

On Subspaces of a Finsler Space with Randers Conformal Metric

M. K. Gupta

Department of Pure and Applied Mathematics,
Guru Ghasidas Vishwavidyalaya, Bilaspur (C.G.), India
Email: mkgiaps@rediffmail.com

P. N. Pandey

Department of Mathematics, University of Allahabad Allahabad, India
Email: pnpiaps@rediffmail.com

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Abstract: The aim of the present paper is to find certain geometrical properties of the subspaces of a Finsler space $(M^n, *L)$ with Randers conformal metric $*L(x, y) = e^{\sigma(x)} L(x, y) + b_i(x)y^i$.

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1. Introduction

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space equipped with the fundamental function $L(x, y)$. For a given function $\sigma(x)$, the transformation $L(x, y) \rightarrow e^{\sigma(x)} L(x, y)$ is called a conformal transformation. The conformal theory of Finsler spaces was studied by M. S. Knebelman¹, M. Hashiguchi², H. Izumi³⁻⁴ etc., while the Randers change of metric $L(x, y) \rightarrow *L(x, y) = L(x, y) + b_i(x)y^i$ was introduced by M. Matsumoto⁵. Recently S. H. Abed⁶⁻⁷ introduced a change generalizing the above changes in the form $L(x, y) \rightarrow *L(x, y)$ where

$$(1.1) \quad *L(x, y) = e^{\sigma(x)} L(x, y) + b_i(x)y^i.$$

This change will be called a Randers conformal change. Abed called this a β -conformal change, but we think that the name Randers conformal change is rather appropriate for such a change.

On the other hand, the theory of Finsler subspaces was studied by S. Kikuchi⁸, H. Rund⁹, T. Sakaguchi¹⁰, H. Yasuda¹¹ etc. The present authors¹² discussed the subspaces of a Finsler space with metric given by an h -vector.

In this paper we find certain geometric properties of the subspaces of a Finsler space with Randers conformal metric. The terminologies and notations are referred to Matsumoto¹³.

2. Preliminaries

Let M^n be an n -dimensional smooth manifold and $F^n = (M^n, L)$ be an n -dimensional Finsler space equipped with a metric function $L(x, y)$ on M^n . The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor are defined by

$$l_i = \dot{\partial}_i L, \quad g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2, \quad h_{ij} = L \dot{\partial}_i \dot{\partial}_j L \quad \text{and} \quad C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij} \quad \text{respectively.}$$

Throughout this paper, we use the symbols $\dot{\partial}_i$ and ∂_i for $\partial / \partial y^i$ and $\partial / \partial x^i$ respectively. The Cartan connection in F^n is given as $CG = (F_{jk}^i, G_j^i, C_{jk}^i)$. The h - and v -covariant derivatives of a covariant vector $X_i(x, y)$ with respect to the Cartan connection are respectively given by

$$(2.1) \quad X_{i|j} = \partial_j X_i - (\dot{\partial}_h X_i) G_j^h - F_{ij}^r X_r$$

and

$$(2.2) \quad X_i | _j = \dot{\partial}_j X_i - C_{ij}^r X_r$$

An m -dimensional subspace M^m ($1 < m < n$) of the underlying manifold M^n may be represented parametrically by the equations $x^i = x^i(u^\alpha)$, where u^α are the Gaussian coordinates on M^m (Latin indices run from 1 to n , while Greek indices take values from 1 to m). We assume that the matrix of projection factors $B_\alpha^i = \partial x^i / \partial u^\alpha$ is of rank m . If the supporting element y^i at a point $u = (u^\alpha)$ of M^m is assumed to be tangent to M^m then we may write $y^i = B_\alpha^i(u) v^\alpha$. Thus $v = (v^\alpha)$ may be supposed as the supporting element of M^m at the point u^α . Since the function $\underline{L}(u, v) = L(x(u), y(u, v))$ gives rise to a Finsler metric on M^m , we get an m -dimensional Finsler space $F^m = (M^m, \underline{L}(u, v))$.

