

Fixed Point Theorems for A-Contractive and E-Expansive Maps in Uniform Space

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Abstract: In this paper we give two common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the concept of an A – distance or an E – distance.

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1. Introduction

In 1968 Kannan¹ proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. Rhoades² find a comparison of various definition of contraction. Then Sessa³ coined the notion of weakly commuting mappings. Jungck⁴ generalized this idea to compatible mapping and then weakly compatible mapping⁵. During this time a number of authors established fixed point theorems for pair of maps^{6,7}.

Definition⁵ 1.1: Let T and S be two self mappings of a metric space (X, d) . S and T are said to be compatible if $\lim_{n \rightarrow \infty} d(STx_n, TSx_n) = 0$ whenever (x_n) is a sequence in X such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t, \text{ for some } t \in X.$$

This notion was frequently used to prove existence theorems in the theory of common fixed point for contractive or expansive self mappings of complete metric spaces. Further, in 1998, Jungck and Rhoades⁸ introduced the following concept of weakly compatible.

Definition⁴ 1.2: Two selfmappint T and S of a metric space X are said to be weakly compatible if they commute at their coincidence points, i.e. if $Tu = Su$ for some $u \in X$, then $TSu = STu$.

Kada, Suzuki and Takahashi⁹ have introduced the concept of a W -distance on metric spaces and have generalized some important results in non-convex minimizations and in fixed point theorems to uniform spaces. Following the ideas of Kada, Suzuki and Takahashi⁹, Montes and Charris¹⁰ established some results on fixed and coincidence points of maps by means of appropriate W -contractive or W -expansive assumptions in uniform spaces. For more information we refer the reader to the book by N. Bourbaki¹¹.

A uniform space $(X, \mathfrak{V})$ a nonempty set X endowed of an uniformity $\mathfrak{V}$, the latter being a special kind of filter on $X \times X$, all whose elements contain the diagonal $\Delta = \{(x, x) / x \in X\}$. If $V \in \mathfrak{V}$ and $(x, y) \in V$, $(y, x) \in V$, x and y are said to be V -close, and a sequence (x_n) in X is a Cauchy sequence for $\mathfrak{V}$ if for any $V \in \mathfrak{V}$, there exists $N \geq 1$ such that (x_n) and (x_m) are V -close for $n, m \geq N$. An uniformity $\mathfrak{V}$ defines a unique topology $T(\mathfrak{V})$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X / (x, y) \in V\}$. Since each $V \in \mathfrak{V}$ contains a symmetrical $W \in \mathfrak{V}$ and if $(x, y) \in W$ then x and y are both W and V -close.

2. Preliminaries

Definition 2.1: Let $(X, \mathfrak{V})$ be a uniform space. A function $p: X \times X \rightarrow \mathbb{R}^+$ is said to be an A -distance if for any $V \in \mathfrak{V}$ there exists $\delta > 0$ such that if $p(z, x) \leq \delta$, and $p(z, y) \leq \delta$, for some $z \in X$, then $(x, y) \in V$.

Definition 2.2: Let $(X, \mathfrak{V})$ be a uniform space. A function $p: X \times X \rightarrow \mathbb{R}^+$ is said to be an E -distance, if

- (p₁) p is an A -distance,
- (p₂) $p(x, y) \leq p(x, z) + p(z, y)$, $\forall x, y, z \in X$.

Example 2.1: Let $(X, \mathfrak{V})$ be a uniform space and let d be a distance on X . Clearly $(X, \mathfrak{V}_d)$ is a uniform space where $\mathfrak{V}_d$ is the set of all subsets of $X \times X$ containing a “band” $B_\varepsilon = \{(x, y) \in X \times X / d(x, y) < \varepsilon\}$ for some $\varepsilon > 0$. Moreover, if $\mathfrak{V} \subseteq \mathfrak{V}_d$, then d is an E -distance on $(X, \mathfrak{V})$.

Example 2.2: Recently, Montes and Charris introduced the concept of W -distance on uniform spaces. Every W -distance p is an E -distance, since it

satisfies (p_1) , (p_2) and the condition: for all $x \in X$, the function $p(x, \cdot)$ is lower semi-continuous.

Example 2.2: Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ the usual metric. Consider the function p defined as follows

$$p(x, y) = \begin{cases} y, & y \in [0, 1) \\ 2y, & y \in [1, +\infty) \end{cases}$$

It is easy to see that the function p is an E-distance on $(X, \mathfrak{T}_d)$ but it is not an W-distance on $(X, \mathfrak{T}_d)$ since the function $p(x, \cdot) : X \times X \rightarrow \mathbb{R}^+$ is not lower semi-continuous at 1.

The following lemma contains some useful properties of A-distances. It is stated⁹ for metric spaces and for uniform spaces. The proof is straightforward.

Lemma 2.1: Let $(X, \mathfrak{T})$ be a Hausdorff uniform space and p be an A-distance (respectively E-distance) on X . Let $(x_n), (y_n)$ be arbitrary sequences in X and $(\alpha_n), (\beta_n)$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following holds:

(a) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.

