

Hypersurface of a Finsler Space with Randers Change of (α, β) -Metric

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Abstract: The aim of the present paper is to discuss different kinds of hypersurfaces of the Finsler space with Randers change of special

(α, β) -metric of type $L^* = \alpha + \mu\beta + \nu\frac{\beta^5}{\alpha^4} + \beta$ (where μ and ν are constants). We have obtained conditions for the hypersurface to be a hyperplane of first, second or third kinds.

Keywords: Finsler space, hypersurface, (α, β) - metric, hyperplane of 1st, 2nd and 3rd Kinds.

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1. Introduction

M. Matsumoto¹ introduced the concept of (α, β) -metric on a differentiable manifold with local coordinates x^i , where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M^n . M. Hashiguchi and Y. Ichijyo² studied some special (α, β) -metrics and obtained interesting results. B. N. Prasad³ obtained the Cartan connection for the Finsler space whose metric is transformed by an h -vector. M. Matsumoto⁴ discussed the properties of special hypersurface of a Randers space with $b_i(x)$ as gradient of a scalar function $b(x)$. M. Kitayama⁵ obtained certain results related to Finslerian hypersurface given by β - change. P. N. Pandey and Suresh K. Shukla⁶ discussed the Hypersurfaces of a Recurrent Finsler Space. In 2011,

H. Wosoughi⁷ discussed hypersurface of special Finsler space with an exponential (α, β) -metric and obtained certain results.

The hypersurface of a Finsler space with some given special metrics has been studied by authors⁸⁻⁹. A change of a Finsler metric $L(x, y) \rightarrow L^* = L(x, y) + b_i(x)y^i$ is called Randers change of metric. The notion of a Randers change was introduced by Matsumoto, named by Hashiguchi and Ichijyo¹⁰ and studied in detail by Shibata¹¹. A hypersurface M^{n-1} of the (M^n, L) may be represented parametrically by the equation $x^i = x^i(u^\alpha)$, $\alpha=1, 2, \dots, n-1$ where u^α are Gaussian coordinate on M^{n-1} . Since the function $L^* = L(x(u), y(u, v))$ gives rise to a Finsler metric of M^{n-1} , we get an $(n-1)$ -dimensional Finsler space $F^{n-1} = (M^{n-1}, L^*(u, v))$. The aim of the present paper is to discuss different kinds of hypersurfaces of the Finsler space with Randers change of special (α, β) -metric of type $L^* = \alpha + \mu\beta + \nu\frac{\beta^5}{\alpha^4} + \beta$ (where μ and ν are constants). We have obtained conditions for the hypersurface to be a hyperplane of first, second or third kinds.

2. Preliminaries

Let M^n be a real smooth manifold of dimension n and let $F^{*n} = (M^n, L^*)$ be a Finsler space on the differentiable manifold M^n endowed with a fundamental function $L^*(x, y)$, where

$$(2.1) \quad L^* = \alpha + \mu\beta + \nu\frac{\beta^5}{\alpha^4} + \beta.$$

Differentiating (2.1) partially with respect to α and β , we get

$$(2.2) \quad \left\{ \begin{array}{l} a) L_{\alpha}^* = \frac{\alpha^5 - 4v\beta^5}{\alpha^5}, \\ b) L_{\beta}^* = \frac{(\mu+1)\alpha^4 + 5v\beta^4}{\alpha^4}, \\ c) L_{\alpha\alpha}^* = \frac{20v\beta^5}{\alpha^6}, \\ d) L_{\beta\beta}^* = \frac{20v\beta^3}{\alpha^4}, \\ e) L_{\alpha\beta}^* = -\frac{20v\beta^4}{\alpha^5}, \end{array} \right.$$

where $L_{\alpha}^* = \frac{\partial L^*}{\partial \alpha}$, $L_{\beta}^* = \frac{\partial L^*}{\partial \beta}$, $L_{\alpha\alpha}^* = \frac{\partial L_{\alpha}^*}{\partial \alpha}$, $L_{\beta\beta}^* = \frac{\partial L_{\beta}^*}{\partial \beta}$, $L_{\alpha\beta}^* = \frac{\partial L_{\alpha}^*}{\partial \beta}$.