A covariant vector Y_i is said to be normal to F^m if it satisfies

$$Y_i B_\alpha^i = 0.$$

These are m equations for the determination of n functions Y_i . Since the rank of the matrix of $B_\alpha^i = \partial x^i / \partial u^\alpha$ is m , we may thus choose $(n-m)$ linearly independent and mutually orthogonal unit vectors $\overset{a}{N}_i$ ($a = m+1, \dots, n$) such that

$$(2.3) \quad \overset{a}{N}_i B_\alpha^i = 0.$$

With respect to a given direction y^j , we may thus choose a set of normals satisfying

$$(2.4) \quad \overset{a}{N}_i = g^{ij} \overset{a}{N}_j, \quad g_{ij} B_\alpha^i \overset{a}{N}_j = 0, \quad g_{ij} \overset{a}{N}_i \overset{a}{N}_j = \delta_{ab}.$$

The inverse projection factors $B_i^\alpha(u, v)$ of B_α^i are defined as

$$(2.5) \quad B_i^\alpha = g^{\alpha\beta} g_{ij} B_\beta^j,$$

where $g^{\alpha\beta}$ is the inverse of the metric tensor $g_{\alpha\beta}$ of F^m .

From (2.4) and (2.5), it follows that

$$(2.6) \quad B_\alpha^i B_i^\beta = \delta_\alpha^\beta, \quad B_\alpha^i \overset{a}{N}_i = 0, \quad \overset{a}{N}_i B_i^\alpha = 0, \quad \overset{a}{N}_i \overset{b}{N}_i = \delta_a^b,$$

and further

$$(2.7) \quad B_\alpha^i B_j^\alpha + \overset{a}{N}_i \overset{a}{N}_j = \delta_j^i.$$

For the induced Cartan connection $IC\Gamma = (F_{\beta\gamma}^\alpha, G_\beta^\alpha, C_{\beta\gamma}^\alpha)$ on F^m , the normal curvature vector $\overset{a}{H}_\alpha$ in a normal direction $\overset{a}{N}_i$ is given by

$$(2.8) \quad \overset{a}{H}_\alpha = \overset{a}{N}_i (B_{0\alpha}^i + G_j^i B_\alpha^j),$$

where $B_{\alpha\beta}^i = \partial^2 x^i / \partial u^\alpha \partial u^\beta$ and $B_{0\beta}^i = v^\alpha B_{\alpha\beta}^i$.

3. The Finsler Space $*F^n = (M^n, *L)$

Throughout this paper, the geometric objects associated with $*F^n$ will be asterisked $*$.

If we write $L_{ij} = \partial_i \partial_j L$, $L_{ijk} = \partial_k L_{ij}$, ... etc. then we get

$$*L_{ij} = e^\sigma L_{ij}, \quad *L_{ijk} = e^\sigma L_{ijk}$$

and so on. The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor of $*F^n$ are given by

$$(3.1) \quad *l_i = e^\sigma l_i + b_i,$$

$$(3.2) \quad *h_{ij} = \tau h_{ij},$$

$$(3.3) \quad *g_{ij} = \tau g_{ij} + b_i b_j + e^\sigma (l_i b_j + b_i l_j) + (e^{2\sigma} - \tau) l_i l_j,$$

$$(3.4) \quad *C_{ijk} = \tau C_{ijk} + \frac{e^\sigma}{2L} (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j),$$

where $\tau = e^\sigma *L / L$, $m_i = b_i - \beta l_i / L$ and $\beta = b_i y^i$. The reciprocal tensor $*g^{ij}$ of $*g_{ij}$ is obtained in the form

$$(3.5) \quad *g^{ij} = \frac{1}{\tau} g^{ij} - \frac{1}{\tau^2} (l^i b^j + b^i l^j) + \mu l^i l^j,$$

where $\mu = (e^\sigma L b^2 + \beta) / *L \tau^2$, $b^2 = b_i b^i$, $b^i = g^{ij} b_j$.