(b) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then (y_n) converges to z .

(c) If $p(x_n, x_m) \leq \alpha_n$ for all $m > n$, then (x_n) is a Cauchy sequence in $(X, \mathfrak{T})$.

Definition 2.3: Let $(X, \mathfrak{T})$ be a uniform space with an A-distance (respectively E-distance) p . A sequence in X is p -Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting as follows:

- i. X is S-complete if for every p -Cauchy sequence (x_n) , there exists x in X with $\lim_{n \rightarrow \infty} p(x_n, x) = 0$.
- ii. X is p -Cauchy complete if for every p -Cauchy sequence (x_n) , there exists x in X with $\lim_{n \rightarrow \infty} x_n = x$, with respect to $T(\mathfrak{T})$.
- iii. $f: X \rightarrow X$ is p -continuous if $\lim_{n \rightarrow \infty} p(x_n, x) = 0$ implies $\lim_{n \rightarrow \infty} p(f(x_n), f(x)) = 0$.
- iv. $f: X \rightarrow X$ is $T(\mathfrak{T})$ -continuous if $\lim_{n \rightarrow \infty} x_n = x$ with respect to $T(\mathfrak{T})$ implies $\lim_{n \rightarrow \infty} f(x_n) = f(x)$,
- v. X is said to be p -bounded if $\delta_p(X) = \sup\{p(x, y) / x, y \in X\} < \infty$.

Remark 2.1: Let $(X, \mathfrak{T})$ be a Hausdorff uniform space and let (x_n) be a p -Cauchy sequence. Suppose that X is p -complete, then there exists $x \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, x) = 0$. From lemma 2.1(b) it gives $\lim_{n \rightarrow \infty} x_n = x$ with respect to the topology $T(\mathfrak{T})$. Therefore p -completeness implies p -Cauchy completeness.

Lemma 2.2: Let (X, v) be a Hausdorff uniform space and p be an A – distance (respectively E -distance) on X such that X is p – bounded. Let (S, f) and (T, g) be point wise p – weakly commuting pairs of self mappings of X such that

(i) $S(X) \subseteq g(X)$ and $T(X) \subseteq f(X)$,

(ii) $p(S(x), T(y)) \leq \psi [p(S(x), f(x)), p(f(x), g(y)), p(T(y), g(y)), \{p(T(y), f(x)) + p(S(x), g(y))\}/2]$

for all $x, y \in X$. The continuity of one of the mapping in p – compatible pair (S, f) or (T, g) on $(X, \mathfrak{T})$ implies their reciprocal continuity.

Proof: Suppose that S and f are compatible and f is continuous. We will prove that the pair (S, f) is reciprocally continuous. Let $\{x_n\}$ be a sequence in X such that $S(x_n) \rightarrow x$ and $f(x_n) \rightarrow x$ for some $x \in X$, as $n \rightarrow \infty$. Since f is continuous, so $f(S(x_n)) \rightarrow f(x)$ and $f(f(x_n)) \rightarrow f(x)$ as $n \rightarrow \infty$.

Since (S, f) is p – compatible pair so that $\lim_{n \rightarrow \infty} p(S(f(x_n)), f(S(x_n))) = 0$, which implies that $\lim_{n \rightarrow \infty} p(S(f(x_n)), f(x)) = 0$ or $S(f(x_n)) \rightarrow f(x)$, as $n \rightarrow \infty$.

By (i), for each n , there exists a sequence $\{y_n\}$ in X such that $S(f(x_n)) = g(y_n)$. Thus for each $n \rightarrow \infty$, we have

$$f(f(x_n)) \rightarrow f(x), S(f(x_n)) \rightarrow f(x) \text{ and } g(y_n) \rightarrow f(x)$$

whenever $S(f(x_n)) = g(y_n)$. Again we claim that $T(y_n) \rightarrow f(x)$, as $n \rightarrow \infty$.

If it is not, then there exists subsequence $\{T(y_m)\}$ of $\{T(y_n)\}$ such that for each $\varepsilon > 0$, and a positive integer n_0 such that for all $m > n_0$, $p(T(y_m), f(x)) \leq \varepsilon$ and $p(S(f(x_m)), T(y_m)) \leq \varepsilon$.

From (ii), we have

$$\begin{aligned} p(S(f(x_n)), T(y_m)) &\leq \psi [p(S(f(x_m)), f(f(x_m))), p(f(f(x_m)), g(y_m)), p(T(y_m), \\ &\quad g(y_m)), \{p(T(y_m), f(f(x_m))), p(S(f(x_m), g(y_m))\}/2] \\ &= \psi [p(f(x)), f(x)), p(f(x), f(x)), p(T(y_m), f(x)), \\ &\quad \{p(T(y_m), f(x)), p(f(x), f(x))\}/2] \\ &< p(T(y_m), f(x_m)) \leq \varepsilon \end{aligned}$$

which is a contradiction.

Hence $T(y_m) \rightarrow f(x)$ as $n \rightarrow \infty$.

