

## Four Dimensional C3 – Like Finsler Space

**H. S. Shukla and S. K. Mishra**

Department of Mathematics and Statistics,  
D. D. U. Gorakhpur University, Gorakhpur  
Email: profhsshuklagkp@rediffmail.com,  
kumar.sandip62@yahoo.com

**B. N. Prasad**

C-10, Surajkund Colony, Gorakhpur, Gorakhpur  
Email: baijnath\_prasad2003@yahoo.com

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**Abstract:** In this paper the properties of Finsler space, whose (h)hv-torsion tensor  $C_{ijk}$  is of the form

$$LC_{ijk} = Pa_i a_j a_k + Q \sum_{(i,j,k)} \{a_i, a_j, b_k\} + R \sum_{(i,j,k)} \{a_i, b_j, b_k\} + S b_i b_j b_k,$$

have been studied. Here  $\sum_{(i,j,k)} \{.....\}$  denotes the cyclic sum of  $i, j, k$ .

Since  $C_{ijk}$  of any three dimensional Finsler space is of the above form, any  $n$ -dimensional Finsler space  $F^n (n \geq 4)$  will be called C3- like Finsler space, if its  $C_{ijk}$  is of the form.

**Key words:** Finsler space, C3-like,  $C^v$  – reducible.

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### 1. Introduction

There are three kinds of torsion tensors in the Cartan's theory of Finsler space, two of them are (v)hv- torsion tensor  $P_{ijk}$  and (h)hv- torsion tensor  $C_{ijk}$  which are symmetric in all the indices.

In the year 1971, M. Matsumoto<sup>1</sup>, paid attention to the form of  $C_{ijk}$  in two dimensional Finsler space given by

$$(1.1) \quad LC_{ijk} = I m_i m_j m_k,$$

where  $I$  is the main scalar and  $m_i$  are the component of unit vector orthogonal to the element of support  $y^i$ .

Moor studied the properties of Finsler space in which  $C_{ijk}$  is of the form (1.1). For such a Finsler space  $v$ -curvature tensor  $S_{hijk}$  vanishes, so that  $F^n$  is reduced to a Riemannian space in view of Brickell's theorem, provided that the metric is symmetric. It is well known that the torsion tensor  $C_{ijk}$  in three dimensional Finsler space  $F^3$  (strongly non Riemannian Finsler space), is always written in the form

$$(1.2) \quad LC_{ijk} = H m_i m_j m_k - J \sum_{(i,j,k)} \{m_i m_j n_k\} + I \sum_{(i,j,k)} \{m_i n_j n_k\} + J n_i n_j n_k,$$

where  $(I_i, m_i, n_i)$  is the moor's frame and they three scalars  $H, I, J$  are called main scalars.

B. N. Prasad and J. N. Singh<sup>2</sup> have defined C3-like Finsler space characterized by

$$(1.3) \quad C_{i,j,k} = \sum_{(i,j,k)} \{h_{ij} a_k + C_i C_j b_k\}.$$

Prasad and Singh<sup>2</sup> have shown that under different values of  $a_k$  and  $b_k$ , the C3-like Finsler space reduces to various interesting special Finsler space. These special Finsler space and corresponding forms of  $C_{ijk}$  are

$$(1.4) \quad \text{Generalized C-reducible Finsler space: } C_{i,j,k} = \sum_{(i,j,k)} \{P_{ij} Q_k\}.$$

$$(1.5) \quad \text{Quasi C-reducible Finsler space: } C_{i,j,k} = \sum_{(i,j,k)} \{P_{ij} C_k\}.$$

$$(1.6) \quad \text{Semi C-reducible Finsler space:}$$

$$C_{i,j,k} = \frac{p}{(n+1)} \sum_{(i,j,k)} \{h_{ij} C_k\} + \frac{q}{C^2} C_i C_j C_k,$$

where  $p + q = 1$ .

$$(1.7) \quad \text{C-reducible Finsler space: } C_{i,j,k} = \frac{1}{(n+1)} \sum_{(i,j,k)} \{h_{ij} C_k\},$$

$$(1.8) \quad \text{C2-like Finsler space: } C_{i,j,k} = \frac{1}{C^2} C_i C_j C_k.$$

T. N. Pandey and B. K. Tripathi<sup>3</sup> have studied the properties of  $F^n$  ( $n \geq 4$ ) in which  $C_{ijk}$  is of the form

$$(1.8) \quad C_{i,j,k} = \sum_{(i,j,k)} \{h_{ij}a_k + L^2 C_i C_j b_k\}.$$

For  $a_k = 0$  the above form (1.7) gives  $C_{ijk}$  of two dimensional Finsler space  
For  $a_k = 1$

$$m_k + \frac{J}{3} n_k, \quad b_k = \frac{C^2}{L^2} \left( \frac{H}{3} - I \right) m_k - \frac{4JC^2}{3L^2} n_k,$$

(1.7) gives  $C_{ijk}$  of three dimensional Finsler space.

