

Subspace of a Finsler Space with Kropina Conformal Change

Akansha

Department of Mathematics
Invertis University, Bareilly, India
Email: akanshashukla311@gmail.com

P. N. Pandey

Nehru Gram Bharati (Deemed To Be University), Allahabad
Email: pnpiaps@gmail.com

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Abstract: In the present paper, we have studied a Finsler space whose metric is obtained from the metric of another Finsler space by generalised Kropina conformal change and obtained a necessary and sufficient condition for these Finsler spaces to be projectively related. Apart from other results, the relation between the hypersurfaces of the two Finsler spaces has been discussed.

Key Words: Finsler space, hypersurfaces, generalized Kropina conformal change, projective change, fundamental tensors.

AMS Subject Classification: 53B40, 53C60.

1. Introduction

In 1929, M. S. Knebelman¹ discussed the conformal geometry of generalised metric spaces. In 1980, Makoto Matsumoto² discussed projective changes of Finsler metrics. In 1984, C. Shibata³ dealt with a change of Finsler metric known as β -change of metric. In 1985, Makoto Matsumoto⁴ discussed the induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry. In 1986, T. Yamada⁵ studied Finsler hypersurfaces satisfying certain conditions. In 2008, M. K. Gupta and P. N. Pandey^{6,7} studied hypersurface of a Finsler space with Randers conformal metric and generalised Randers conformal metric. In 2009, these authors⁸ studied hypersurfaces of conformally and h-conformally related Finsler

spaces. In 2013, M. K. Gupta, Abhay Singh and P. N. Pandey⁹ studied a hypersurface of a Finsler space with Randers change of Matsumoto metric.

The aim of the present paper is to study a Finsler space whose metric is obtained from the metric of the Finsler space by generalised Kropina conformal change and to obtain a necessary and sufficient condition for these Finsler spaces to be projectively related. It is also planned to study the relation between the hypersurface of a Finsler space and the hypersurface of a Finsler space whose metric is obtained by the projective generalised Kropina conformal change.

2. Preliminaries

Suppose $F^n = (M^n, L)$ be an n -dimensional Finsler space, where M^n is an n -dimensional smooth manifold and $L(x^j, y^i)$ is a fundamental metric. Suppose $l_i, g_{ij}, g^{ij}, h_j, C_{ijk}, C_{jk}^i$ and G_{jk}^i denote the components of the corresponding normalized supporting element, metric tensor, inverse metric tensor, angular metric tensor, Cartan tensor, associate Cartan tensor and Berwald connection coefficients respectively. Then they satisfy the following relations

$$(2.1) \quad \begin{cases} (a) l_i = \dot{\partial}_i L, & (b) g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2, & (c) C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}, \\ (d) C_{jk}^i = C_{jkh} g^{ih}, & (e) h_j = L \dot{\partial}_i \dot{\partial}_j L, & (f) h_{ij} = g_{ij} - l_i l_j. \end{cases}$$

where g_{ij}, h_j, C_{ijk} and C_{jk}^i are symmetric in their lower indices. Throughout the paper, we use the symbols $\dot{\partial}^i$ and ∂^i for the partial derivatives with respect to y^i and x^i .

The Cartan connection for a Finsler space F^n is given by $C\Gamma = (F_{jk}^i, G_j^i, C_{jk}^i)$. The h -covariant and v -covariant derivatives of a covariant vector $X_i(x, y)$ with respect to Cartan connection are given by

$$(2.2) \quad (a) X_{i|j} = \partial_i X_j - (\dot{\partial}_h X_i) G_j^h, \quad (b) X_i|_j = \dot{\partial}_j X_i - C_{ij}^r X_r.$$

Let $\alpha(x,y)=\sqrt{a_{ij}(x)y^jy^i}$ and $\beta(x,y)=b_iy^i$. Then the metric $L=\frac{\alpha^2}{\beta}$ is called a Kropina metric¹⁰.

Suppose the symmetric and skew symmetric parts of the tensor b_{ij} be denoted by r_{ij} and s_{ij} respectively. Then, we have

$$(2.3) \quad (a) \ 2r_{ij} = b_{ij} + b_{ji} \quad (b) \ 2s_{ij} = b_{ij} - b_{ji}.$$

Let $F^{*n}=(M^n, L^*)$ be another Finsler space over the same manifold M^n . If $L^*(x,y)=e^{\sigma(x)}L(x,y)$, then the change of metric is a conformal change and the function $\sigma(x)$ is conformal factor¹.