Now due to p -continuity, we have for $n \rightarrow \infty$, $p(S(x_n), T(y_n)) \leq \varepsilon$, whenever $p((x), y_n) = 0$. On letting $n \rightarrow \infty$, $p(S(x), f(x)) = 0$, it follows that $S(x) = f(x)$.

Thus we have $S(f(x_n)) \rightarrow S(x)$ and $f(S(x_n)) \rightarrow S(x) = f(x)$ as $n \rightarrow \infty$.

Therefore S and f are reciprocally continuous on X . If the pair (T, g) is assumed to be compatible and T is continuous, the proof is similar.

Definition 2.4: Let (X, v) be a uniform space and p be an A – distance (respectively E -distance) on X .

Before we state our main results, we give a formulation of the concept of compatibility and weak compatibility in the setting of uniform spaces as follows:

Definition 2.5: Let (X, v) be a Hausdorff uniform space and p be an A –distance (respectively E -distance) on X . Two self mappings f and g of X are said to be p – compatible if, for each sequence (x_n) of X such that

$$\lim_{n \rightarrow \infty} p(f(x_n), u) = \lim_{n \rightarrow \infty} p(g(x_n), u) = 0,$$

for some $u \in X$, one has $\lim_{n \rightarrow \infty} p(fg(x_n), g(f(x_n))) = 0$.

Definition 2.6: Let (X, v) be a Hausdorff uniform space and p be an A – distance (respectively E -distance) on X . Two self mappings f and g of X are said to be weak compatible if they commute at their coincidence points.

3. Common Fixed Point Result

In the sequel, we involve a non-decreasing function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying:

(i) For each $t \in]0, +\infty[$, $0 < \psi(t)$.

(ii) $\lim_{n \rightarrow \infty} \psi^n(t) = 0$, $\forall t \in]0, +\infty[$

It is easy to see that under the above properties, ψ satisfies also, $\psi(t) < t$, for each $t > 0$.

Amri and Matakawaski¹² proved a theorem for two weakly compatible maps as follows:

Theorem 3.1: Let (X, v) be a Hausdorff uniform space and p be an A – distance on X such that X is p –bounded. Let f and g be two weakly compatible self mappings of X such that

$$(1) \quad f(X) \subseteq g(X)$$

$$(2) \quad p(f(x), f(y)) \leq \psi\{p(g(x), g(y))\}$$

for all $x, y \in X$. If the range of f or g is a S – complete space of X , then f and g have a common fixed point.

We extend the above theorem for four mappings using p – reciprocally continuity.

Theorem 3.2: Let (X, v) be a Hausdroff uniform space and p be an A – distance on X such that X is p – bounded. Let f, g, S and T four self mappings with pairs (f, S) and (g, T) be point wise weakly commuting pair of X such that

$$(\psi_1) \bullet \quad S(X) \subseteq g(X) \text{ and } T(X) \subseteq f(X)$$

$$(\psi_2) \bullet \quad p(S(x), T(y)) \leq \psi [p(S(x), f(x)), p(f(x), g(y)), p(T(y), g(y)), \{p(T(y), f(x)) + p(S(x), g(y))\}/2]$$

for all $x, y \in X$. Then the p – continuity of one of the mappings in p – compatible pairs (S, f) or (T, g) implies the unique common fixed point of f, g, S and T .

Proof: Suppose that the mapping in compatible pair (S, f) or (T, g) is continuous. Then by lemma 2.2, S, f are reciprocally continuous. Let $x_0 \in X$. From condition (i), there exists $x_1, x_2 \in X$ such that $S(x_0) = g(x_1) = y_1$ and $T(x_1) = f(x_2) = y_2$. In general, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$(3.1) \quad S(x_{2n}) = g(x_{2n+1}) = y_{2n+1} \quad \text{and} \quad T(x_{2n+1}) = f(x_{2n+2}) = y_{2n+2}$$

for $n = 0, 1, 2, \dots$

Let us define $P_{2n+1} = p(y_{2n+1}, y_{2n+2})$ and putting $x = x_{2n}$ and $y = x_{2n+1}$, in (ii), we have

$$\begin{aligned} P_{2n+1} &= p(S(x_{2n}), T(x_{2n+1})) \leq \psi [p(S(x_{2n}), f(x_{2n})), p(f(x_{2n}), g(x_{2n+1})), \\ &\quad p(T(x_{2n+1}), g(x_{2n+1})), \{p(T(x_{2n+1}), f(x_{2n})) + p(S(x_{2n}), g(x_{2n+1}))\}/2] \\ &= \psi [p(y_{2n+1}, y_{2n}), p(y_{2n}, y_{2n+1}), p(y_{2n+2}, y_{2n+1}), \\ &\quad \{p(y_{2n+2}, y_{2n}) + p(y_{2n+1}, y_{2n+1})\}/2] \\ &= \psi [P_{2n}, P_{2n}, P_{2n+1}, \{p(y_{2n}, y_{2n+2}) + p(y_{2n+1}, y_{2n+1})\}/2] \\ (3.2) \quad &= \psi [P_{2n}, P_{2n+1}], \end{aligned}$$