Abed [2] obtained the relation between the Cartan connection coefficients F_{jk}^i and $*F_{jk}^i$ as

$$(3.6) \quad *F_{jk}^i = F_{jk}^i + D_{jk}^i.$$

The expression of $D_{00}^i (= D_{jk}^i y^j y^k)$ is given by

$$(3.7) \quad D_{00}^i = 2L e^{-\sigma} F_0^i - L^2 \sigma^i + \frac{L}{*L} (E_{00} - 2L e^{-\sigma} F_{r0} b^r + 2L \sigma_0 e^\sigma + L^2 \sigma_r b^r) l^i,$$

where

$$2E_{ij} = b_{i|j} + b_{j|i}, \quad 2F_{ij} = b_{i|j} - b_{j|i},$$

$$\sigma_k = \partial_k \sigma, \quad \sigma^i = g^{ir} \sigma_r, \quad F_j^i = g^{ir} F_{rj}.$$

The suffix '0' denotes the transvection by supporting element y^i .

Transvecting (3.6) by y^i and using $F_{jk}^i y^j = G_k^i$, we get

$$(3.8) \quad *G_k^i = G_k^i + D_{0k}^i.$$

Further transvecting (3.8) by y^k and using $G_k^i y^k = 2G^i$, we get

$$(3.9) \quad 2 *G^i = 2G^i + D_{00}^i.$$

4. Finsler Subspaces given by Randers conformal change

Let us consider a Finsler subspace $F^m = (M^m, \underline{L}(u, v))$ of F^n and a subspace $*F^m = (M^m, * \underline{L}(u, v))$ of $*F^n$ given by Randers conformal change. Let N_a^i be unit normal vectors at a point of F^m . The functions $B_\alpha^i(u)$ may be considered as the components of m linearly independent vectors tangent to F^m and they are invariant under the Randers conformal change. Then the unit normal vectors $*N_a^i(u, v)$ of $*F^m$ are uniquely determined by

$$(4.1) \quad *g_{ij} B_\alpha^i *N_a^j = 0, \quad *g_{ij} *N_a^i *N_b^j = \delta_{ab}.$$

Transvecting (2.4) by v^α , we get

$$(4.2) \quad y_j N_a^j = 0.$$

Transvecting (3.3) by $N_a^i N_b^j$ and paying attention to (2.4) and (4.2), we have

$$(4.3) \quad *g_{ij} N_a^i N_b^j = \tau \delta_{ab} + (b_i N_a^i) (b_j N_b^j).$$

Putting $a=b$, we obtain

$$(4.4) \quad *g_{ij} \left(N_a^i / \sqrt{\tau + (b_k N_a^k)^2} \right) \left(N_a^j / \sqrt{\tau + (b_k N_a^k)^2} \right) = 1,$$

which shows that $N_a^i / \sqrt{\tau + (b_k N_a^k)^2}$ are unit vectors. Again transvecting

(3.3) by $B_\alpha^i N_a^j$ and using (2.4) and (4.2), we get

$$(4.5) \quad *g_{ij} B_\alpha^i N_a^j = (b_j N_a^j) (b_i B_\alpha^i + e^\sigma l_i B_\alpha^i).$$

This shows that the vector N_a^j is normal to $*F^m$ if and only if

$$(b_j N_a^j) (b_i B_\alpha^i + e^\sigma l_i B_\alpha^i) = 0.$$

This implies at least one of the following:

$$(a) \quad b_i B_\alpha^i + e^\sigma l_i B_\alpha^i = 0, \quad (b) \quad b_j N_a^j = 0.$$

The transvection of $b_i B_\alpha^i + e^\sigma l_i B_\alpha^i = 0$ by v^α gives $b_i y^i + e^\sigma L = 0$, i.e.

$*L = 0$, which is not true. Therefore $b_i B_\alpha^i + e^\sigma l_i B_\alpha^i \neq 0$. Hence

$$(4.6) \quad b_j N_a^j = 0.$$

Thus the vector N_a^j is normal to $*F^m$ if and only if b_j is tangent to F^m .

