Properties of $Q^*$ Sets↑

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Abstract: In this paper, we study the properties of $Q^*$ open sets. In particular, we investigate the properties and theorems in affine spaces and irreducible spaces. Also we define Gd set, contra $Q^*$ closure and study some of these properties.

1. Introduction

We defined $Q^*$ closed sets and $Q^*$ open sets in an affine space↑ in the year 2010. Affine space is a topological space which characterizes most of the geometrical objects.

We need the following definitions:

Definition 1.1. Let $(X, T)$ be a topological space. Let $A \subset X$. $A$ is said to be $Q^*$closed if $A$ is closed and $\text{int} \ A = \Phi$. Then the complement of $Q^*$ closed set is $Q^*$ open.

Definition 1.2↑. Let $C^n$ be a complex $n$–space. Let $I$ be a collection of some complex polynomials of $C^n$. Let $V_I = \{x \in C^n/ f (x) = 0 \text{ for all } f \in I\}$. That is common zero set of $I$. Then $V_I$ is called affine algebraic variety.

Definition 1.3↑. The set of all complements of affine algebraic varieties satisfies the four axioms defining a topology on $C^n$. This topology is called Zariski topology on $C^n$.

Definition 1.4↑. The set $C^n$ considered as a topological space with its Zariski topology is called affine $n$–space. We denote this affine $n$-space by $A^n$.

2. $Q^*$ Open Sets in Various Spaces

In this section we discuss the properties of $Q^*$ open sets in some particular spaces namely affine spaces and irreducible spaces.

Theorem 2.1. In $A^n$, every non-empty open set is $Q^*$ open set.

Proof. Let $U$ be any non-empty open set with respect to Zariski topology. Let $U \neq X$. The $U = V_{I_1}^C$, for some $I_1$, where $I_1$ has at least one non-zero polynomial. Let $G$ be any open set. Let $G \neq X$. Then $G = V_{I_2}^C$, for some $I_2$ where $I_2$ has at least one nonzero polynomial.

Now $U \cap G = V_{I_1}^C \cap V_{I_2}^C$. Then $U \cap G = (V_{I_1} \cup V_{I_2})^C$. Therefore $U \cap G = (V_{I_1I_2})^C$ since $I_1I_2$ has at least one nonzero polynomial, $(V_{I_1I_2})^C \neq \Phi$. Therefore $U \cap G \neq \Phi$. $U$ intersects every nonempty open set. Therefore $U$ is dense. Hence every nonempty open set is $Q^*$ open set in $A^n$.

Definition 2.2. A topological space $X$ is called irreducible if for any decomposition $X = A_1 \cup A_2$ with closed subsets $A_i \subseteq X$ $(i = 1, 2)$ then we have $X = A_1$ or $X = A_2$.

A subset $X'$ of a topological space $X$ is called irreducible if $X'$ is irreducible as a subspace.

Example 1. Let $X = \{1, 2, 3\}$ and $\tau = \{\Phi, \{1\}, \{1, 2\}, X\}$. Closed sets are $\Phi$, $\{2, 3\}$, $\{3\}$, $X$. Then $X$ is irreducible.

Example 2. Let $X = \mathbb{N}$ and $\tau = \{\Phi, \{1\}, \{1, 2\}, \ldots, X\}$ Then $X$ is irreducible.

Example 3. Let $X = [1,100]$. Let $U_a = [1,a]$ and $\tau = \{U_a / a \in X\}$. Then $X$ is irreducible.

Lemma 2.3. The topological space $X$ is irreducible if and only if every nonempty open set is $Q^*$ open.

Proof. Let $X$ be irreducible. Let $U$ be any nonempty open set. If $U = X$ then nothing to prove. Let $U \neq X$. Then $cl U \neq X$. If possible suppose that $U$ is not $Q^*$ open. Then there exits an open set $V$ such that $U \cap V = \Phi$. This implies $U^C \cap V^C = X$, where $U^C$ and $V^C$ are proper closed sets. This is a contradiction to $X$ is irreducible.

Conversely, suppose that every open set is $Q^*$ open. We claim that $X$ is irreducible. Suppose $X$ is reducible. Then $X = A \cup B$, where $A$ and $B$ are proper nonempty closed sets. This implies $A^C \cap B^C = \Phi$. Then $A^C$ is not dense. Then $A^C$ is an open set but not $Q^*$ open, a contradiction. Hence $X$ is irreducible.