in view of (p_2) . Now consider two cases of $P_{2n+1} \geq P_{2n}$, for some n , then from (3.2),

$$P_{2n+1} \leq \psi(P_{2n+1}) < P_{2n+1}, \text{ which is a contradiction.}$$

Thus $P_{2n} > P_{2n+1}$ for all n , and so from (3.2), we have

$$(3.3) \quad P_{2n+1} \leq \psi(P_{2n}).$$

Similarly

$$(3.4) \quad P_{2n} \leq \psi(P_{2n-1}).$$

In general, we have for all $n = 1, 2, \dots$

$$(3.5) \quad P_n \leq \psi(P_{n-1}).$$

Now from (3.5) $P_n \leq \psi(P_{n-1}) \leq \psi^2(P_{n-2}) \dots \psi^n(P_0)$. Thus

$$(3.6) \quad p(S(x_n), T(x_{n+1})) \leq \psi[p(S(x_0), T(x_1))],$$

or

$$(3.7) \quad p_n \leq \psi(P(X)), \text{ where } P(X) = \sup \{p(x, y) / x, y \in X\}.$$

Then by (ii) and lemma 2.1(a), we deduce that sequence $\{y_n\}$ is a p -Cauchy sequence. For this it is sufficient to show that $\{y_{2n}\}$ is also a p -Cauchy sequence. Since X is p -complete, then there exists a point z in X such that $y_n \rightarrow z$ as $n \rightarrow \infty$. Moreover

$$(3.8) \quad y_{2n+1} = S(x_{2n}) = g(x_{2n+1}) \rightarrow z$$

and

$$y_{2n+2} = T(x_{2n+1}) = f(x_{2n+2}) \rightarrow z.$$

Since f and S are p -compatible and reciprocally p -continuous mappings, then $S(f(x_{2n})) \rightarrow f(z)$ and $f(S(x_{2n})) \rightarrow S(z)$. The p -compatibility of S and f yields

$$(3.9) \quad \lim_{n \rightarrow \infty} p(S(f(x_n)), f(S(x_n))) = 0.$$

This implies for each sequence $\{x_n\}$ in X ,

$$\lim_{n \rightarrow \infty} p(S(x_n), z) = \lim_{n \rightarrow \infty} p(f(x_n), z) = 0.$$

Hence from (3.9), $S(z) = f(z)$.

Since $S(x) \subseteq g(x)$, there exists a point w in X such that $S(z) = g(w)$. Now we have in (ψ_2) ,

$$\begin{aligned} p(S(z), T(w)) &\leq \psi[p(S(z), f(z)), p(f(z), g(w)), p(T(w), g(w)), \{p(T(w), \\ &\quad f(z)) + p(S(z), g(w))\}/2] \\ &= \psi[p(S(z), S(z)), p(S(z), S(w)), p(T(w), S(z)), \{p(T(w), \\ &\quad S(z)) + p(S(z), S(z))\}/2] \\ &\leq \psi[0, 0, p(T(w), S(z))] \\ (3.10) \quad &= \psi[p(T(w), S(z))] < p(T(w), S(z)) \end{aligned}$$

That is in view of (ψ_2) , $S(z) = T(w)$. Thus

$$(3.11) \quad S(z) = f(z) = g(w) = T(w).$$

Since S and f are point-wise p -weakly commuting mappings such that

$$(3.12) \quad p(S(f(z)), f(S(z))) \leq p(S(z), f(z))$$

i.e.,

$$(3.13) \quad S(f(z)) = f(S(z)) \text{ and } S(S(z)) = S(f(z)) = f(S(z)) = f(f(z)).$$

Similarly since (T, g) is a p – compatible pair, so that

$$(3.14) \quad T(T(w)) = T(g(w)) = g(T(w)) = g(g(w)).$$

Now show that $S(z) = S(S(z))$, for which we take $x = S(z)$ and $y = w$ in (ii), we have

$$\begin{aligned} p(S(S(z)), T(w)) &\leq \psi [p(S(S(z)), f(S(z)), p(f(S(z)), g(w)), p(T(w), f(S(z)), \\ &\quad \{p(T(w), f(S(z)) + p(S(S(z), g(w))\}/2] \\ &= \psi [p(S(S(z), S(S(z))), p(S(S(z), S(z))), p(S(z), S(S(z))), \\ &\quad \{p(S(z), S(S(z))) + p(S(S(z), S(z))\}/2] \\ (3.15) \quad &\leq p(S(S(z), S(z)), \end{aligned}$$

in view of (ψ_2) . That is $S(z) = S(S(z))$ and so $S(z) = S(S(z) = f(S(z))$. Thus $S(z)$ is a common fixed point of S and f . Similarly we can prove that $T(w)$ is a common fixed point of B and g . Since $T(w) = S(z)$, hence $S(z)$ is a common fixed point of f, g, S and T . Thus $S(z)$ is a common fixed point of f, g, S and T . For uniqueness suppose that $S(u) (\neq S(z))$. Then from (ii)•, we have