If the conformal change is given by

$$(2.4) \quad L^*(x,y)=e^{\sigma(x)}\frac{L^2}{\beta}, \text{ where } \beta(x,y)=b_i(x)y^i,$$

then it is called a Kropina conformal change.

If $\sigma(x)$ is constant then (2.4) reduces to Kropina homothetic change

$$(2.5) \quad L^*(x,y)=k\frac{L^2}{\beta},$$

where $k=e^{\sigma(x)}$ is a constant.

A subspace M^m , ($1 < m < n$) of dimension m of the underlying manifold M^n may be represented parametrically by the equations $x^j = x^j(u^\alpha)$, where u^α are the Gaussian coordinates on M^m (Latin indices run from 1 to n , while Greek indices take values from 1 to m). The rank of the matrix of projection factors $B_\alpha^j = \partial x^j / \partial u^\alpha$ is supposed to be m . If the supporting element y^i at a point $u=(u^\alpha)$ of M^m is assumed to be tangential to M^m then y^i may be written as $y^i = B_\alpha^i(u)v^\alpha$ so that $v=(v^\alpha)$ is thought of as the supporting element of M at the point u^α . The function $\underline{L}=L(x(u), y(u,v))$ gives rise to a Finsler metric on M^m . Thus, we get an m - dimensional Finsler space $F^m = (M^m, \underline{L}(u,v))$.

Since the rank of the matrix of projection factors is m , we may choose $(n-m)$ linearly independent and mutually orthogonal unit vectors $\overset{a}{N}_i$ such that

$$(2.6) \quad \overset{a}{N}_i B_\alpha^i = 0, \quad \alpha = m+1, m+2, \dots, n.$$

We may choose a set of normals with respect to the given directions y^j satisfying

$$(2.7) \quad (a) \overset{a}{N}_i = g^{ij} \overset{a}{N}_j, \quad (b) g_{ij} B_\alpha^i N_a^j = 0, \quad (c) g_{ij} N_a^i N_b^j = \delta_{ab}, \quad a, b = m+1, \dots, n.$$

Let us define $B_i^\alpha = B_i^\alpha(u, v)$ by

$$(2.8) \quad B_i^\alpha = g^{\alpha\beta} g_{ij} B_\beta^j.$$

This, in view of $g_{ij} B_\beta^j B_\gamma^i = g^{\beta\gamma}$, implies

$$(2.9) \quad B_\alpha^i B_i^\beta = \delta_\alpha^\beta.$$

From (2.7), (2.8) and (2.9), we have

$$(2.10) \quad (a) N_a^i B_i^\alpha = 0, \quad (b) N_a^i N_i^b = \delta_a^b, \quad (c) B_\alpha^i B_j^\alpha + N_a^i N_j^a = \delta_j^i.$$

The second fundamental h -tensor $\overset{a}{H}_{\alpha\beta}$ and the normal curvature vector $\overset{a}{H}_\alpha$ for the induced Cartan connection $IC\Gamma = (F_{\beta\gamma}^\alpha, G_\beta^\alpha, C_{\beta\gamma}^\alpha)$ in a normal direction N_a^i are given by

$$(2.11) \quad \overset{a}{H}_{\alpha\beta} = \overset{a}{N}_i \left(B_{\alpha\beta}^i + F_{jk}^i B_\alpha^j B_\beta^k \right) + \overset{a}{M}_{\beta\alpha}^b \overset{b}{H}_\beta$$

and

$$(2.12) \quad \overset{a}{H}_\alpha = \overset{a}{N}_i \left(B_{0\alpha}^i + G_j^i B_\alpha^j \right),$$

where $\overset{a}{M}_{\beta\alpha}^b = C_{jk}^i N_i^a N_b^j B_\alpha^k$, $B_{\alpha\beta}^i = \partial^2 x^i / \partial u^\alpha \partial u^\beta$ and $B_{0\alpha}^i = B_{\beta\alpha}^i v^\beta$.