S. Numata<sup>4</sup> has studies various properties of C3-like Finsler space. He has given more precise definition of C3-like Finsler space and cited some example of such a space.

## 2. Relation satisfied by P, Q, R and S of C3-like Finsler space

**Definition (2.1).** A Finsler space  $F^n$  of dimension  $n$  ( $\geq 4$ ) is called C3-like Finsler space if its (h)hv- torsion tensor  $C_{ijk}$  can be written in the form

$$(2.1) \quad LC_{ijk} = P a_i a_j a_k + Q \sum_{(i,j,k)} \{a_i a_j b_k\} + R \sum_{(i,j,k)} \{a_i b_j b_k\} + S b_i b_j b_k,$$

where  $P, Q, R$  and  $S$  are scalars. It is obvious that  $a_i$  and  $b_i$  may be taken as unit vector by suitable modification of scalars  $P, Q, R$  and  $S$ . Further it may be supposed that  $b_i$  is orthogonal to  $a_i$ . if  $b_i$  is not orthogonal to  $a_i$ , then we take the vector  $w_i = b_i - (a, b) a_i$ ,  $(a, b) = g^{ij} a_i b_j$ .

It is clear that this vector is orthogonal to  $a_i$  and  $LC_{ijk}$  is written in a form similar to (2.1) in terms of  $a_i$  and  $w_i$ . Consequently we may take

$$(2.2) \quad g^{ij} a_i b_j = 0, \quad g^{ij} a_i a_j = g^{ij} b_i b_j = 1.$$

Transvecting (2.1) by  $g^{ij}$  and using (2.2), we get

$$(2.3) \quad LC_k = (P + R)a_k + (Q + S)b_k.$$

Again transvecting (2.3) by  $C^k$  we have

$$(2.4) \quad LC^2 = (P + R)u + (Q + S)v,$$

where  $u = C^k a_k$ ,  $v = C^k b_k$  and  $C = C^k C_k$ .

Transvecting (2.3) by  $b^k$  and  $a^k$  respectively we get

$$(2.5) \quad (a) \quad L v = Q + S, \quad (b) \quad L u = P + R.$$

Putting the values of  $(Q + S)$  and  $(P + Q)$  from (2.5), we get  $C_k = u a_k + v b_k$ . If  $a_i$  is orthogonal to  $C_i$  then  $u = 0$  and (2.5 b) leads to  $P + R = 0$ .

Similarly if  $b_i$  is orthogonal to  $C^i$  then  $v = 0$  and (2.5 a) leads to  $Q + S = 0$ . Thus we have the following:

**Theorem (2.1).** *In a C3-like Finsler space  $a_i$  is orthogonal to  $C^i$  if and only if  $P + R = 0$  and  $b_i$  is orthogonal to  $C^i$  if and only if  $Q + S = 0$ .*

### 3. C3-like four dimensional Finsler space

The scalar components  $C_{\alpha\beta\gamma}$  of  $LC_{ijk}$  with respect to orthogonal frame  $\{e_\alpha\}$ :  $\alpha = 1, 2, 3, 4\}$  of four dimensional Finsler space is given by [5]

$$(3.1) \quad LC_{ijk} = C_{\alpha\beta\gamma} e_{\alpha i} e_{\beta j} e_{\gamma k}.$$

Contracting (3.1) with  $g^{jk}$  we have  $LC_i = C_{\alpha\beta\gamma} e_{\alpha i}$ . Writing scalar components of both the sides of (2.1), we get

$$(3.2) \quad C_{\alpha\beta\gamma} = Pa_\alpha a_\beta a_\gamma + Q \sum_{(\alpha\beta\gamma)} \{a_\alpha a_\beta b_\gamma\} + R \sum_{(\alpha\beta\gamma)} \{a_\alpha b_\beta b_\gamma\} + S b_\alpha b_\beta b_\gamma,$$

where  $a_\alpha$  and  $b_\alpha$  are scalar components of  $a_i$  and  $b_i$  respectively i.e.