$$\begin{aligned} p(S(S(z), T(S(u))) &\leq \psi [p(S(S(z), f(S(z))), p(f(S(z), g(S(u))), p(T(S(u), \\ &\quad g(S(u))), \{p(T(S(u), f(S(z))) + p(S(S(z), g(S(u)))\}/2] \\ &= \psi [p(S(z), S(z)), p(S(z), S(z)), p(S(z), S(z)), \\ &\quad \{p(S(z), S(S(z))) + p(S(S(z), S(z))\}/2] \\ &\leq \psi [p(S(z), S(u))] \end{aligned}$$

implies that $p(S(S(z), T(S(u))) = 0$, in view of (ψ_2) , or

$$(3.16) \quad p(S(z), S(u)) = 0 \Rightarrow S(z) = S(u).$$

Since an E – distance function p is an A – distance, then if f, g, S and T have a unique common fixed point $S(z)$, then there exists $z \in X$ such that $S(z) = z$. Thus z is a common fixed point of f, g, S and T .

If we take $f = g = I$ (identity mapping on X) in our theorem, we have the following result

Corollary 3.1: *Let (X, v) be a Hausdroff uniform space and p be an A – distance on X such that X is p – bounded. Let S and T be two weakly compatible self mappings of X such that*

- i. $S(X) \subseteq T(X)$
- ii. $p(S(x), T(y)) \leq \psi [p(S(x), x), p(x, y), p(T(y), y)\{p(T(y), x) + p(S(x), y)\}/2]$

for all $x, y \in X$. If S and T are reciprocally continuous mappings, then S and T have a unique common fixed point.

Thus S and T have a unique common fixed point. Thus our theorem generalizes the theorem of Amri and Matakawaski¹².

Example 3.1: $X = [0, 1]$ and $d(x, y) = |x - y|$, the usual metric. Let f, g, S and T defined as

$$\begin{aligned} f(x) &= \begin{cases} x & , \quad 0 \leq x < \frac{1}{2} \\ 1 & , \quad \frac{1}{2} \leq x \leq 1 \end{cases} \\ g(x) &= \begin{cases} x & , \quad 0 \leq x < \frac{1}{2} \\ 1 & , \quad \frac{1}{2} \leq x \leq 1 \end{cases} \\ S(x) &= \begin{cases} x^2 & , \quad 0 \leq x < \frac{1}{2} \\ 0 & , \quad \frac{1}{2} \leq x \leq 1 \end{cases} \\ T(x) &= \begin{cases} x^2 & , \quad 0 \leq x < \frac{1}{2} \\ 0 & , \quad \frac{1}{2} \leq x \leq 1 \end{cases}. \end{aligned}$$

Consider the function p and ψ defined as follows

$$\psi(x) = \begin{cases} x^2 & , \quad 0 \leq x < \frac{1}{2} \\ \frac{x}{2} & , \quad \frac{1}{2} \leq x \leq 1 \end{cases}$$

and

$$p(x, y) = \begin{cases} y & , \quad 0 \leq y < \frac{1}{2} \\ 1 & , \quad \frac{1}{2} \leq y \leq 1. \end{cases}$$

Then f, g, S and T satisfies all the conditions of our Theorem 3.2. Also, we have

$$d\left(S\left(\frac{1}{4}\right), T\left(\frac{1}{3}\right)\right) = \frac{7}{144} > \psi\left(d\left(S\left(\frac{1}{4}\right), f\left(\frac{1}{4}\right)\right), d\left(f\left(\frac{1}{4}\right), g\left(\frac{1}{3}\right)\right), d\left(T\left(\frac{1}{3}\right), g\left(\frac{1}{3}\right)\right)\right),$$

$$\left\{d\left(T\left(\frac{1}{3}\right), f\left(\frac{1}{4}\right)\right) + d\left(S\left(\frac{1}{4}\right), g\left(\frac{1}{3}\right)\right)\right\}/2 \Big] = \frac{9}{256},$$

which implies that $d(S(x), T(y)) \leq \psi(N(x, y))$ does not hold for all $x, y \in X$, where

$$\begin{aligned} N(x, y) &= [d(S(x), f(x)), d(f(x), g(y)), d(T(y), g(y)), \\ &\quad \{d(T(y), f(x)) + d(S(x), g(y))\}/2]. \end{aligned}$$

However, we have

$$p(S(x), T(y)) \leq \psi [p(S(x), f(x)), p(f(x), g(y)), p(T(y), g(y)), \\ \{p(T(y), f(x)) + p(S(x), g(y))\}/2],$$

$\forall x, y \in X$ and 0 is the unique common fixed point of f, g, S and T .

