

Unit Root Test of Autoregressive Time Series Model with Partially Linear Time Trend: A Bayesian Approach

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Abstract: The present paper considers the Bayesian analysis of an autoregressive model involving partially linear time trend. The posterior odds ratio for testing the unit root hypothesis is obtained under appropriate prior assumptions. A simulation study is carried out with the objective of observing the impact of misspecifying time trend on the posterior odds ratio of unit root hypothesis.

Key words and phases: autoregressive models; partially linear time trend; unit root hypothesis; prior distribution; posterior odds ratio.

1. Introduction

A partially linear model is a particular econometric model which is hybrid of a linear regression model and a completely non linear model. In the context of time series model if dependency on time is non linear, it can be modeled using partially linear time trend. The presence of unit root in economic time series has profound implications from both statistical analysis and economic theorizing point of view. Dickey and Fuller^{1,2} considered a test for testing the unit root hypothesis for first order autoregressive time series model. The test for trend stationarity against difference stationarity in autoregressive models have been proposed by Nelson and Plosser³. Juhl and Xiao⁴ studied non-stationarity of time series involving partially linear time trend and tested the unit root hypothesis using semi parametric approach. Sims⁵ and Sims and Uhling⁶, Dejong and Whiteman⁷, Schotman and van Dijk⁸, Silvapulle and Maddock⁹, Andrews¹⁰, Hasegawa *et al*¹¹, Chaturvedi and Kumar^{12,13} considered Bayesian approach for testing unit root hypothesis for various time series models.

The objective of present paper is to develop Bayesian unit root test for autoregressive model involving partially linear time trend. The posterior odds ratio for the unit root hypothesis has been derived under appropriate prior assumptions for the model with and without augmentation term. The simulation study is carried out for observing impact on posterior odds ratio when (i) partially linear time trend is taken into account and (ii) when it is ignored. Some trend functions have been selected from Juhl and Xiao⁵ for simulation purpose.

2. Model with partially linear time trend

Let us consider the time series model

$$(2.1) \quad y_t = \mu + \delta t + u_t$$

where $\{y_t; t = 1, 2, \dots, T\}$ are the time series observations and u_t is the disturbance term following AR(1) process $u_t = \rho u_{t-1} + \varepsilon_t$.

We can write the model (2.1) as

$$(2.2) \quad (y_t - \phi) = \rho(y_{t-1} - \phi) + \beta t + \varepsilon_t$$

where
$$\phi = \mu + \frac{\rho}{1 - \rho} \delta \quad \beta = (1 - \rho)\delta$$

Here $\{\varepsilon_t\}$ is a white noise process with mean zero and variance $\frac{1}{\tau}$.

We consider a more general form of time series model (2.2) involving a partially non-linear time trend $g(t)$:

$$(2.3) \quad (y_t - \phi) = \rho(y_{t-1} - \phi) + \beta t + g(t) + \varepsilon_t$$

For the real series first we test whether $g(t)$ is associated with the series or not. If yes perform the unit root testing with this model, otherwise one may use model with linear time trend.

We are interested in testing the unit root hypothesis $H_0 : \rho = 1$ against the alternative hypothesis $H_1 : \rho \in S; S = \{a < \rho < 1; a > -1\}$. Under the unit root hypothesis model (2.3) reduces to

$$(2.4) \quad \Delta y_t = \delta + g(t) + \varepsilon_t$$

where $\Delta y_t = y_t - y_{t-1}$

Let us assume the following prior distributions of the parameters of the model (see Schotman and van Dijk⁸)

$$(2.5) \quad \begin{aligned} p(\phi|\rho, \tau) &\sim N\left(y_0, \frac{1}{(1-\rho^2)\tau}\right), \\ p(\beta|\rho, \tau) &\sim N\left(0, \frac{(1-\rho)^2}{v\tau}\right), \\ p(\delta|\tau) &\sim N\left(0, \frac{1}{v\tau}\right), \\ p(\rho) &\sim U(1, a), a > -1, \\ p(\tau) &\propto \frac{1}{\tau}, \end{aligned}$$

where y_0 is the initial observation.