Theorem 2.4. Let $(X, \tau)$ be a topological space. Let $W \subset X$. Every nonempty open set of $W$ is $Q^*$ open in $W$ if and only if every nonempty open set of $cl W$ is $Q^*$ open in $cl W$. 


**Proof.** Let every nonempty open set of W be Q* open set in W. We claim that every nonempty open set of cl W is Q* open in cl W.

Let A be any nonempty open set in cl W. Then A \( \cap \) W is open in W. By hypothesis A \( \cap \) W is dense in W. It is enough to prove that every open set intersects A. Let U be any open set of cl W. Take \( x \in U \). Now \( x \in cl W \).

Therefore every open set of x intersects W.

Also U \( \cap \) W is a nonempty open set in W. Since A \( \cap \) W is dense in W, (A \( \cap \) W) \( \cap \) (U \( \cap \) W) \( \neq \) \( \Phi \). This implies A \( \cap \) U \( \neq \) \( \Phi \). Hence A is dense in cl W and hence A is Q* open in cl W.

Conversely, suppose that every nonempty open set of cl W is Q* open in cl W. We claim that every nonempty open set of W is Q* open in W.

Let U be any nonempty open set of W. Then there exists an open set G in cl W such that U = G \( \cap \) W. By hypothesis G is Q* open in W. That implies G \( \cap \) W is Q* open in W. Therefore U is Q* open in W. Hence the theorem.

**Theorem 2.5.** Let f: \( \mathbb{C}^n \rightarrow \mathbb{C} \). Let \( A \subset \mathbb{C}^n \). Then \( x_0 \in cl A \), for any \( x_0 \) if and only if f is identically zero in A implies \( f(x_0) = 0 \).

**Proof.** Let \( x_0 \in cl A \) and f be identically zero in A.

Let I = \{f\}. Since f is identically zero in A, A \( \subset \) V_1. Since V_1 is closed, cl A \( \subset \) V_1. Since \( x_0 \in cl A \subset V_1 \), f(\( x_0 \)) = 0. Conversely, let V_1 be any closed set containing A. We claim that \( x_0 \in V_1 \). We have V_1 = \{x \in \mathbb{C}^n/ f(x) = 0, \forall f \in I\}. Since A \( \subset \) V_1, f(x) = 0 \( \forall x \in A, \forall f \in I \). Therefore f(\( x_0 \)) = 0 \( \forall f \in I \).

Hence \( x_0 \in V_1 \). But cl A is the smallest closed set containing A. Therefore \( x_0 \in cl A \). Thus the Lemma.

**Theorem 2.6.** If A is Q* open then there exists I such that \( f(A^c) = 0 \) \( \forall f \in I \) and \( f(A) = 0 \) implies \( f \equiv 0 \).

**Proof.** Let A be Q* open. Then \( A^c \) is closed and cl A = X. Then there exists I such that V_1 = \( A^c \) and cl A = X. If \( x \in V_1 = \) \( A^c \), then f(\( x \)) = 0 \( \forall f \in I, x \in A^c \). Therefore f(\( A^c \)) = 0 \( \forall f \in I \). Let us take f(\( A \)) = 0. We claim that f \( \equiv 0 \). Let \( x_0 \in cl A \). Since f(\( A \)) = 0 \( \forall x \in A \) and by Lemma (2.5), f(\( x_0 \)) = 0. Therefore f(\( x \)) = 0 \( \forall x \in cl A \). But cl A = X. Therefore f(\( x \)) = 0 \( \forall x \in X \). Hence f \( \equiv 0 \).

**3. General Properties**

Let \((X, \tau)\) be a topological space. We have proved that the collection of all Q* open sets together with \( \Phi \) is a topology \( \tau \). Let \( \tau_1 = \tau_{O^*} \). We find \((\tau_1)_{O^*}\), which is denoted by \( \tau_2 \) and so on.
Theorem 3.1. Let \((X, \tau)\) be a topological space. Then the union of all proper open sets is \(Q^*\) open.