$$a_i = a_\alpha e_{\alpha i} \quad \text{and} \quad b_i = b_\alpha e_{\alpha i}.$$

Since  $C_{ijk}$  is indicatory tensor,  $a_i$  and  $b_i$  are indicatory tensor i.e.  $a_i y^i = 0$ ,  $b_i y^i = 0$

which imply that  $a_1 = 0$ ,  $b_1 = 0$ . Then (2.2) leads to

$$(3.3) \quad a_2^2 + a_3^2 + a_4^2 = 1, b_2^2 + b_3^2 + b_4^2 = 1, a_2 b_2 + a_3 b_3 + a_4 b_4 = 0.$$

Taking particular values of  $\alpha, \beta, \gamma$  in (3.2) we get

$$(3.4) \quad (a) \quad C_{222} = Pa_2^3 + 3Qa_2^2 b_2 + 3Ra_2 b_2^2 + Sb_2^3,$$

$$(b) \quad C_{233} = Pa_2 a_3^2 + Q(2a_2 a_3 b_3 + a_3^2 b_2) + R(2b_2 a_3 b_3 + a_2 b_3^2) + Sb_2 b_3^2,$$

$$(c) \quad C_{333} = Pa_3^3 + 3Qa_3^2 b_3 + 3Ra_3 b_3^2 + Sb_3^3,$$

$$(d) \quad C_{244} = Pa_2 a_4^2 + Q(2a_2 a_4 b_4 + a_4^2 b_2) + R(2b_2 a_4 b_4 + a_2 b_4^2) + Sb_2 b_4^2,$$

$$(e) \quad C_{234} = Pa_2 a_3 a_4 + Q(a_2 a_3 b_4 + a_3 a_4 b_2 + a_4 a_2 b_3) \\ + R(b_2 b_3 b_4 + a_3 b_4 b_2 + a_4 b_2 b_3) + Sb_2 b_3 b_4,$$

$$(f) \quad C_{223} = Pa_2^2 a_3 + Q(a_2^2 b_3 + 2a_2 a_3 b_2) + R(2a_2 b_2 b_3 + a_3 b_2^2) + Sb_2^3 b_3,$$

$$(g) \quad C_{344} = Pa_3 a_4^2 + Q(2a_3 a_4 b_4 + a_4^2 b_3) + R(2b_3 a_4 b_4 + a_3 b_4^2) + Sb_3 b_4^2,$$

$$(h) \quad C_{334} = Pa_3^2 a_4 + Q(a_3^2 b_4 + 2a_3 a_4 b_3) + R(2a_3 b_3 b_4 + a_4 b_3^2) + S b_3^2 b_4,$$

$$(i) \quad C_{444} = P a_4^3 + 3Q a_4^2 b_4 + 3R a_4 b_4^2 + S b_4^3,$$

$$(j) \quad C_{224} = Pa_2^2 a_4 + Q(a_2^2 b_4 + 2a_2 a_4 b_2) + R(2a_2 b_2 b_4 + a_4 b_2^2) + S b_2^3 b_4,$$

From equation (3.3) and (3.4), we have

$$(3.5) \quad \begin{aligned} (P + Q)a_2 + (Q + S)b_2 &= LC, & (P + R)a_3 + (Q + S)b_3 &= 0, \\ (P + R)a_4 + (Q + S)b_4 &= 0, \end{aligned}$$

**Theorem (3.1).** *Necessary conditions for a four dimensional Finsler space to be C3-like are given by (3.5).*

#### 4. $C^v$ -reducible, Semi $C^v$ – reducibility of C3-like Finsler space

**Definition (4.1).** A Finsler space  $F^n$  of dimensional  $n > 3$  is called  $C^v$ -reducible if the following conditions are satisfied

(1)  $F^n$  is not Riemannian,

(2) Cartan's v-covariant derivative of (h)hv-torsion tensor  $C_{ijk}$  is of the form

$$(4.1) \quad LC_{ijk}|_h = h_{ij}B_{kh} + h_{kh}B_{ij} + h_{jk}B_{li} + h_{li}B_{jk} + h_{ik}B_{lj} + h_{lj}B_{ik},$$

$$\text{where } B_{ij} = \frac{1}{2(n+1)} [\alpha h_{ij} - 2(1_i C_j + 1_j C_i)] \quad \text{and} \quad \alpha = \frac{L}{(n-1)} C^i|_i.$$