From (2.11) and (2.12), we have

$$(2.13) \quad (a) \quad \overset{a}{H}_{0\alpha} = \overset{a}{H}_{\beta\alpha} v^\beta = \overset{a}{H}_\alpha, \quad (b) \quad \overset{a}{H}_{\alpha 0} = \overset{a}{H}_{\alpha\beta} v^\beta = \overset{a}{H}_\alpha + \overset{a}{M}_{\alpha}^b \overset{b}{H}_0.$$

In 1985, Matsumoto⁴ proved that

Lemma 2.1: *A totally geodesic subspace is characterised by $\overset{a}{H}_\alpha = 0$ or equivalently $\overset{a}{H}_0 = 0$.*

3. Finsler Space with Kropina Conformal Change Metric

Let $F^{*n} = (M^n, L^*)$ be an n -dimensional Finsler space on the differentiable manifold M^n whose metric L^* is obtained from the fundamental metric L of the Finsler space F^n by generalized Kropina conformal change (2.4).

Throughout the paper, the geometric objects associated with F^{*n} will be asterisked *.

Differentiating (2.4) partially with respect to y^i , we get

$$(3.1) \quad l_i^* = 2e^{\sigma(x)} \frac{L}{\beta} l_i - e^{\sigma(x)} \frac{L^2}{\beta^2} b_i,$$

where $l_i^* = \dot{\partial}_i L^*$.

Differentiating (3.1) partially with respect to y^j and using (2.1)(e) and (2.1)(f), we have

$$(3.2) \quad h_{ij}^* = 2e^{2\sigma(x)} \frac{L^2}{\beta^2} \left\{ g_{ij} - \frac{L}{\beta} (l_i b_j + l_j b_i) + \frac{L^2}{\beta^2} b_i b_j \right\}, h^*$$

where $h_{ij}^* = L^* \dot{\partial}_j l_i^*$. From (3.2) and $g_{ij}^* = h_{ij}^* + l_i^* l_j^*$, we have

$$(3.3) \quad g_{ij}^* = 2\tau \frac{L}{\beta} g_{ij} + 3\tau \frac{L^3}{\beta^3} b_i b_j - 4\tau \frac{L^2}{\beta^2} (l_i b_j + l_j b_i) + 4\tau \frac{L}{\beta} l_i l_j,$$

where $\tau = e^{2\sigma(x)} \frac{L}{\beta}$.

The inverse metric tensor g^{*ij} is given by

$$(3.4) \quad g^{*ij} = \frac{\beta}{2\tau L} g^{ij} - \frac{\beta}{2\tau L b^2} b^i b^j + \frac{\beta^2}{\tau L^2 b^2} (l^i b^j + l^j b^i) - \frac{\beta}{\tau L} \left\{ \frac{2\beta^2 - L^2 b^2}{L^2 b^2} \right\} l^i l^j,$$

where $b^i = g^{ij} b_j$ and $b^2 = b^i b_i$.

Differentiating (3.3) partially with respect to y^k , we find

$$(3.5) \quad \begin{aligned} C_{ijk}^* = & \frac{2\tau L}{\beta} \left\{ C_{ijk} - \frac{1}{\beta} (g_{jk} b_i + g_{ki} b_j + g_{ij} b_k) - \frac{3L^2}{\beta^3} b_i b_j b_k \right. \\ & + \frac{1}{L} (g_{jk} l_i + g_{ki} l_j + g_{ij} l_k) + \frac{3L}{\beta^2} (b_i b_j l_k + b_j b_k l_i + b_k b_i l_j) \\ & \left. - \frac{2}{\beta} (b_i l_j l_k + b_j l_k l_i + b_k l_i l_j) \right\}. \end{aligned}$$

Transvecting (3.5) with g^{*jh} , we have

$$(3.6) \quad \begin{aligned} C_{ik}^{*h} = & C_{ik}^h + \frac{C_{ijk} b^j}{L b^2} (2\beta l^h - L b^h) - \frac{1}{\beta} (\delta_k^h b_i + \delta_i^h b_k) - \frac{\beta}{L^2 b^2} g_{ik} b^h \\ & + \frac{1}{L} (\delta_k^h l_i + \delta_i^h l_k) + \frac{1}{L^2 b^2} (2\beta^2 - L^2 b^2) g_{ik} l^h - \frac{7\beta^2}{b^2} b^h b_i b_k \\ & - \frac{2}{L^2 b^2} (\beta^2 + L^2 b^2) (b_i l_k + b_k l_i) l^h - \frac{1}{L b^2} (5\beta^2 + 3L^2 b^2) l^h b_i b_k \\ & + \frac{\beta^2}{L b^2} b^h (b_i l_k + b_k l_i) - 2l^h l_i l_k. \end{aligned}$$