Further we define the following notations

$$(2.6) \quad \begin{aligned} k(\rho) &= \left[\frac{T(T+1)(2T+1)}{6} + \frac{v}{(1-\rho)^2} \right], \\ h(\rho) &= \left\{ T(1-\rho)^2 + (1-\rho^2) - \frac{(1-\rho)^2 \{T(T+1)\}^2}{4k(\rho)} \right\}, \\ \eta(\rho) &= \sum_{t=1}^T (y_t - \rho y_{t-1} - g(t))^2 - \frac{1}{k(\rho)} \left\{ \sum_{t=1}^T t(y_t - \rho y_{t-1} - g(t)) \right\}^2 \\ &\quad (1-\rho^2)y_0^2 - h(\rho)\hat{\phi}^2, \\ \hat{\phi} &= \frac{1}{h(\rho)} \left[(1-\rho) \sum_{t=1}^T (y_t - \rho y_{t-1} - g(t)) + y_0(1-\rho^2) - \frac{T(T+1)(1-\rho)}{2k(\rho)} \right. \\ &\quad \left. \sum_{t=1}^T t(y_t - \rho y_{t-1} - g(t)) \right], \\ k &= [T + v], \\ \eta &= \sum_{t=1}^T (\Delta y_t - g(t))^2 - \frac{1}{k} \left[\sum_{t=1}^T (\Delta y_t - g(t)) \right]^2, \end{aligned}$$

Theorem 1: The posterior odds ratio for testing the unit root hypothesis for a time series model involving partial linear time trend, with prior odds ratio $p_0/(1-p_0)$, is given by

$$(2.7) \quad \beta_{01} = \frac{p_0}{1-p_0} \frac{(1-a)k^{-\frac{1}{2}}(\eta)^{-\frac{T}{2}}}{\int_a^1 \frac{(1-\rho^2)^{\frac{1}{2}}}{(1-\rho)(k(\rho))^{\frac{1}{2}}(h(\rho))^{\frac{1}{2}}(\eta(\rho))^{\frac{T}{2}}} d\rho}.$$

Proof: The Likelihood function under the alternative hypothesis is given by

$$(2.8) \quad p(y|\phi, \beta, \rho, \tau) = \frac{\tau^{\frac{T}{2}}}{(2\pi)^{\frac{T}{2}}} \exp \left[-\frac{\tau}{2} \left\{ \sum_{t=1}^T (y_t - \rho y_{t-1} - \phi(1-\rho) - \beta t - g(t))^2 \right\} \right]$$

Combining the likelihood function (2.8) with the prior distributions (2.5), we get the marginal distribution of y under H_1 as

$$p(y|H_1) = \int_a^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\tau^{\frac{T}{2}} v^{\frac{1}{2}} (1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{T}{2}+1} (1-a)} \exp \left[-\frac{\tau}{2} \left\{ \sum_{t=1}^T (y_t - \rho y_{t-1} - \phi(1-\rho) - \beta t - g(t))^2 + \frac{v\beta^2}{(1-\rho)^2} + (1-\rho^2)(\phi - y_0)^2 \right\} \right] d\beta d\phi d\tau d\rho.$$

We observe that

$$\begin{aligned} \sum (\beta t)^2 - 2 \sum \beta t (y_t - \rho y_{t-1} - \phi(1-\rho) - g(t)) + \frac{v\beta^2}{(1-\rho)^2} \\ = (\beta - \hat{\beta})^2 k(\rho) - \hat{\beta}^2 k(\rho). \end{aligned}$$

where

$$(2.9) \quad \hat{\beta} = \frac{1}{k(\rho)} \sum t(y_t - \rho y_{t-1} - \phi(1-\rho) - g(t))$$

Hence, we have

(2.10)