T. N. Pandey and M. K. Gupta<sup>6</sup> have shown that every  $C^v$ -reducible Finsler space is C-reducible and Converse is also true. Numata<sup>4</sup> has shown that there is no C-reducible C3-like Finsler space. Hence from the above two results, we have

**Theorem (4.1).** *There is no  $C^v$  – reducible C3-like Finsler space.*

**Definition (4.2).** A non-Riemannian Finsler space  $F^n$  is called semi- $C^v$  – reducible Finsler space, if the Carton's v-covariant derivative of h(hv)-torsion tensor can be written as

$$(4.2) \quad LC_{ijk}|_h = P \sum_{i,j,k,h} \{h_{ij}B_{kh}\} + \frac{q}{C^2} \sum_{i,j,k,h} \{C_i C_j B_{kh}\}$$

$$\text{where } B_{ih} = \frac{1}{2(n+1)} [\alpha h_{ih} - 2(1_i C_h + 1_h C_i)] \quad \text{and} \quad \alpha = \frac{L}{(n-1)} C^i|_i.$$

$p + q = 1$ ;  $p, q$  do not vanish and  $\sum_{i,j,k,h} \{ \}$  denote the cyclic sum of  $i, j, k, h$ .

Taking v-covariant derivative of (2.1) we have

$$\begin{aligned}
 (4.3) \quad LC_{ijk} |_h + 1_h C_{ijk} &= P |_h a_i a_j a_k + P(a_i |_h a_j a_k + a_i a_j |_h a_k + a_i a_j a_k |_h) \\
 &+ Q |_h \sum_{(i,j,k)} \{a_i a_j b_k\} + Q \sum_{(i,j,k)} \{a_i |_h a_j b_k + a_i a_j |_h b_k + a_i a_j b_k |_h\} \\
 &+ R |_h \sum_{(i,j,k)} \{a_i b_j b_k\} + R \sum_{(i,j,k)} \{a_i |_h b_j b_k + a_i b_j |_h b_k + a_i b_j b_k |_h\} \\
 &+ S |_h b_i b_j b_k + S(b_i |_h b_j b_k + b_i b_j |_h b_k + b_i b_j b_k |_h).
 \end{aligned}$$

Transvecting (4.4) by  $b^i b^j a^k$  and using (2.2) we have

$$\begin{aligned}
 (4.4) \quad LC_{ijk} |_h b^i b^j a^k + 1_h C_{ijk} b^i b^j a^k &= Q(a_j |_h b^j + a_i |_h b^i + R |_h) \\
 &+ R(a_k |_h a^k + 2b_i |_h b^i) + Sb_k |_h a^k.
 \end{aligned}$$

Putting the value of  $LC_{ijk}|_h$  from (4.2) in (4.4) we have

$$\begin{aligned}
 (4.5) \quad &\left[ p \sum_{i,j,k,h} \{h_{ij} B_{kh}\} - \frac{q}{C^2} \sum_{i,j,k,h} \{C_i C_j B_{kh}\} \right] b^i b^j a^k + 1_h C_{ijk} b^i b^j a^k \\
 &= Q(a_j |_h b^j + a_i |_h b^i + R |_h) + R(a_k |_h a^k + 2b_i |_h b^i) + Sb_k |_h a^k.
 \end{aligned}$$

So we have

**Theorem (3.2).** *If a C3-like Finsler space in semi  $C^\nu$  –reducible then (4.5) holds.*

## References

1. M. Matsumoto, On C-reducible Finsler space, *Tensor, N. S.*, **24** (1972)29-37.
2. B. N. Prasad and J. N. Singh, On C3-like Finsler space, *Indian J. Pure appl. Math.*, **19** (1988) 423-428.
3. T. N. Pandey and B. K. Tripathi, Four dimensional C3-like Finsler space, *Tensor, N. S.*, **68** (2007).
4. S. Numata, *On C3-like Finsler space*, Reports on Mathematical Physics, **18** (1980) 1-10.
5. M. Matsumoto, *Foundation of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Saikawa Otsu 520 Japan, 1986.
6. M. K. Gupta,  *$C^\nu$  –reducible Finsler space*, Ph.D. Thesis, D. D. U. Gorakhpur University, Gorakhpur, 2005.
7. M. Matsumoto and C. Shabita, S4-likeness of Finsler spaces, *J. Math Kyoto University Press. Saikawa Otsu 520 Japan*, (1986).
8. T. N. Pandey, A theory of four dimensional Finsler space in terms of scalars, *J. Nat. Acad. Math.*, **11** (1997) 1-5.