The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor are defined by¹²

$$(2.3) \quad \left\{ \begin{array}{l} a) l_i = \dot{\partial}_i L, \\ b) g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2, \\ c) h_{ij} = L \dot{\partial}_i \dot{\partial}_j L, \\ d) C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}, \end{array} \right.$$

respectively, where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

In the Finsler space $F^{*n} = (M^n, L^*)$ the normalized element of support (2.3a) and the angular metric tensor (2.3c) are given by⁶

$$(2.4) \quad l_i^* = \alpha^{-1} L_{\alpha}^* y_i + L_{\beta}^* b_i,$$

$$(2.5) \quad h_{ij}^* = p a_{ij} + q_0 b_i b_j + q_1 (b_i Y_j + b_j Y_i) + q_2 Y_i Y_j,$$

where

$$(2.6) \quad Y_i = a_{ij} y^j,$$

$$(2.7) \quad p = L^* L_{\alpha}^* \alpha^{-1}$$

$$= \frac{1}{\alpha^{10}} \left[\alpha^{10} - 3\nu \alpha^5 \beta^5 + (\mu+1) \beta \alpha^9 - 4\nu (\mu \alpha^4 + \nu \beta^4 + \alpha^4) \beta^6 \right],$$

$$(2.8) \quad q_0 = L^* L_{\beta\beta}^*$$

$$= \frac{20\nu}{\alpha^8} \left[\alpha^5 \beta^3 + (\mu+1) \alpha^4 \beta^4 + \nu \beta^8 \right],$$

$$(2.9) \quad q_1 = L^* L_{\alpha\beta}^* \alpha^{-1}$$

$$= -\frac{20\nu \beta^4}{\alpha^{10}} \left[\alpha^5 + \mu \beta \alpha^4 + \nu \beta^5 + \beta \alpha^4 \right],$$

$$(2.10) \quad q_2 = L^* \alpha^{-2} (L_{\alpha\alpha}^* - L_{\alpha}^* \alpha^{-1})$$

$$= \frac{1}{\alpha^{12}} \left[23\nu \alpha^5 \beta^5 + 24\nu \{ (\mu+1) \alpha^4 + \nu \beta^4 \} \beta^6 - (\mu+1) \beta \alpha^9 - \alpha^{10} \right].$$

In the Finsler space $F^{*n} = (M^n, L^*)$ the fundamental metric tensor (2.3b) is given by⁷

$$(2.11) \quad g_{ij}^* = p a_{ij} + p_0 b_i b_j + p_1 (b_i Y_j + b_j Y_i) + p_2 Y_i Y_j,$$

Where

$$(2.12) \quad p_0 = q_0 + L_{\beta}^{*2}$$

$$= \frac{1}{\alpha^8} \left[30\nu (\mu+1) \alpha^4 \beta^4 + (\mu+1)^2 \alpha^8 + 20\nu \alpha^5 \beta^5 + 45\nu^2 \beta^8 \right],$$

$$(2.13) \quad p_1 = q_1 + L^{*-1} p L_{\beta}^*$$

$$= \frac{1}{\alpha^{10}} \left[-\nu \alpha^4 \beta^4 \{ 15\alpha + 24\beta (\mu+1) \} - 40\nu^2 \beta^9 + (\mu+1) \alpha^9 \right],$$

$$(2.14) \quad p_2 = q_2 + p^2 L^{*-2}$$

$$= \frac{1}{\alpha^{12}} \left[15\nu \alpha^5 \beta^5 + 40\nu^2 \beta^{10} + (\mu+1) \{ 24\nu \alpha^4 \beta^6 - \beta \alpha^9 \} \right].$$