The relation between Cartan connection coefficients F_{jk}^i of the Finsler space F^n and Cartan connection coefficients F_{jk}^{*i} of the Finsler space F^{*n} is given by

$$(3.7) \quad F_{jk}^{*i} = F_{jk}^i + D_{jk}^i,$$

where D_{jk}^i is the difference tensor and satisfy the following

$$(3.8) \quad (a) D_{jk}^i y^j = D_{0k}^i, \quad (b) D_{0j}^i y^j = D_{00}^i.$$

Transvecting (3.7) by y^k and using (3.8)(a), we get

$$(3.9) \quad G_j^{*i} = G_j^i + D_{0j}^i,$$

where $G_j^{*i} = F_{jk}^{*i} y^k$ and $G_j^i = F_{jk}^i y^k$.

Transvecting (3.9) by y^j and using (3.8)(b), we get

$$(3.10) \quad 2G^{*i} = 2G^i + D_{00}^i.$$

Differentiating (3.10) with respect to y^j , we get

$$(3.11) \quad 2G_j^{*i} = 2G_j^i + \dot{\partial}_j D_{00}^i.$$

From (3.9) and (3.11), we have

$$(3.12) \quad \dot{\partial}_j D_{00}^i = 2D_{0j}^i.$$

The expressions for D_{00}^i , D_{0j}^i and D_{jk}^i are given by

$$(3.13) \quad D_{00}^j = \frac{y^j}{2\beta} \left\{ -r_{00} + 2S + \beta\sigma_0 \right\} + \frac{Lb^j}{2\beta^2} \left\{ Lr_{00} + 2\beta R - 2LS - \frac{L\beta}{2}\sigma_0 \right\} - \sigma^j - \frac{L^2}{2\beta} s_0^j.$$

$$(3.14) \quad \begin{aligned} D_{0k}^j &= \frac{\delta_k^j}{2} \left\{ \frac{-r_{00}}{2\beta} + \frac{S}{\beta} + \frac{\sigma_0}{4} \right\} + \frac{y^j}{2} \left\{ \frac{-r_{0k}}{\beta} + \frac{r_{00}}{2\beta^2} b_k - \frac{S}{\beta^2} b_k + \frac{1}{\beta} s_k - \frac{s_{0k}}{2\beta} \right\} \\ &+ \frac{b^j}{2} \left\{ \frac{Lr_{00}}{\beta^2} l_k + \frac{L^2 r_{0k}}{\beta^2} - \frac{L^2 r_{00}}{\beta^3} b_k + \frac{R}{\beta} l_k + \frac{L}{\beta} R_k - \frac{LR}{\beta^2} b_k \right. \\ &\quad \left. - \frac{2LS}{\beta^2} l_k + \frac{2L^2 S}{\beta^3} b_k - \frac{L^2}{\beta^2} S_k - \frac{L}{2\beta} l_k \sigma_0 + \frac{L^2}{4\beta^2} b_k \sigma_0 \right\} \\ &- \frac{y^j}{8} \sigma_k + \frac{L^2}{4\beta^2} b_k s_0^j - \frac{L^2}{4\beta} s_k^j, \end{aligned}$$