Theorem 3.3: Let (X, ν) be a Hausdroff uniformly space and p be an E – distance on X such that X is p – bounded. Let (S, f) and (T, g) be point wise p – weakly commuting pairs of self mappings of X , such that

$$(i)^\dagger \quad S(X) \subseteq g(X) \text{ and } T(X) \subseteq f(X),$$

$$(ii)^\dagger \quad p(S(x), T(y)) \leq \psi [p(S(x), f(x)), p(f(x), g(y)), p(T(y), g(y)), \{p(T(y), f(x)) + p(S(x), g(y))\}/2]$$

for all $x, y \in X$. The continuity of one of the mapping in p – compatible pair (S, f) or (T, g) on

(X, ν) implies unique common fixed point of f, g, S and T .

Proof: Since an E – distance function p is an A – distance function, so that f, g, S and T have a unique common fixed point. Suppose z and w are two common fixed points of f, g, S and T such that

$$f(z) = g(z) = S(z) = T(z) = z \quad \text{and} \quad f(w) = g(w) = S(w) = T(w) = w.$$

If $p(z, w) \neq 0$, then

$$\begin{aligned} p(z, w) &= p(S(z), T(w)) \\ &\leq \psi [p(S(z), f(z)), p(f(z), g(w)), p(T(w), g(w)) \\ &\quad \{p(T(w), f(z)) + p(S(z), g(w))\}/2] \\ &= \psi [p(z, z), p(z, w), p(w, w), \{p(w, z) + p(z, w)\}/2] \\ &< p(z, w), \end{aligned}$$

which is a contradiction. Thus $p(z, w) = 0$, similarly we can show that

$p(w, z) = 0$. Consequently by (p_2) , we have $p(z, z) \leq p(z, w) + p(w, z)$ and therefore $p(z, z) = 0$.

Now we have $p(z, z) = 0$ and $p(z, w) = 0$, which implies $z = w$. Hence the theorem.

Corollary 3.2: Let (X, ν) be a Hausdroff uniformly space and p be an E – distance on X . Suppose that X is p – bounded and S – complete. Let $f = g$ and $S = T = I$ (an identity mapping on X) such that

$$p(x, y) \leq \psi [p(f(x), f(y))] \text{ if } x, y \in X.$$

Then f has a unique fixed point.

Thus our generalize theorem of Montes and Chris¹⁰.

4. Common Fixed Point Result for E –Expansive

Weakly Compatible Maps

In this section we involve a non-decreasing function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying the following conditions:

(ϕ_1) For each $t > 0$, $\phi(t) > t$,

(ϕ_2) For any decreasing sequence (t_n) in $\mathbb{R}^+$, if $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} \phi(t_n) = t$, some $t \in \mathbb{R}^+$, then $t = 0$.

Theorem 4.1: Let (X, ν) be a Hausdroff uniform space and p be an E – distance on X . Let f, g, S and T be four mappings with pairs (f, S) and (g, T) be point wise weakly commuting self mappings of X such that

(i) $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$

(ii) $p(S(x), T(y)) \geq \phi[p(S(x), f(x)), p(f(x), g(y)), p(T(y), g(y)), \{p(S(x), g(y)) + p(T(y), f(x))\}/2]$

for all $x, y \in X$. Then the p – continuity of one of the mappings in p – compatible pairs (f, S) or (g, T) implies the unique common fixed point of f, g, S and T .

Proof: Suppose that S and f are compatible and f is continuous. We will prove that the pair (f, S) or (g, T) is reciprocally continuous. Let $x_0 \in X$, from condition (ii) † , there exists x_1, x_2, ε, X such that $f(x_0) = T(x_1) = y_1$ and $g(x_1) = S(x_2) = y_2$. In general we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$y_{2n+1} = f(x_{2n}) = T(x_{2n+1})$ and $y_{2n+2} = g(x_{2n+1}) = S(x_{2n+2})$
for $n = 0, 1, 2, \dots$.

Let us define $M_{2n+1} = p(y_{2n+1}, y_{2n+2})$ and putting $x = x_{2n}$ and $y = x_{2n+1}$ in, we have

$p(S(x_{2n}), T(x_{2n+1})) \geq \phi [p(S(x_{2n}), f(x_{2n})), p(f(x_{2n}), g(x_{2n+1})), p(T(x_{2n+1}), g(x_{2n+1})), \{p(S(x_{2n}), g(x_{2n+1})) + p(T(x_{2n+1}), f(x_{2n}))\}/2]$

or, $p(y_{2n}, y_{2n+1}) \geq \phi [p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2}), p(y_{2n+1}, y_{2n+2}), \{p(y_{2n}, y_{2n+2}) + p(y_{2n+1}, y_{2n+1})\}/2]$

or,

(4.1) $M_{2n} \geq \phi [M_{2n}, M_{2n+1}].$

Case I: If $M_{2n} \leq M_{2n+1}$ for some n , then from (4.1), $M_{2n} \geq \phi(M_{2n}) > M_{2n}$, which is contradiction, so we have $M_{2n} > M_{2n+1}$ for all n , and so from (4.1), we have

