly Compatible Mappings and Common Fixed Points, *Fixed Point Theory* (in press).
5. R. P. Pant, Common Fixed Points of Non commuting Mappings, *J. Math. Anal. Appl.*, **188** (1994) 436-440.
6. G. Jungck, Common Fixed Points for Non continuous Non self Maps on Non metric Spaces, *Far East Journal of Mathematical Sciences*, **4** (1996) 199-215.
7. G. Jungck and B. E. Rhoades, Fixed Point Theorems for Occasionally Weakly Compatible Mappings, *Fixed Point Theory*, **7 (2)** (2006) 287–296; *Fixed Point Theory*, **9** (2008) 383–384 (erratum).