$$\begin{aligned}
(3.15) \quad & \dot{\partial}_h D_{0k}^j = \frac{\delta_k^j}{2} \left\{ \frac{-r_{0h}}{\beta} + \frac{r_{00}}{2\beta^2} b_h - \frac{S}{\beta^2} b_h + \frac{1}{\beta} S_h \right\} + \frac{\delta_h^j}{2} \left\{ \frac{-r_{0k}}{\beta} + \frac{r_{00}}{2\beta^2} b_k - \frac{S}{\beta^2} b_k + \frac{1}{\beta} S_k - \frac{s_{0k}}{\beta} \right\} \\
& + \frac{y^j}{2} \left\{ \frac{-r_{hk}}{\beta} + \frac{2b_r y^s}{\beta} \dot{\partial}_h F_{ks}^r + \frac{1}{\beta^2} (r_{0k} b_h + r_{0h} b_k) - \frac{1}{\beta^3} b_k b_h (r_{00} - 2S) + \frac{1}{\beta} S_{kh} \right. \\
& - \frac{1}{\beta^2} (S_k b_h + S_h b_k) + \frac{1}{2\beta^2} (s_{0k} b_h + s_{0h} b_k) - \frac{s_{hk}}{\beta} \left. \right\} + \frac{b^j}{2} \left\{ \frac{2L}{\beta^2} (r_{0h} l_k + r_{0k} l_h) + \frac{L^2 r_{kh}}{\beta^2} \right. \\
& - \frac{2L^2}{\beta^2} b_r y^s \dot{\partial}_h F_{ks}^r + \frac{L}{\beta} R_{kh} - \frac{2L^2}{\beta^3} (r_{0k} b_h + r_{0h} b_k) + \frac{L}{\beta^4} b_k b_h \left(3Lr_{00} + 2\beta R - 6LS - \frac{L\beta\sigma_0}{2} \right) \\
& + \frac{l_k l_h}{\beta^2} \left(r_{00} - 2S - \frac{\beta\sigma_0}{2} \right) + \frac{L_{kh}}{\beta^2} \left(Lr_{00} + \beta R - 2LS - \frac{L\beta\sigma_0}{2} \right) + \frac{(l_k b_h + l_h b_k)}{\beta^3} \left(-2Lr_{00} - \beta R \right. \\
& + 4LS + \frac{L\beta\sigma_0}{2} \left. \right) + \frac{1}{\beta} (R_k l_h + R_h l_k) - \frac{L}{\beta^2} (R_k b_h + R_h b_k) - \frac{2L}{\beta^2} (S_k l_h + S_h l_k) + \frac{2L^2}{\beta^3} (S_k b_h \\
& + S_h b_k) - \frac{L^2}{\beta^2} S_{kh} \left. \right\} - \frac{\delta_h^j}{8} \sigma_k - \frac{L^2}{2\beta^3} b_k b_h s_0^j + \frac{L^2}{4\beta} (b_k s_h^j + b_h s_k^j),
\end{aligned}$$

where

$$(3.16) \quad S = \frac{[L^2 b^2 - 2\beta^2] r_{00} - 2L^2 \beta s_0 - L^2 \beta^2 \sigma_r b^r + \beta^3 \sigma_0}{2L^2 b^2}$$

and

$$(3.17) \quad R = \frac{-2L\beta r_{00} - 2L^3 s_0 - L^3 \beta \sigma_r b^r + L[2\beta^2 + L^2 b^2] \sigma_0}{4L^2 b^2}.$$

$$R_k = \dot{\partial}_k R, \quad S_k = \dot{\partial}_k S, \quad R_{kh} = \dot{\partial}_h R_k, \quad S_{kh} = \dot{\partial}_h S_k,$$

$$(3.18) \quad \begin{cases} (a) \frac{\partial \sigma(x)}{\partial x^i} = \sigma_i, & (b) \sigma_r y^r = \sigma_0, & (c) \sigma_i g^{ij} = \sigma^j(x), & (d) r_{0k} = r_{jk} y^j, \\ (e) r_{00} = r_{0k} y^k, & (f) s_{0k} = s_{jk} y^j, & (g) s_0 = s_{0k} b^k, & (h) s_0^j = s_{0i} g^{ij}. \end{cases}$$

Differentiating (3.9) with respect to y^k and using (3.7) and $G_{jk}^i = (\partial_k F_{jr}^i) y^r + F_{jk}^i$, we have

$$(3.19) \quad \partial_j D_{0k}^i = (\partial_k D_{jr}^i) y^r + D_{jk}^i.$$

Differentiating b_{ij} partially with respect to y^k and using the definition of h -covariant derivative, we get

$$(3.20) \quad \dot{\partial}_k b_{ij} = -b_r \dot{\partial}_k F_{ij}^r.$$

Using (2.3)(b) in (3.19), we have

$$(3.21) \quad \dot{\partial}_k s_{ij} = \dot{\partial}_k (b_{ij} - b_{ji}) = 0.$$