$$\begin{aligned}
p(y|H_1) &= \int_{a=-\infty}^1 \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\tau^{\frac{T}{2}} v^{\frac{1}{2}} (1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{T}{2}+1} (1-a)} \exp \left[-\frac{\tau}{2} \left\{ \sum_{t=1}^T (y_t - \rho y_{t-1} - \phi(1-\rho) - g(t))^2 \right. \right. \\
&\quad \left. \left. + (1-\rho^2)(\phi - y_0)^2 + (\beta - \hat{\beta})^2 k(\rho) \right. \right. \\
&\quad \left. \left. - \frac{1}{k(\rho)} \left(\sum_{t=1}^T t(y_t - \rho y_{t-1} - \phi(1-\rho) - g(t)) \right)^2 \right\} \right] d\beta d\phi d\tau d\rho \\
&= \int_{a=-\infty}^1 \int_{-\infty}^{\infty} \frac{\tau^{\frac{T-1}{2}} v^{\frac{1}{2}} (1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{T}{2}+1} (1-a)(1-\rho)(k(\rho))^{\frac{1}{2}}} \exp \left[-\frac{\tau}{2} \left\{ \sum_{t=1}^T (y_t - \rho y_{t-1} - g(t))^2 \right. \right. \\
&\quad \left. \left. - \frac{1}{k(\rho)} \left\{ \sum_{t=1}^T t(y_t - \rho y_{t-1} - g(t)) \right\}^2 + (1-\rho^2)y_0^2 - h(\rho) \left[(\phi - \hat{\phi})^2 - \hat{\phi}^2 \right] \right\} \right] d\phi d\tau d\rho \\
&= \int_{a=0}^1 \int_0^{\infty} \frac{\tau^{\frac{T-2}{2}} v^{\frac{1}{2}} (1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{\frac{T}{2}} (1-a)(1-\rho)(k(\rho))^{\frac{1}{2}} (h(\rho))^{\frac{1}{2}}} \exp \left[-\frac{\tau}{2} \eta(\rho) \right] d\tau d\rho \\
&= \Gamma\left(\frac{T}{2}\right) \frac{v^{\frac{1}{2}}}{\pi^{\frac{T}{2}}} \int_a^1 \frac{(1-\rho^2)^{\frac{1}{2}}}{(1-a)(1-\rho)(k(\rho))^{\frac{1}{2}} (h(\rho))^{\frac{1}{2}}} \left(\frac{1}{\eta(\rho)}\right)^{\frac{T}{2}} d\rho
\end{aligned}$$

Again the likelihood function under the null hypothesis H_0 is

$$(2.11) \quad p(y|\tau) = \frac{\tau^{\frac{T}{2}}}{(2\pi)^{\frac{T}{2}}} \exp \left[-\frac{\tau}{2} \left\{ \sum_{t=1}^T (\Delta y_t - \delta - g(t))^2 \right\} \right]$$

Combining the likelihood function (2.11) with the prior distributions, we obtain the following marginal distribution under the unit root hypothesis:

$$(2.12) \quad p(y|H_0) = \frac{v^{\frac{1}{2}}}{(\pi)^{\frac{T}{2}} k^{\frac{1}{2}}} \left(\frac{1}{\eta}\right)^{\frac{T}{2}} \Gamma\left(\frac{T}{2}\right)$$

Utilizing the marginal posterior distributions (2.10) and (2.12), the posterior odds ratio with the prior odds ratio $p_0/(1-p_0)$ is given by

$$\beta_{01} = \frac{p_0}{1-p_0} \frac{(1-a)k^{\frac{1}{2}}(\eta)^{-\frac{T}{2}}}{\int_a^1 \frac{(1-\rho^2)^{\frac{1}{2}}}{(1-\rho)(k(\rho))^{\frac{1}{2}}(h(\rho))^{\frac{1}{2}}(\eta(\rho))^{\frac{T}{2}}} d\rho},$$

which is the required result.

3. Model with augmentation term

Let we consider more general form of the model (2.3) involving augmentation term of order k

$$(3.1) \quad (y_t - \phi) = \rho(y_{t-1} - \phi) + \beta t + g(t) + \sum_{j=1}^k \theta_j \Delta y_{t-j} + \varepsilon_t$$

Under the unit root hypothesis model becomes

$$(3.2) \quad \Delta y_t = \delta + g(t) + \sum_{j=1}^k \theta_j \Delta y_{t-j} + \varepsilon_t$$

We assume same prior assumptions for the parameters of the model and non-informative prior for the parameters of augmentation term

$$(3.3) \quad p(\theta) \propto 1 \quad ; \quad \theta = (\theta_1 \dots \theta_k)' \in R^k$$