From (4.4), (4.5) and (4.6), we can say that $N_a^j / \sqrt{\tau}$ is a unit normal vector of $*F^m$. Therefore in view of (4.1), we get

$$(4.7) \quad *N_a^i = N_a^i / \sqrt{\tau},$$

which, in view of (3.3), (4.2) and (4.6), gives

$$(4.8) \quad *N_i^a = *g_{ij} *N_j^a = \sqrt{\tau} N_i^a.$$

Thus, we have

Theorem 4.1. *Let $*F^n$ be the Finsler space obtained from F^n by a Randers conformal change (1.1). If $*F^m$ and F^m are m -dimensional subspaces of these spaces then the vector b_i is tangential to the subspace F^m if and only if every vector normal to F^m is also normal to $*F^m$.*

From (2.8), (3.8) and (4.8), we get

$$*H_\alpha^a = \sqrt{\tau} (H_\alpha^a + N_i^a D_{0k}^i B_\alpha^k).$$

Transvecting by v^α and using $v^\alpha B_\alpha^k = y^k$, we get

$$(4.9) \quad *H_0^a = \sqrt{\tau} (H_0^a + N_i^a D_{00}^i).$$

Transvecting (3.7) by N_i^a and using (4.2), we get

$$(4.10) \quad D_{00}^i N_i^a = 2Le^{-\sigma} F_0^i N_i^a - L^2 \sigma^i N_i^a.$$

If the vector b_i is gradient, i.e. $b_{i|j} = b_{j|i}$, i.e. $F_{ij} = 0$ then (4.10) reduces to

$$D_{00}^i N_i^a = -L^2 \sigma^i N_i^a.$$

Therefore (4.9) becomes

$$*H_0^a = \sqrt{\tau} (H_0^a - L^2 \sigma^i N_i^a).$$

If each geodesic of the subspace F^m with respect to the induced metric is also a geodesic of the enveloping space F^n , then F^m is called a totally geodesic subspace. A totally geodesic subspace is characterized by $H_\alpha^a = 0$

or equivalently $H_0^a = 0$.

Thus, we have

Theorem 4.2. *Let the vector b_i in the Randers conformal change (1.1), be gradient and the subspace F^m be totally geodesic. Then the subspace $*F^m$ is totally geodesic if and only if the vector σ_i is tangential to the subspace F^m .*

References

1. M. S. Knebelman, Conformal geometry of generalized metric spaces, *Proc. Nat. Acad. Sci. USA*, **15** (1929) 376-379.
2. M. Hashiguchi, On conformal transformation of Finsler metrics, *J. Math. Kyoto Univ.*, **16** (1976) 25-50.
3. H. Izumi, Conformal transformations of Finsler spaces I. Conccircular transformations of a curve with Finsler metric, *Tensor N. S.*, **31** (1977) 33-41.
4. H. Izumi, Conformal transformations of Finsler spaces II. An h -conformally flat Finsler space, *Tensor N.S.*, **34** (1980) 337-359.
5. M. Matsumoto, On Finsler spaces with Randers metric and special forms of special tensors, *J. Math. Kyoto Univ.*, **14-3** (1974) 477-498.
6. S. H. Abed, Conformal β -changes in Finsler spaces, *Proc. Math. Phys. Soc. Egypt.*, **86** (2008) 79-89.
7. S. H. Abed, Cartan connections associated to a β -conformal change in Finsler geometry, *Tensor N.S.*, **70** (2008) 146-158.
8. S. Kikuchi, On the theory of subspace in a Finsler space, *Tensor N.S.*, **2** (1952) 67-79.
9. H. Rund, *The differential geometry of Finsler spaces*, Springer - Verlag, Berlin / Gottingen / Heidelberg, 1959.
10. T. Sakaguchi, Subspaces in Finsler spaces, *Memoirs of the National Defence Academy, Japan*, **28** (1988) 1-37.
11. H. Yasuda, A theory of subspaces in a Finsler space, *Ann. Rep. Asahikawa Med. Coll.*, **8** (1987) 1-43.
12. M. K. Gupta and P. N. Pandey, On subspaces of a Finsler space with a special metric, *Bull. All. Math. Soc.*, **23-2** (2008) 263-272.
13. M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha press, Saikawa, Otsu, 520 Japan, 1986.