In view of (2.5), (3.13) reduces to

$$(3.22) \quad D_{00}^j = \frac{y^j}{2\beta} \{-r_{00} + 2S\} + \frac{Lb^j}{2\beta^2} \{Lr_{00} + 2\beta R - 2LS\} - \frac{L^2}{2\beta} s_0^j,$$

where

$$(3.23) \quad S = \frac{[L^2 b^2 - 2\beta^2] r_{00} - 2L^2 \beta s_0}{2L^2 b^2}$$

and

$$(3.24) \quad R = \frac{-L\beta r_{00} - L^3 s_0}{2L^2 b^2}$$

From (3.10) and (3.21), we have

$$(3.25) \quad G^{*i} = G^i - \frac{y^j}{2\beta} \left\{ \frac{\beta r_{00} + L^2 s_{0r} b^r}{L^2 b^2} \right\} + \frac{L^2 b^j}{4\beta} \left\{ \frac{\beta r_{00} + L^2 s_{0r} b^r}{L^2 b^2} \right\} - \frac{L^2}{4\beta} s_0^j.$$

Now we propose a lemma:

Lemma 3.1: *The tensor $s_0^j (\neq 0)$ is not parallel to l^i .*

Proof: Let s_0^i is parallel to l^i then

$$(3.26) \quad s_0^i = \mu l^i$$

for some scalar $\mu \neq 0$. Transvecting (3.26) with l_i , we get

$$(3.27) \quad \mu = s_0^i l_i = \frac{s_{00}}{L}.$$

Since s_{ij} is skew-symmetric in i and j , (3.27) implies

$$(3.28) \quad \mu = 0.$$

Hence,

$$(3.29) \quad s_0^i = 0,$$

which is a contradiction. This completes the proof.

In view of above lemma and (3.24), the Kropina homothetic change is projective if and only if

$$(3.30) \quad s_{0i} = 0.$$

Differentiating (3.30) partially with respect to y^j and using (3.20), we have

$$(3.31) \quad s_{ij} = 0,$$

which implies

$$(3.32) \quad b_{ij} = b_{ji}.$$

Thus, we have

Theorem 3.2: *The Kropina homothetic change (2.5) becomes a projective change if and only if the vector b_i is gradient.*

Suppose the vector b_i in a Kropina homothetic change (2.5) is parallel with respect to the Cartan connection, i.e. $b_{ij} = 0$. Thus we have $r_{ij} = 0 = s_{ij}$ and

$\frac{\partial \sigma(x)}{\partial x^j} = 0$. Thus in view of (3.13), (3.14), (3.15), (3.16), (3.17), (3.18) and (3.19), we get

$$(3.33) \quad D_{jk}^i = 0,$$

which in view of (3.9), gives

$$(3.34) \quad G_{jk}^{*i} = G_{jk}^i.$$

Conversely, if $D_{jk}^i = 0$ then (3.25) and $b^i = \frac{y^i}{L^2 \beta}$ implies

$$(3.35) \quad \frac{L^2}{4\beta} s_0^i = \frac{y^i}{2\beta^2} \left\{ \frac{\beta r_{00} + L^2 s_{0r} b^r}{L^2 b^2} \right\} (1 - 2\beta),$$

which in view of Lemma 3.1 gives $s_{0i} = 0$ and then $r_{00} = 0$. This implies $b_{ij} = 0$. Thus, we have

Theorem 3.3: *For the Kropina homothetic change (2.5), the difference tensor D_{jk}^i vanishes if and only if the vector b_i is parallel with respect to the Cartan connection of F^n and then the Berwald connection coefficients for both the spaces F^n and F^{*n} are the same.*

4. Subspaces of a Finsler Space given by Kropina Conformal Change

Suppose $F^m = (M^m, \underline{L}(u, v))$ be a subspace of the Finsler space F^n and $F^{*m} = (M^m, \underline{L}^*(u, v))$ be a subspace of the Finsler space F^{*n} given by Kropina conformal change. The functions $B_\alpha^i(u)$ may be considered as the components of m linearly independent vectors tangent to F^m and they are invariant under Kropina conformal change. Let N_a^i be the unit normal vector at a point $u^\alpha \in F^m$, then unit normal vector $N_a^{*i}(u, v)$ of F^{*m} and the inverse projection factor $B_i^{*\alpha}$ along F^{*m} are uniquely determined by

$$(4.1) \quad (a) g_{ij}^* B_\alpha^i N_a^{*j} = 0, \quad (b) g_{ij}^* N_a^{*i} N_b^{*j} = \delta_{ab}$$

and

$$(4.2) \quad B_i^{*\alpha} = g^{*\alpha\beta} g_{ij}^* B_j^\beta,$$

where $g^{*\alpha\beta}$ is the inverse of metric tensor $g_{\alpha\beta}^*$ of F^{*m} .