Let us define the following notations

$$(3.4) \quad \Delta Y_t = (\Delta y_{t-1} \quad \Delta y_{t-2} \quad \dots \quad \Delta y_{t-k})'$$

$$\psi_{AUG} = \left(\sum_{t=1}^T \Delta Y_t \sum_{t=1}^T \Delta Y_t' \right)$$

$$A_{AUG} = \left(\sum_{t=1}^T \Delta Y_t \right)' (\psi_{AUG})^{-1} \left(\sum_{t=1}^T \Delta Y_t \right)$$

$$k_{AUG}(\rho) = \left[\frac{T(T+1)(2T+1)}{6} + \frac{v}{(1-\rho)^2} - \left(\frac{T(T+1)}{2} \right)^2 A_{AUG}(\rho) \right]$$

$$\begin{aligned}
x_{AUG}(\rho) &= \left[\sum t(y_t - \rho y_{t-1} - g(t)) - \left(\frac{T(T+1)}{2} \right) \sum (y_t - \rho y_{t-1} - g(t)) A_{AUG}(\rho) \right] \\
w_{AUG}(\rho) &= (1 - \rho) \left(\frac{T(T+1)}{2} \right) [1 - T A_{AUG}(\rho)] \\
h_{AUG}(\rho) &= \left[T(1 - \rho^2) - T^2(1 - \rho)^2 A_{AUG}(\rho) - \frac{1}{k_{AUG}(\rho)} (x_{AUG}(\rho))^2 \right] \\
\hat{\phi}_{AUG} &= \frac{1}{h_{AUG}(\rho)} \left[(1 - \rho) \sum_{t=1}^T (y_t - \rho y_{t-1} - g(t)) (1 - T A_{AUG}(\rho)) \right. \\
&\quad \left. + (1 - \rho^2) y_0 \frac{1}{k_{AUG}(\rho)} w_{AUG}(\rho) x_{AUG}(\rho) \right] \\
\eta_{AUG}(\rho) &= \left[\sum_{t=1}^T (y_t - \rho y_{t-1} - g(t))^2 + (1 - \rho^2) y_0^2 - \left(\sum_{t=1}^T (y_t - \rho y_{t-1} - g(t)) \right)^2 \right. \\
&\quad \left. A_{AUG}(\rho) - \frac{1}{k_{AUG}(\rho)} (w_{AUG}(\rho))^2 - h_{AUG}(\rho) \hat{\phi}_{AUG}^2 \right] \\
k_{AUG} &= (T + v - T^2 A_{AUG}) \\
\hat{\delta}_{AUG} &= \frac{1}{k_{AUG}} \left[\sum_{t=1}^T (\Delta y_t - g(t)) - T A_{AUG} \left(\sum_{t=1}^T (y_t - g(t)) \right) \right] \\
\eta_{AUG} &= \sum_{t=1}^T (\Delta y_t - g(t))^2 - \left(\sum_{t=1}^T (y_t - g(t)) \right)^2 A_{AUG} - k_{AUG} \hat{\delta}_{AUG}^2
\end{aligned}$$

The following theorem gives the posterior odds ratio for the testing the unit root hypothesis in model (3.1) and can be easily proved along the lines of theorem 1:

Theorem 2: *The Posterior odds ratio for unit root hypothesis in model involving partially linear time trend and augmentation term with prior odds ratio $p_0/(1-p_0)$ is given by*

$$(3.4) \quad \beta_{01(AUG)} = \frac{p_0}{1-p_0} \frac{(1-a)k_{AUG}^{-\frac{1}{2}}(\eta_{AUG})^{-\frac{T-k}{2}}}{\int_a^1 \frac{(1-\rho^2)^{\frac{1}{2}}}{(1-\rho)(k_{AUG}(\rho))^{\frac{1}{2}}h_{AUG}(\rho)^{\frac{1}{2}}(\eta_{AUG}(\rho))^{\frac{T-k}{2}}} d\rho}$$

4. Numerical Illustration

For observing the impact of presence of partial time trend on posterior odds ratio, we have generated series from the time series model with partial linear time trend:

$$y_t = \rho y_{t-1} + [(1-\rho)\mu + \rho\delta] + (1-\rho)\delta t + g(t) + \varepsilon_t; \quad \varepsilon_t \sim N(0,1)$$

Where $g(t)$ is the function which accounts for the non-linearity in time trend. For simulation purpose some non-linear forms of $g(t)$ taken from Juhl and Xiao⁵ are $g(t) = t^2 - 1$, $g(t) = 2(t-1)(t-2)$ and $g(t) = (t-1)(t-2)$. We have generated parameters δ and μ from the normal distribution $N(10,1)$ and $N(50,1)$ respectively. Further we have taken $T = 35, 45, 55$; $\rho = 0.90, 0.95, 0.98$ and experiment is replicated 100 times for a particular set of parameters.