The reciprocal tensor g^{*ij} of g_{ij}^* is given by

$$(2.15) \quad g^{*ij} = p^{-1} a^{ij} - S_0 b^i b^j - S_1 (b^i y^j + b^j y^i) - S_2 y^i y^j,$$

$$(2.16) \quad \left\{ \begin{array}{l} a) \quad b^i = a^{ij} b_j; \quad b^2 = a_{ij} b^i b^j, \\ b) \quad S_0 = \frac{p p_0 + (p_0 p_2 - p_1^2) \alpha^2}{\tau p} \\ c) \quad S_1 = \frac{p p_1 + (p_0 p_2 - p_1^2) \beta}{\tau p} \\ d) \quad S_2 = \frac{p p_2 + (p_0 p_2 - p_1^2) b^2}{\tau p} \\ e) \quad \tau = p(p + p_0 b^2 + p_1 \beta) + (p_0 p_2 - p_1^2)(\alpha^2 b^2 - \beta^2). \end{array} \right.$$

For the Finsler space F^{*n} the $h\nu$ -torsion tensor is given by

$$(2.17) \quad C_{ijk}^* = p_1 (h_{ij}^* m_k + h_{jk}^* m_i + h_{ki}^* m_j) + \gamma_1 m_i m_j m_k,$$

where

$$(2.18) \quad (a) \quad \gamma_1 = p \frac{\partial p_0}{\partial \beta} - 3p_1 q_0, \quad (b) \quad m_i = b_i - \alpha^{-2} \beta Y_i.$$

Here m_i is a non-vanishing covariant vector orthogonal to the element of support y^i .

Let $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ be the components of Christoffel symbols of the associated Riemannian space R^n and ∇_k denote the covariant differential operator with respect to x^k relative to the Christoffel symbols. We will use the following tensors:

$$(2.19) \quad (a) \quad 2E_{ij}^* = b_{ij} + b_{ji}, \quad (b) \quad 2F_{ij}^* = b_{ij} - b_{ji},$$

where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma^* = (F_{jk}^{*i}, G_j^{*i}, C_{jk}^{*i})$ be the Cartan connection of F^{*n} . The difference tensor $D_{jk}^{*i} = F_{jk}^{*i} - \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ of the Finsler space F^{*n} is given by

$$\begin{aligned}
 (2.20) \quad D_{jk}^{*i} = & B^{*i} E_{jk}^* + F_k^{*i} B_j^* + F_j^{*i} B_k^* + B_j^{*i} b_{0k} + B_k^{*i} b_{0j} \\
 & - b_{0m} g^{*im} B_{jk}^* - C_{jm}^{*i} A_k^{*m} - C_{km}^{*i} A_j^{*m} + C_{jkm}^* A_s^{*m} g^{*is} \\
 & + \lambda^{*s} \left(C_{jm}^{*i} C_{sk}^{*m} + C_{km}^{*i} C_{sj}^{*m} - C_{jk}^{*m} C_{ms}^{*i} \right),
 \end{aligned}$$

where

$$(2.21) \quad \begin{cases} a) B_k^* = p_0 b_k + p_1 Y_k, & B^{*i} = g^{*ij} B_j^*, \\ b) B_{ij}^* = \frac{p_1 \left(a_{ij} - \alpha^{-2} Y_i Y_j \right) + \left(\frac{\partial p_0}{\partial \beta} \right) m_i m_j}{2}, \\ c) A_k^{*m} = B_k^{*m} E_{00}^* + 2B_0^* F_0^{*m}, & F^{*k} = g^{*kj} F_{ji}^*, \\ d) B_i^{*k} = g^{*kj} B_{ji}^*, & B_0^* = B_i^* y^i, \end{cases}$$

where 0 denote the contraction with y^i except for the quantities p_0 , q_0 and S_0 .