From (4.1)(a), (4.1)(b) and (4.2), we have

$$(4.3) \quad (a) B_\alpha^i B_i^{*\beta} = \delta_\alpha^\beta, \quad (b) B_\alpha^i N_i^{*a} = 0 \quad (c) N_a^{*i} B_i^{*\alpha} = 0, \quad (d) N_a^{*i} N_i^{*b} = \delta_a^b.$$

From (4.3), we have

$$(4.4) \quad B_\alpha^i B_j^{*\alpha} + N_a^{*i} N_j^{*a} = \delta_j^i.$$

Transvection of (2.10)(a) with v^α gives

$$(4.5) \quad y_j N_a^j = 0.$$

Transvecting (3.3) with $N_a^i N_b^j$ and using (2.7)(c) and (4.5), we have

$$(4.6) \quad g_{ij}^* N_a^i N_b^j = \frac{2\tau L}{\beta} + \frac{3\tau L^3}{\beta^3} (b_k N_a^k)^2.$$

Putting $a = b$ in (4.6), we obtain

$$(4.7) \quad g_{ij}^* N_a^i N_a^j = \frac{2\tau L}{\beta} + \frac{3\tau L^3}{\beta^3} (b_k N_a^k)^2,$$

which implies that $\frac{N_a^i}{\sqrt{\frac{2\tau L}{\beta} + \frac{3\tau L^3}{\beta^3} (b_k N_a^k)^2}}$ is a unit vector.

Transvecting (3.3) with $B_\alpha^i N_a^j$ and using (2.7)(b) and (4.5), we have

$$(4.8) \quad g_{ij}^* B_\alpha^i N_a^j = (b_j N_a^j) \left\{ \frac{3\tau L^3}{\beta^3} (b_i B_\alpha^i) - \frac{4\tau L^2}{\beta^2} (l_i B_\alpha^i) \right\},$$

which shows that N_a^j is normal to F^{*m} if and only if

$$(4.9) \quad \left(b_j N_a^j \right) \left\{ \frac{3\tau L^3}{\beta^3} (b_i B_\alpha^i) - \frac{4\tau L^2}{\beta^2} (l_i B_\alpha^i) \right\} = 0.$$

This implies that either $\left\{ \frac{3\tau L^3}{\beta^3} (b_i B_\alpha^i) - \frac{4\tau L^2}{\beta^2} (l_i B_\alpha^i) \right\} = 0$ or $b_j N_a^j = 0$.

Transvecting $\left\{ \frac{3\tau L^3}{\beta^3} (b_i B_\alpha^i) - \frac{4\tau L^2}{\beta^2} (l_i B_\alpha^i) \right\} = 0$ with v^α and using $y^j = B_\alpha^j v^\alpha$, we have

$$(4.10) \quad \left\{ \frac{3\tau L^3}{\beta^3} b_i y^j - \frac{4\tau L^2}{\beta^2} l_i y^j \right\} = 0,$$

which gives

$$(4.11) \quad -\frac{\tau L^3}{\beta^2} = 0,$$

which is not possible. Hence we have

$$(4.12) \quad b_j N_a^j = 0.$$

Thus, the vector N_a^j is normal to F^{*m} if and only if b_j is tangent to F^m .

From (4.5), (4.7) and (4.12), we can say that $N_a^i / \sqrt{\frac{2\tau L}{\beta}}$ is a unit normal vector of F^{*m} .

Therefore, in view of (4.1)(a) and (4.1)(b), we have

$$(4.13) \quad N_a^{*i} = \frac{N_a^i}{\sqrt{\frac{2\tau L}{\beta}}}$$

Transvecting (3.3) with N_a^{*j} and using (4.5), (4.12) and (4.13), we have

$$(4.14) \quad N_i^* = g_{ij}^* N_a^{*j} = \sqrt{\frac{2\tau L}{\beta}} N_i^a$$

Thus, we conclude:

Theorem 4.1: Let F^{*n} be a Finsler space obtained from the Finsler space F^n by a Kropina conformal change (2.4). If F^{*m} and F^m are the subspaces of these spaces then the vector $b_i b^i$ is tangential to the subspace F^m if and only if every vector normal to F^m is also normal to F^{*m} .