The posterior odds ratio is obtained for generated series setting the model (i) with partially linear time trend and (ii) with linear time trend. We accept the unit root hypothesis if the posterior odds ratio is more than 1 leading to the conclusion that series is difference stationary. If posterior odds ratio is less than 1, unit root hypothesis is rejected leading to the conclusion that series is trend stationary. The computation of posterior odds ratio requires numerical integration, which has been performed using Monte Carlo integration technique. The computations have been done using MATLAB 7 software. We take the prior odds ratio as 1, i.e., assign equal prior probabilities to the null hypothesis of difference stationarity and the alternative of trend stationarity. For integrating purpose 10,000 values of autoregressive parameters have been drawn from uniform distribution $(1, a)$. The value of a is obtained following the approach of Schotman and van Dijk⁸.

The numerical results in table 1-3 provide posterior odds ratio for various parametric values and selected functions $g(t)$. The posterior odds ratio is evaluated for two different cases viz (i) incorporating non-linear component of trend $g(t)$ in the model denoted by POR_{nl} and (ii) ignoring non-linear part $g(t)$ and considering linear trend in the model, denoted by POR_l . The ratio POR_l / POR_{nl} is given in the last column of each table 1-3. We observe drastic increase in posterior odds ratio when non-linear

component $g(t)$ is ignored. The maximum change in posterior odds ratio is observed for the function $g(t)=2(t-1)(t-2)$. Further the ratio POR_l / POR_{nl} is increases with increasing sample size and/or increasing ρ .

Table-1

$g(t)$	Size of series (T)	$\rho = 0.9$		
		POR_{nl}	POR_l	$Ratio = POR_l / POR_{nl}$
t^2-1	35	5.51E-63	1.80E-06	3.27E+56
	45	7.97E-94	2.11E-09	2.65E+84
	55	8.99E-125	1.08E-12	1.20E+112
$(t-1)*(t-2)$	35	8.91E-60	3.27E-06	3.67E+53
	45	8.51E-87	4.59E-09	5.39E+77
	55	6.29E-127	2.79E-12	4.44E+114
$2(t-1)*(t-2)$	35	3.13E-71	3.54E-06	1.13E+65
	45	2.67E-105	5.12E-09	1.91E+96
	55	3.39E-142	3.19E-12	9.42E+129

Table-2

$g(t)$	Size of series (T)	$\rho = 0.95$		
		POR_{nl}	POR_l	$Ratio = POR_l / POR_{nl}$
t^2-1	35	1.69E-51	3.07E-06	1.81E+45
	45	1.25E-80	6.80E-09	5.44E+71
	55	4.38E-112	8.18E-12	1.86E+100
$(t-1)*(t-2)$	35	2.42E-48	5.34E-06	2.21E+42
	45	4.32E-78	1.43E-08	3.31E+69
	55	8.83E-110	1.99E-11	2.26E+98
$2(t-1)*(t-2)$	35	2.10E-58	5.66E-06	2.70E+52
	45	3.08E-89	1.52E-08	4.95E+80
	55	1.06E-116	2.15E-11	2.04E+105

Table-3

g(t)	Size of series (T)	$\rho = 0.98$		
		POR_{nl}	POR_l	$Ratio = POR_l / POR_{nl}$
t^2-1	35	7.17E-35	1.47E-03	2.05E+31
	45	4.51E-60	4.24E-07	9.39E+52
	55	1.36E-89	1.48E-10	1.09E+79
$(t-1)*(t-2)$	35	6.71E-33	1.33E-03	1.98E+29
	45	1.90E-58	5.39E-07	2.84E+51
	55	7.72E-88	2.64E-10	3.42E+77
$2(t-1)*(t-2)$	35	3.53E-43	1.83E-03	5.18E+39
	45	9.10E-73	6.66E-07	7.32E+65
	55	1.23E-102	2.96E-10	2.42E+92

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