3. Induced Cartan Connection

Let F^{*n-1} be a hypersurface of F^{*n} given by the equation $x^i = x^i(u)$. The element of support y^i of F^{*n} is to be taken tangential to F^{*n-1} , i.e.

$$(3.1) \quad y^i = B_{\alpha}^{*i}(u) v^{\alpha}.$$

The metric tensor $g_{\alpha\beta}^*$ and v -torsion tensor $C_{\alpha\beta\gamma}^*$ are given by

$$(3.2) \quad (a) \quad g_{\alpha\beta}^* = g_{ij}^* B_{\alpha}^{*i} B_{\beta}^{*j}, \quad (b) \quad C_{\alpha\beta\gamma}^* = C_{ijk}^* B_{\alpha}^{*i} B_{\beta}^{*j} B_{\gamma}^{*k}.$$

At each point u^{α} of F^{*n-1} , a unit normal vector $N^{*i}(u, v)$ is defined by

$$(3.3) \quad (a) \quad g_{ij}^* B_{\alpha}^{*i} N^{*j} = 0, \quad (b) \quad g_{ij}^* N^{*i} N^{*j} = 1.$$

The angular metric tensor h_{ij}^* of F^{*n} , satisfies the following relations

$$(3.4) \quad (a) \quad h_{\alpha\beta}^* = h_{ij}^* B_{\alpha}^{*i} B_{\beta}^{*j}, \quad (b) \quad h_{ij}^* B_{\alpha}^{*i} N^{*j} = 0, \quad (c) \quad h_{ij}^* N^{*i} N^{*j} = 1.$$

The inverse projection factor $B_i^{*\alpha}(u, v)$ of $B_i^{*\alpha}$ is given by

$$(3.5) \quad B_i^{*\alpha} = g^{*\alpha\beta} g_{ij}^* B_\beta^{*j},$$

where $g^{*\alpha\beta}$ is the inverse of the metric tensor $g_{\alpha\beta}^*$ of F^{*n-1} .

From (3.3) and (3.5), we get

$$(3.6) \quad B_\alpha^{*i} B_i^{*\beta} = \delta_\alpha^\beta, \quad B_\alpha^{*i} N_i^* = 0, \quad N^{*i} B_i^{*\alpha} = 0, \quad N^{*i} N_i^* = 1,$$

and further

$$(3.7) \quad B_\alpha^{*i} B_j^{*\alpha} + N_j^* N^{*i} = \delta_j^i.$$

For induced cartan connection $ICT = (F_{\beta\gamma}^{*\alpha}, G_{\beta\gamma}^{*\alpha}, C_{\beta\gamma}^{*\alpha})$ on F^{*n-1} , the second fundamental h -tensor $H_{\alpha\beta}^*$ and the normal curvature vector H_α^* are given by

$$(3.8) \quad H_{\alpha\beta}^* = N_i^* (B_{\alpha\beta}^{*i} + F_{jk}^{*i} B_\alpha^{*j} B_\beta^{*k}) + M_\alpha^* H_\beta^*, \quad H_\alpha^* = N_i^* (B_{0\alpha}^{*i} + G_j^{*i} B_\alpha^{*j}),$$

where $M_\alpha^* = C_{ijk}^* B_\alpha^{*i} N^{*j} N^{*k}$, $B_{\alpha\beta}^{*i} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}$ and $B_{0\alpha}^{*i} = B_{\beta\alpha}^{*i} v^\beta$.

It is clear that $H_{\alpha\beta}^*$ is not symmetric

$$(3.9) \quad H_{\alpha\beta}^* - H_{\beta\alpha}^* = M_\alpha^* H_\beta^* - M_\beta^* H_\alpha^*.$$

Equation (3.8) yields

$$(3.10) \quad H_{\alpha 0}^* = H_{\alpha\beta}^* v^\beta = H_\alpha^* + M_\alpha^* + M_\alpha^* H_0^*.$$