(4.2) $M_{2n} \geq \phi(M_{2n+1}).$

Similarly, we can take

$$(4.3) \quad M_{2n-1} \geq \phi(M_{2n}).$$

In general we can see for all $n = 1, 2, 3, \dots$

$$(4.4) \quad M_n \geq \phi(M_{n+1}).$$

Now from (4.4), we have

$$(4.5) \quad M_n \geq \phi(M_{n+1}) \geq \phi^2(M_{n+2}) \dots \phi^m(M_{n+m}),$$

which implies that $\{M_n\}$ and $\{\phi(M_n)\}$ are decreasing sequences and the $\lim_{n \rightarrow \infty} M_n$ and $\lim_{n \rightarrow \infty} \phi(M_n)$ exist. Therefore on letting $n \rightarrow \infty$, we obtain $M_n = \phi(M_n) = t$ for some $t \in \mathbb{R}^+$. From condition (ϕ_2) , it gives $t = 0$. Hence $\lim_{n \rightarrow \infty} p(y_{2n}, y_{2n+1}) = 0$. Now we show that the sequence $\{\phi(M_{2n})\}$ is a p -Cauchy sequences. Suppose that $\{\phi(M_{2n})\}$ is not a p -Cauchy sequence. Then there exists a positive number ε such that for each positive integer $2k$, there exists integer $2n(k)$ and $2m(k)$ such that $2k \leq 2n(k) \leq 2m(k)$, and

$$(4.6) \quad p(y_{2n(k)}, y_{2m(k)}) \geq \varepsilon.$$

For each integer $2k$, let $2m(k)$ denotes the smallest integer satisfying the last two inequalities. Then

$$(4.7) \quad p(y_{2n(k)}, y_{2m(k)-2}) < \varepsilon.$$

From (p_1) , we get

$$\begin{aligned} \varepsilon &\leq p(y_{2n(k)}, y_{2m(k)}) \leq p(y_{2n(k)}, y_{2m(k)-2}) \\ &\quad + p(y_{2m(k)-2}, y_{2m(k)-1}) + p(y_{2m(k)-1}, y_{2m(k)}), \end{aligned}$$

which in view of (4.7) gives $p(y_{2n(k)}, y_{2m(k)}) = \varepsilon$, since

$$\lim_{n \rightarrow \infty} p(y_{2n}, y_{2n+1}) = 0.$$

Again, we have

$$\begin{aligned} p(y_{2n(k)-1}, y_{2m(k)-1}) &= p(S(x_{2n}), T(x_{2n+1})) \\ &\geq \phi[p(S(x_{2n}), f(x_{2n})), p(f(x_{2n}), g(x_{2n+1})), p(T(x_{2n+1}), \\ &\quad g(x_{2n+1})), \\ &\quad \{p(S(x_{2n}), g(x_{2n+1})) + p(T(x_{2n+1}), f(x_{2n}))\}/2] \\ &= \phi[p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2}), p(y_{2n+1}, y_{2n+2}), \{ \\ &\quad p(y_{2n}, y_{2n+2}) + p(y_{2n+1}, y_{2n+1})\}/2] \\ &\geq \phi[p(y_{2n}, y_{2n+1}) > p(y_{2n(k)}, y_{2m(k)})] \end{aligned}$$

$$\begin{aligned} \text{and} \quad p(y_{2n(k)-1}, y_{2m(k)-1}) &\leq p(y_{2n(k)-1}, y_{2n(k)}) + p(y_{2n(k)}, y_{2m(k)-2}) \\ &\quad + p(y_{2m(k)-2}, y_{2m(k)-1}). \end{aligned}$$

On letting $k \rightarrow \infty$, we obtain,

$$(4.8) \quad \lim_{n \rightarrow \infty} \phi[p(y_{2n(k)}, y_{2m(k)})] = \varepsilon.$$

Now we have $\lim_{k \rightarrow \infty} p(y_{2n(k)}, y_{2m(k)}) = \lim_{k \rightarrow \infty} \phi[p(y_{2n(k)}, y_{2m(k)})] = \varepsilon$
 By (ϕ_2) , we get $\varepsilon = 0$, which is a contradiction. Thus the sequence $\{\phi(M_n)\}$ is a p -Cauchy sequence. Since X is p -complete, then there exists a point u in X such that $\phi(M_n) \rightarrow u$ as $n \rightarrow \infty$. Thus we show that $S(u) = T(u)$.
 Moreover

$$(4.9) \quad y_{2n+1} = f(x_{2n}) = T(x_{2n+1}) = u$$

$$\text{and} \quad y_{2n+2} = g(x_{2n+1}) = S(x_{2n+2}) = u.$$

Since f and S are p -compatible and reciprocally p -continuous mappings, then $f(S(x_{2n})) \rightarrow S(u)$ and $S(f(x_{2n})) \rightarrow f(u)$.

The p -compatibility of f and S yields, $\lim_{n \rightarrow \infty} (f(S(x_n)), S(f(x_n))) = 0$, which implies for each sequence $\{x_n\}$ in X ,

$$(4.10) \quad \lim_{n \rightarrow \infty} p(f(x_n), u) = \lim_{n \rightarrow \infty} p(S(x_n), u) = 0.$$

Hence from (4.10), we have $f(u) = S(u)$.