Proof. Let \(A = \bigcup A_i\), where \(A_i\) is proper open set (with respect to \(\tau\)). Clearly \(A\) is open. Always \(\text{cl } A \subseteq X\). We claim that \(X \subseteq \text{cl } A\). Let \(x_0 \in X\). Let \(U\) be any open set containing \(x_0\). Therefore \(U \cap A - \{x_0\} \neq \emptyset\). Therefore \(x_0 \in \text{cl } A\). Then \(\text{cl } A = X\). Hence \(A\) is \(Q^*\)open.

Result 3.2. Let \((X, \tau)\) be a topological space. Let \(A\) be union of all proper open subsets of \(X\). (Let \(\tau_{Q^*}\) denote the collection of all \(Q^*\)open sets with respect to \(\tau\)). If \(\tau_1 = \tau_{Q^*}, \tau_2 = (\tau_1)_{Q^*}\) etc, then \(A \in \tau_i, \forall i = 1,2,\ldots\)

Proof. By Theorem 3.1, \(A\) is \(Q^*\)open. Let \(\tau_1 = \tau_{Q^*}\). Clearly \(A \in \tau_1\). If \(B \in \tau_1(B\) is \(Q^*\)open with respect to \(\tau)\) then \(B\) is open in \(\tau\). Then \(B \subseteq A\). Then union of all proper open sets with respect to \(\tau_1\) is \(A\). Therefore \(A\) is \(Q^*\)open with respect to \(\tau_1\). Hence \(A \in \tau_2\). Similarly \(A \in \tau_1\), for all \(i\).

Converse is not true. Consider the example
Let \(X = \{a,b,c,d\}\) and \(\tau = \{\emptyset,\{a,b\}\},\{a,b,c\}, X\}\). Also \(\tau_{Q^*} = \tau\). Let \(B = \{a,b\}\). \(\text{cl } B = X\). Also \(\tau = \tau_{Q^*}\). \(B \in \tau_1\) for all \(i\). But \(B \neq \text{union of all proper open subsets of } X\).

Result 3.3. If \(A\) and \(B\) are open sets with \(A \cap B = \emptyset\), then \(A\) and \(B\) are not \(Q^*\)open.

Proof: Since \(A \cap B = \emptyset\), the points of \(B\) can’t be limit points of \(A\). Then \(\text{cl } A \neq X\). Hence \(A\) is not \(Q^*\)open. Similarly \(B\) is not \(Q^*\)open.

Theorem 3.4. Let \((X, \tau)\) be a topological space. If \(\tau_1 = \tau_{Q^*}\) and \(\tau_2 = (\tau_1)_{Q^*}\) then \(\tau_1 = \tau_2\).

Proof: Clearly \(\tau_1\) is finer than \(\tau_2\). We have to prove that \(\tau_2\) is finer than \(\tau_1\). Let \(A \in \tau_1\). Since \(\tau_1 \subseteq \tau\) and \(A\) is dense with respect to \(\tau\), \(A\) is dense with respect to \(\tau_1\). Then \(A\) is \(Q^*\) open with respect to \(\tau_1\), that is, \(A \in \tau_2\). Therefore \(\tau_2\) is finer than \(\tau_1\). Hence \(\tau_1 = \tau_2\).

Theorem 3.5. Let \((X, \tau)\) be a topological space. (Let \((\tau_A)\) denote the subspace topology on \(A\)). If \(B \subseteq A \subseteq X\), where \(A\) is open with respect to \(\tau\) and \(B\) is \(Q^*\)open in \(X\) then \(B\) is \(Q^*\)open in \(A\).

Proof: Given \(B\) is open in \(X\) and \(\text{cl } B = X\). Then \(B \cap A\) is open in \(A\). Also \(B \cap A = B\) is open in \(A\). Claim \(\text{cl } B\) with respect to \(\tau_A\) is \(A\). Let \(U\) be open in
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A. Since A is open in X, U is open in X. Then U ∈ τ. Since cl B = X, U ∩ B ≠ ∅. Hence every open set U in A intersects B. Therefore B is Q*open in A.

**Theorem 3.6.** Let (X, τ) be a topological space. If B ⊂ A ⊂ X, where A is Q* open and B is Q*open in A then B is Q*open in X.