The second fundamental v -tensor $M_{\alpha\beta}^*$ is given by

$$(3.11) \quad M_{\alpha\beta}^* = C_{ijk}^* B_\alpha^{*i} B_\beta^{*j} N^{*k}.$$

The relative h and v -covariant derivatives of B_α^{*i} and N^{*i} are given by

$$(3.12) \quad \begin{cases} a) \ B_{\alpha|\beta}^{*i} = H_{\alpha\beta}^* N^{*i}, \\ b) \ B_\alpha^{*i} | \beta = M_{\alpha\beta}^* N^{*i}, \\ c) \ N_{|\beta}^{*i} = -H_{\alpha\beta}^* B_j^{*\alpha} g^{*ij}, \\ d) \ N^{*i} | \beta = -M_{\alpha\beta}^* B_j^{*\alpha} g^{*ij}. \end{cases}$$

$$(3.13) \quad M_{\beta}^* = N_i^* C_{jk}^{*i} B_{\beta}^{*j} N^{*k}.$$

Let $X_i(x, y)$ be a vector field of F^{*n} , the relative h and v -covariant derivatives of X_i are given by

$$(3.14) \quad X_{i|\beta} = X_{i|j} B_{\beta}^{*j} + X_{i|j} N^{*j} H_{\beta}^*, \quad X_{i|\beta} = X_{i|j} B_{\beta}^{*j}.$$

M. Matsumoto¹³ defined different kinds of hypersurfaces and obtained their characteristic conditions, which are given in the following lemmas.

Lemma 3.1: A Finslerian hypersurface F^{n-1} is a hyperplane of the first kind if and only if $H_{\alpha} = 0$.

Lemma 3.2: A Finslerian hypersurface F^{n-1} is a hyperplane of the second kind if and only if $H_{\alpha} = 0$ and $H_{\alpha\beta} = 0$.

Lemma 3.3: A Finslerian hypersurface F^{n-1} is a hyperplane of the third kind if and only if $H_{\alpha\beta} = 0 = M_{\alpha\beta}$ and $H_{\alpha} = 0$.

4. Hypersurface $F^{*n-1}(c)$ of the Finsler space F^{*n}

Let us consider a special Finsler metric $L^* = \alpha + \mu\beta + \nu\frac{\beta^5}{\alpha^4} + \beta$ with gradient $b_i(x) = \partial_i b$. From parametric equation $x^i = x^i(u^{\alpha})$ of $F^{*n-1}(c)$, we get $\partial_{\alpha} b(x(u)) = 0 = b_i B_{\alpha}^{*i}$ so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{*n-1}(c)$. Therefore along the $F^{*n-1}(c)$, we have

$$(4.1) \quad (a) \quad b_i B_{\alpha}^{*i} = 0, \quad (b) \quad b_i y^i = 0.$$

Hence the induced metric $\underline{L}^*(u, v)$ of $F^{*n-1}(c)$ is given by

$$(4.2) \quad \underline{L}^*(u, v) = \sqrt{a_{\alpha\beta} v^{\alpha} v^{\beta}}, \quad a_{\alpha\beta} = a_{ij} B_{\alpha}^{*i} B_{\beta}^{*j},$$

which is a Riemannian metric.

At a point of $F^{*n-1}(c)$, from (2.7), (2.8), (2.9), (2.10), (2.12), (2.13), (2.14) and (2.16), we get

$$(4.3) \quad \begin{cases} a) & p=1, q_0=0, q_1=0, q_2=-\frac{1}{\alpha^2}, \tau=1, \\ b) & p_0=(\mu+1)^2, p_1=\frac{\mu+1}{\alpha}, p_2=0, \\ d) & S_0=0, S_1=\frac{\mu+1}{\alpha}, S_2=-\frac{(\mu+1)^2}{\alpha^2}b^2. \end{cases}$$

Using (4.3) in (2.15), we find

$$(4.4) \quad g^{*ij} = a^{ij} - \frac{\mu+1}{\alpha} (b^i y^j + b^j y^i) + \frac{(\mu+1)^2}{\alpha^2} b^2 y^i y^j.$$

Transvecting (4.4) with $b_i b_j$ and using (4.1b), we have

$$(4.5) \quad g^{*ij} b_i b_j = b^2,$$

which gives

$$(4.6) \quad b_i(x) = b N_i^*,$$

where b is the length of vector b^i .