Again since $g(X) \subseteq S(X)$, then there exists a point w in X , such that $g(w) = S(u)$.

From (ii) ‡ , we have

$$\begin{aligned} p(S(u), T(w)) &\geq \phi[p(S(u), f(w)), p(f(u), g(w)), p(T(w), \\ &\quad g(w)), \{p(S(u), g(w)) + p(T(w), f(u))\}/2] \\ &= \phi[0, p(S(u), S(u)), p(T(w), S(u)), \\ &\quad \{p(S(u), S(u)) + p(T(w), S(u))\}/2] \\ (4.11) \quad &\geq \phi[p(S(u), T(w))] > p(S(u), T(w)) \end{aligned}$$

which is contradiction. Then $p(S(u), T(w)) = 0$, which implies that,

$$S(u) = T(w), \quad \text{i.e.}$$

$$S(u) = f(u) = g(w) = T(w).$$

Since S and f are point wise p -weakly commuting mappings such that,

$$p(f(S(u)), S(f(u))) \leq p(f(u), S(u))$$

i.e. $f(S(u)) = S(f(u))$ and by p -continuity

$$f(f(u)) = f(S(u)) = S(f(u)) = S(S(u)).$$

Similarly (g, T) is a p -compatible pair so that

$$g(g(w)) = g(T(w)) = T(g(w)) = T(T(w)).$$

Now to show that $f(u) = f(f(u))$, we put $x = f(u)$ and $y = w$ in (ii) ‡ , we have

$$\begin{aligned} p(S(f(u)), T(w)) &\geq \phi[p(S(f(u)), f(f(u)), p(f(f(u)), g(w)), p(T(w), g(w)), \\ &\quad \{p(S(f(w)), g(w)) + p(T(w), f(f(u)))\}/2] \end{aligned}$$

$$\begin{aligned}
&= \phi [p(f(f(u)), f(f(u)), p(S(f(u)), T(w)), p(T(w), T(w)), \\
&\quad \{p(S(f(u)), T(w)) + p(T(w), S(f(u)))\}/2] \\
&\geq \phi [p(S(f(u)), T(w))] > p(S(f(u)), T(w))
\end{aligned}$$

in view of (ϕ_2) , we have $S(f(u)) = T(w)$ i.e. $S(f(u)) = f(f(u)) = f(u)$.

Thus $f(u)$ is a common fixed point of f and S . Similarly we prove that $g(w) = f(u)$ is a common fixed point of g and T . Thus $f(u) = u$ is a common fixed point of f, g, S and T . For uniqueness suppose that $f(z) (\neq f(u))$, then from (ii) ‡ , we have

$$\begin{aligned}
p(S(f(u)), T(f(z))) &\geq \phi [p(S(f(u)), f(f(u)), p(f(f(u)), g(f(z))), p(T(f(z)), \\
&\quad g(f(z))), \{p(S(f(u)), g(f(z))) + p(T(f(z)), \\
&\quad f(f(u)))\}/2] \\
&= \phi [p(f(u), f(u)), p(f(u), f(z)), p(f(z), f(z)), \\
&\quad \{p(f(u), f(z)) + p(f(z), f(z))\}/2] \\
(4.12) \quad &\geq \phi [p(f(z), f(u))] > p(f(z), f(u))
\end{aligned}$$

in view of (ϕ_2) , implies that

$$p(S(f(u)), T(f(z))) = 0 \Rightarrow p(f(z), f(u)) = 0 \Rightarrow f(z) = f(u).$$

Thus $f(u)$ is a unique common fixed point of f, g, S and T . Since $f(u) = u$. Thus u is a unique common fixed point of f, g, S and T .

Example 4.1: Let $X = [0, \infty^+)$ and d is a usual metric defined of $d(x, y) = |x - y|$. Let f, g, S and T defined by

$$\begin{aligned}
f(x) &= \begin{cases} \frac{x}{3}, & 0 \leq x < 1, \\ 1, & x \geq 1, \end{cases} & g(x) &= \begin{cases} \frac{x}{2}, & 0 \leq x < 1, \\ 0, & x \geq 1, \end{cases} \\
S(x) &= \begin{cases} \frac{2x}{3}, & 0 \leq x < 1, \\ 2, & x \geq 1, \end{cases} & T(x) &= \begin{cases} \frac{3x}{2}, & 0 \leq x < 1, \\ 1, & x \geq 1 \end{cases}
\end{aligned}$$

consider the function p and ϕ as

$$\phi(x) = \begin{cases} \frac{9}{8}x, & 0 \leq x < 1 \\ x+1, & x \geq 1 \end{cases} \quad \text{and} \quad p(x, y) = \{y, \text{ for all } x \in [0, \infty^+)\}$$

Here we see that all the conditions of our theorem satisfied for each point of X . Here 0 is the unique common fixed point of f, g, S and T . If we put $S = T = f^*$ and $g = f = g^*$, then our theorem generalize theorem of Amari & Matakawaski¹².

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