**Proof:** Since B is open in A and A is open in X, B is open in X. We claim cl B with respect to τ is X. Let U be any open set with respect to τ. Since cl A = X, U ∩ A ≠ ∅. Therefore U ∩ A is an open set with respect to τ A. Since cl B with respect to τ A is A, (U ∩ A) ∩ B ≠ ∅. Then U ∩ (A ∩ B) ≠ ∅. Hence cl B with respect to τ = X and hence B is Q*open in X.

**Definition 3.7.** Let (X, τ) be a topological space. Contra Q* cl A is defined by the intersection of all Q*open sets containing A.

**Theorem 3.8.** Let (X, τ) be a topological space. Then contra cl A ⊂ contra Q* cl A.

**Proof:** Let A ⊂ X. We have contra cl A = ∩{B/ B is open, A ⊂ B} and contra Q*cl A = ∩{ B/ B is Q*open, A ⊂ B}. Since every Q* open set is open, contra cl A ⊂ contra Q* cl A.

**Example.** Let X = \{a,b,c\}. Let τ = \{Φ, \{a,b\}, \{c\},X\}. Also τ Q* = \{X\}. Let A = \{a,b\}. Then contra cl A = \{a,b\}; contra Q*cl A = X. Therefore contra cl A ≠ contra Q*cl A.

**Remark.** It directly follows from definitions that if every Q* open set is Q*closed then Q*cl A = contra Q* cl A.

**Definition 3.9.** Let (X, τ) be a topological space. Let A ⊂ X. A is said to be Gd set if A = U ∪ V, where U is open and V is proper Q*open.

**Example 3.10.** Let X = [1,100]. Let U_a = [1,a] and τ = \{ U_a/ a ∈ X\}. Also τ Q* = τ. Then every set U_a ≠ X is a Gd set.

**Remark 3.10.** Every Gd set is Q* open but its converse is not true as is evident from the following example.

Let X = \{a,b,c\} and τ = \{Φ, \{a,b\}, X\}. Then τ Q* = \{X, \{a,b\}\}. Here \{b,c\} is Q* open but not Gd set.

**Definition 3.11.** Let (X, τ) be a topological space. Let D be any directed set. Let <x_a>, α ∈ D be a net in X. We say that <x_a> Q* converges to x_0 if given any Q* open set U containing x_0 there exists α_0 ∈ D such that x_a ∈ U ∀ α ≥ α_0.
Theorem 3.12. Let $(X, \mathcal{T})$ be a topological space. Let $A \subseteq X$, $x_0 \in Q^* \text{cl } A$ if and only if there exists a net $\langle x_\alpha \rangle$ in $A$, such that $\langle x_\alpha \rangle$ $Q^*$ converges to $x_0$.

Proof. Let $\langle x_\alpha \rangle$ be a net in $A$ such that $\langle x_\alpha \rangle$ $Q^*$ converges to $x_0$. We claim that $x_0 \in Q^* \text{cl } A$. Let $U$ be any $Q^*$ open set containing $x_0$. Since $\langle x_\alpha \rangle Q^*$ converges to $x_0$ there exists $\alpha_0 \in D$ such that $x_\alpha \in U$, $\forall \alpha \geq \alpha_0$. Also $\langle x_\alpha \rangle$ is a net in $A$. Then $x_{\alpha_0} \in U \cap A$. Therefore $U \cap A \neq \emptyset$. Since $U$ is arbitrary, every $Q^*$ open set containing $x_0$ intersects $A$. Hence $x_0 \in Q^* \text{cl } A$. Conversely suppose $x_0 \in Q^* \text{cl } A$. We claim that there exists a net in $A$ such that the net $Q^*$ converges to $x_0$. Let $D = \{U / U$ is $Q^*$ open set containing $x_0\}$. Define $\leq$ in $D$ as follows: $U_1 \leq U_2$ if $U_2 \subseteq U_1$. Clearly $(D, \leq)$ is a directed set. Let $U$ be any $Q^*$ open set containing $x_0$. Since $x_0 \in Q^* \text{cl } A$, there is a point $x_U$ in $U \cap A$. Therefore $\langle x_U \rangle$ is a net in $A$. Given a $Q^*$ open set containing $x_0$, $U \geq G$ implies $U \subseteq G$. Since $x_U \in U \subseteq G$, $x_U \in G$. Therefore $\langle x_U \rangle$ is a net in $A$ such that $\langle x_U \rangle$ $Q^*$ converges to $x_0$. Hence the Theorem.

Reference