Transvecting (4.4) with b_j and using (4.1b) and (4.6), we get

$$(4.7) \quad b^i = a^{ij} b_j = b N^{*i} + (\mu+1) b^2 \alpha^{-1} y^i.$$

This leads to

Theorem 4.1: *In the Finslerian hypersurface $F^{*n-1}(c)$ of a Finsler space with Randers change of special (α, β) -metric, the induced metric is a Riemannian metric given by (4.2) and the scalar function $b(x)$ is given by (4.5) and (4.6).*

The angular metric tensor and metric tensor of F^{*n} are given by

$$(4.8) \quad h_{ij}^* = a_{ij} - \frac{Y_i Y_j}{\alpha^2}$$

and

$$(4.9) \quad g_{ij}^* = a_{ij} + (\mu+1)^2 b_i b_j + \frac{\mu+1}{\alpha} (b_i Y_j + b_j Y_i),$$

respectively.

Transvecting (4.8) with $B_{\alpha}^{*i} B_{\beta}^{*j}$ and using (3.4), we get

$$(4.10) \quad h_{\alpha\beta}^{*} = h_{\alpha\beta}^{*(\alpha)}$$

Where $h_{\alpha\beta}^{*(\alpha)}$ denote the angular metric tensor of induced Riemannian metric. Differentiating (2.12) with respect to β , we get

$$(4.11) \quad \frac{\partial p_0}{\partial \beta} = \frac{20\nu}{\alpha^8} [6(\mu+1)\alpha^4 \beta^3 + 3\alpha^5 \beta^2 + 18\nu \beta^7].$$

Thus along $F^{*n-1}(c)$, $\frac{\partial p_0}{\partial \beta} = 0$. Therefore (2.18) gives $\gamma_1 = 0$, $m_i = b_i$. Then from (2.17), we get

$$(4.12) \quad C_{ijk}^{*} = \frac{\mu+1}{\alpha} (h_{ij}^{*} b_k + h_{jk}^{*} b_i + h_{ki}^{*} b_j).$$

Transvecting (4.12) with $B_{\alpha}^{*i} B_{\beta}^{*j} N^{*k}$ and using (3.4), (3.11) and (4.6), we have

$$(4.13) \quad M_{\alpha\beta}^{*} = \frac{\mu+1}{\alpha} h_{\alpha\beta}^{*} b.$$

From (3.4), (3.13), (4.1b) and (4.12), we have

$$(4.14) \quad M_{\alpha}^{*} = 0.$$

Using (4.14) in (3.9), we have

$$(4.15) \quad H_{\alpha\beta}^{*} = H_{\beta\alpha}^{*}.$$

Thus, we have the following:

Theorem 4.2: The second fundamental ν -tensor $M_{\alpha\beta}^{*}$ of $F^{*n-1}(c)$ is given by (4.13) and the second fundamental h -tensor $H_{\alpha\beta}^{*}$ is symmetric.

Taking h -covariant derivative of (4.1) with respect to the induced connection, we find

$$(4.16) \quad b_{i|\beta} B_{\alpha}^{*i} + b_i B_{\alpha|\beta}^{*i} = 0.$$

Applying (3.14) for the vector b_i , we get

$$(4.17) \quad b_{i|\beta} = b_{i|j} B_{\beta}^{*j} + b_i |_{\beta} N^{*j} H_{\beta}^{*}.$$

In view of (3.12) and (4.17), (4.16) implies

$$(4.18) \quad b_{i|j} B_{\alpha}^{*i} B_{\beta}^{*j} + b_i |_{\beta} B_{\alpha}^{*i} N^{*j} H_{\beta}^{*} + b_i H_{\alpha\beta}^{*} N^{*i} = 0.$$