Transvecting (3.6) with $N_i^* N_b^{*j} B_\alpha^k$ and using (4.5), (4.12) and (4.13), we have

$$(4.15) \quad M_{\alpha}^* = C_{jk}^i N_i^* N_b^{*j} B_\alpha^k - \frac{1}{\beta} b_k B_\alpha^k + \frac{1}{L} l_k B_\alpha^k,$$

which implies

$$(4.16) \quad M_{\alpha}^* = M_{\alpha}^* - \left(\frac{b_i}{\beta} - \frac{l_i}{L} \right) B_\alpha^i.$$

Transvecting (2.1)(a) with y^j , we have

$$(4.17) \quad l_i y^i = L.$$

In view of (4.17) and $\beta = b_i y^i$, (4.16) gives

$$(4.18) \quad M_{\alpha}^* = M_{\alpha}^*$$

Hence, we have

Theorem 4.2: The covariant vector M_{α}^* under the Kropina conformal change is invariant.

The normal curvature vector H_α^* on F^{*m} is given by

$$(4.19) \quad H_\alpha^* = N_i^* (B_{0\alpha}^i + G_j^{*i} B_\alpha^j),$$

which in view of (2.12), (3.9), (3.14) and (4.14), reduces to

$$(4.20) \quad H_{\alpha}^* = \sqrt{\frac{2\tau L}{\beta}} \left(H_{\alpha}^a + N_i^a D_{0j}^i B_{\alpha}^j \right).$$

Transvecting (4.20) with v^{ρ} and using $v^{\rho} B_{\alpha}^k = y^k$, we get

$$(4.21) \quad H_0^* = \sqrt{\frac{2\tau L}{\beta}} \left(H_0^a + N_i^a D_{00}^i \right).$$

Transvecting (3.13) with $H_i^a N_i$ and using (4.5) and (4.13), we have

$$(4.22) \quad N_i^a D_{00}^i = -\frac{L^2}{4} \sigma^j N_i^a.$$

Thus, in view of Lemma 2.1, we have

Theorem 4.3: *Let the vector b_i be in the Kropina conformal change (2.4) be gradient and the subspace F^m is totally geodesic if and only if the vector σ_i is tangential to the subspace F^m .*

References

1. M. S. Knebelman, Conformal Geometry of Generalised Metric Spaces, *Proc. Nat. Acad. Sci.*, **15** (1929), 376-379.
2. Makoto Matsumoto, Projective Changes of Finsler Metrics and Projectively Flat Finsler Spaces, *Tensor N. S.*, **34** (1980), 303-315.
3. C. Shibata, On Invariant Tensors of β - Changes of Finsler Metrics, *J. Math. Kyoto Univ.*, **24(1)** (1984), 163-188.
4. Makoto Matsumoto, The Induced and Intrinsic Finsler Connections of a Hypersurface and Finslerian Projective geometry, *J. Math. Kyoto Univ.*, **25(1)** (1985), 107-144.
5. T. Yamada, On Finsler Hypersurfaces Satisfying Certain Conditions, *Tensor N. S.*, **43** (1986), 56-65.
6. M. K. Gupta and P. N. Pandey, On Hypersurface of a Finsler Space with Randers Conformal Metric, *Tensor N.S.*, **70(3)** (2008), 229-240.
7. M. K. Gupta and P. N. Pandey, On Hypersurface of a Finsler Space with a Special Metric, *Acta. Math. Hungar.*, **120** (2008), 165-177.
8. M. K. Gupta and P. N. Pandey, Hypersurfaces of Conformally and h-conformally Related Finsler Space, *Acta Math. Hungar.*, **123(3)** (2009), 257-264.

9. M. K. Gupta, Abhay Singh and P. N. Pandey, On a Hypersurface of a Finsler Space with Randers Change of Matsumoto Metric, *Geometry*, (2013).
10. Makoto Matsumoto, Theory of Finsler Spaces with $(\alpha\beta)$ -metric, *Reports on Mathematical Physics*, **31(1)** (1992), 43-83.
11. H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, 1959.
12. P. L. Antonelli (ed.), *Handbook of Finsler Geometry*, Kluwer Academic Publishers, Dordrecht, (2003).