From $b_i |_{\beta} = -b_h C_{ij}^{*h}$, (3.13), (4.6) and (4.14) together imply

$$(4.19) \quad b_i |_{\beta} B_{\alpha}^{*i} N^{*j} = -b M_{\alpha}^{*} = 0.$$

Using (4.6) and (4.19) in (4.18), we find

$$(4.20) \quad b H_{\alpha\beta}^{*} + b_{i|j} B_{\alpha}^{*i} B_{\beta}^{*j} = 0.$$

Since $H_{\alpha\beta}^{*}$ is symmetric, $b_{i|j}$ is symmetric. Transvecting (4.20) with v^{β} , we get

$$(4.21) \quad b H_{\alpha}^{*} + b_{i|j} B_{\alpha}^{*i} y^j = 0.$$

Again transvecting with v^{α} , we get

$$(4.22) \quad b H_0^{*} + b_{i|j} y^i y^j = 0.$$

In view of Lemma 3.1, the hypersurface $F^{*n-1}(c)$ is a hyperplane of first kind if and only if $b_{i|j} y^i y^j = 0$. Here $b_{i|j}$ being the covariant derivative with respect to the Cartan connection of F^{*n} may depend on y^i . Since b_i is gradient vector, from (2.19) for induced metric L^{*} , we have $E_{ij}^{*} = b_{i|j}$, $F_{ij}^{*} = 0$. Thus, (2.20) reduces to

$$(4.23) \quad \begin{aligned} D_{jk}^{*i} = & B^{*i} b_{jk} + B_j^{*i} b_{0k} + B_k^{*i} b_{0j} - b_{0m} g^{*im} B_{jk}^{*} - C_{jm}^{*i} A_k^{*m} \\ & - C_{km}^{*i} A_j^{*m} + C_{jkm}^{*} A_s^{*m} g^{*is} + \lambda^{*s} \left(C_{jm}^{*i} C_{sk}^{*m} + C_{km}^{*i} C_{sj}^{*m} - C_{jk}^{*m} C_{ms}^{*i} \right). \end{aligned}$$

Using (4.3) and (4.4) in (2.21), we get

$$(4.24) \quad \left\{ \begin{array}{l} a) \quad B_j^* = (\mu+1)^2 b_j + \frac{\mu+1}{\alpha} Y_j, \\ b) \quad B_{ij}^* = \frac{\mu+1}{2\alpha} (a_{ij} - \alpha^{-2} Y_i Y_j), \\ c) \quad B_j^{*i} = \frac{\mu+1}{2\alpha} (\delta_j^i - \alpha^{-2} y^i Y_j) - \frac{(\mu+1)^2}{2\alpha^2} b_j y^i, \\ d) \quad B^{*i} = \frac{\mu+1}{\alpha} y^i, \quad \lambda^{*m} = B^{*m} b_{00}, \\ e) \quad A_k^{*m} = B_k^{*m} b_{00} + B^{*m} b_{k0}. \end{array} \right.$$

In view of (4.1), we have $B_0^{*i} = 0$ and $B_{i0}^* = 0$, which together with (4.24) gives $A_0^{*m} = B^{*m} b_{00}$. Transvecting (4.23) with y^k , we get

$$(4.25) \quad D_{j0}^{*i} = B^{*i} b_{j0} + B_j^{*i} b_{00} - B^{*m} C_{jm}^{*i} b_{00}.$$

Again transvecting (4.25) with y^j , we find

$$(4.26) \quad D_{00}^{*i} = B^{*i} b_{00} = \frac{\mu+1}{\alpha} y^i b_{00}.$$

Transvecting (4.26) with b_i and using (4.1), we have

$$(4.27) \quad b_i D_{00}^{*i} = 0.$$

Thus, the relation $b_{i|j} = b_{ij} - b_r D_{ij}^{*r}$ and (4.27) gives

$$(4.28) \quad b_{i|j} y^i y^j = b_{00}.$$

Using (4.28) in (4.22), we get

$$(4.29) \quad b H_0^* + b_{00} = 0.$$

From equation (4.29) and Lemmas 3.1 and 3.2, it is clear that the necessary and sufficient condition for $F^{*n-1}(c)$ to be a hyperplane of first kind is that $b_{00} = 0$. Since $b_{ij} = \nabla_j b_i$ does not depend on y^i and satisfy (4.1b), this condition may be written as for some $b_{ij} y^i y^j = (b_i y^i)(c_j y^j) = 0$ for some $c_j(x)$. Therefore

$$(4.30) \quad 2b_{ij} = b_i c_j + b_j c_i.$$

From (4.1) and (4.30) it follows that

$$(4.31) \quad b_{00} = 0, b_{ij} B_\alpha^{*i} B_\beta^{*j} = 0, b_{ij} B_\alpha^{*i} y^j = 0.$$

Hence (4.29) gives $H_0^* = 0$. Again from (4.23), (4.24) and (4.30), we have

$$(4.32) \quad b_{i0} b^i = \frac{1}{2} c_0 b^2, \lambda^{*m} = 0, A_j^{*i} B_\beta^{*j} = 0, B_{ij}^* B_\alpha^{*i} B_\beta^{*j} = \frac{\mu+1}{2\alpha} h_{\alpha\beta}^*.$$

Using (3.11), (4.4), (4.7), (4.13) and (4.32) in (4.23), we get

$$(4.33) \quad b_r D_{ij}^{*r} B_\alpha^{*i} B_\beta^{*j} = -\frac{(\mu+1)c_0 b^2}{2\alpha} h_{\alpha\beta}^*.$$

Also from the $b_{i|j} = b_{ij} - b_r D_{ij}^{*r}$ and (4.31), we get

$$(4.34) \quad b_{i|j} B_\alpha^{*i} B_\beta^{*j} = -b_r D_{ij}^{*r} B_\alpha^{*i} B_\beta^{*j} = \frac{(\mu+1)c_0 b^2}{2\alpha} h_{\alpha\beta}^*.$$

Therefore equation (4.20) reduces to

$$(4.35) \quad b H_{\alpha\beta}^* + \frac{(\mu+1)c_0 b^2}{2\alpha} h_{\alpha\beta}^* = 0.$$

Hence the hypersurface $F^{*n-1}(c)$ is umbilical. Thus, we have

Theorem 4.3: *The necessary and sufficient condition for a Finslerian hypersurface $F^{*n-1}(c)$ of a Finsler space with Randers change of special (α, β) -metric to be a hyperplane of the first kind is that (4.35) holds and $F^{*n-1}(c)$ is umbilical.*

From lemma 3.3, the hypersurface $F^{*n-1}(c)$ is a hyperplane of second kind if and only if $H_\alpha^* = 0$ and $H_{\alpha\beta}^* = 0$. Thus (4.35) gives $c_0 = c_i(x) y^i = 0$. Therefore there exists a function $\phi(x)$ such that

$$(4.36) \quad c_i(x) = \phi(x) b_i(x).$$

Hence (4.30) reduces to. $b_{ij} = \phi b_i b_j$

Theorem 4.4: *The necessary and sufficient condition for a Finslerian hypersurface $F^{*n-1}(c)$ of a Finsler space with Randers change of special (α, β) -metric to be a hyperplane of second kind is $b_{ij} = \phi b_i b_j$.*

In view of (4.13) and (4.14), Lemma 3.3 shows that $F^{*n-1}(c)$ does not become a hyperplane of the third kind. Thus, we have

Theorem 4.5: *The Finslerian hypersurface $F^{*n-1}(c)$ of a Finsler space with Randers change of special (α, β) -metric is not a hyperplane of the third kind.*

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