

On Submanifolds of an Almost Para Complex Manifold

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Abstract: In the present paper, I have studied curvature tensor in an almost para complex manifold. In the second section, projective curvature tensor has been studied in para Kähler manifold. Third section deals with a submanifold of an almost para complex manifold while para invariant sub-manifold is the study matter of the fourth section.

Key words & Phases: Almost para complex manifold, para Kähler manifold, para invariant submanifold.

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1. Preliminaries

Let $\tilde{M}$ be an almost para complex manifold of dimension m and $T_p(\tilde{M})$ be the tangent space at $p \in \tilde{M}$. Then there exists an endomorphism

$\tilde{F}: T_p(\tilde{M}) \longrightarrow T_p(\tilde{M})$, such that

$$(1.1) \quad \tilde{F}^2 = I$$

Let there be a Riemannian metric $\tilde{g}$ such that

$$(1.2) \quad \tilde{g}(\tilde{F}\tilde{X}, \tilde{F}\tilde{Y}) = -\tilde{g}(\tilde{X}, \tilde{Y}),$$

then $(\tilde{F}, \tilde{g})$ is called almost para complex structure. Let us put

$$\tilde{F}(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{F}\tilde{X}, \tilde{Y}), \text{ then } \tilde{F}(\tilde{X}, \tilde{Y}) + \tilde{F}(\tilde{Y}, \tilde{X}) = 0.$$

Let $\tilde{\nabla}$ be Riemannian connection. If

$$(1.3) \quad \tilde{\nabla}_{\tilde{X}} \tilde{F} = 0$$

then $\tilde{M}$ is called almost para-Kähler manifold.

2. Curvature tensor

Let $\tilde{M}$ be an almost Para-Kähler manifold with structure $(\tilde{F}, \tilde{g}, \tilde{\nabla})$.

For such space we have the following:

Theorem 2.1: *If $\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z})$ stands for curvature tensor in $\tilde{M}$, then*

$$(2.1) \quad \tilde{R}(\tilde{X}, \tilde{Y}, \tilde{\tilde{Z}}) = \tilde{F}(\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}))$$

where $\tilde{\tilde{Z}} = \tilde{F}(\tilde{Z})$.

Proof: We have $\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{\nabla}_{\tilde{X}} \tilde{\nabla}_{\tilde{Y}} \tilde{Z} - \tilde{\nabla}_{\tilde{Y}} \tilde{\nabla}_{\tilde{X}} \tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{Z}$.

Putting $\tilde{F}\tilde{Z}$ in place of $\tilde{Z}$, we have

$$\begin{aligned} \tilde{R}(\tilde{X}, \tilde{Y}, \tilde{\tilde{Z}}) &= \tilde{\nabla}_{\tilde{X}} \tilde{\nabla}_{\tilde{Y}} \tilde{F}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}} \tilde{\nabla}_{\tilde{X}} \tilde{F}\tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{F}\tilde{Z} \\ &= \tilde{\nabla}_{\tilde{X}} \left\{ (\tilde{\nabla}_{\tilde{Y}} \tilde{F}) \tilde{Z} + \tilde{F} \tilde{\nabla}_{\tilde{Y}} \tilde{Z} \right\} - \tilde{\nabla}_{\tilde{Y}} \left\{ (\tilde{\nabla}_{\tilde{X}} \tilde{F}) \tilde{Z} + \tilde{F} \tilde{\nabla}_{\tilde{X}} \tilde{Z} \right\} \\ &\quad - \left(\tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{F} \right) \tilde{Z} - \tilde{F} \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{Z} \\ &= \tilde{F} \left(\tilde{\nabla}_{\tilde{X}} \tilde{\nabla}_{\tilde{Y}} \tilde{Z} - \tilde{\nabla}_{\tilde{Y}} \tilde{\nabla}_{\tilde{X}} \tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{Z} \right) \\ &= \tilde{F} \left(\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}) \right). \end{aligned}$$

Corollary 2.1: *If $\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = \tilde{g}(\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}), \tilde{W})$*

then $\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{\tilde{Z}}, \tilde{\tilde{W}}) = -\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{\tilde{W}})$

which implies $\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{\tilde{W}}) = -\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{\tilde{Z}}, \tilde{\tilde{W}})$

and $\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = -\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{\tilde{Z}}, \tilde{\tilde{W}})$.

Proof: It follows from (1.1), (1.2) and (2.1).

Let Weyl projective curvature tensor be

$$(2.2) \quad \tilde{W}(\tilde{X}, \tilde{Y}, \tilde{Z}) = \tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}) - \frac{\tilde{K}}{m(m-1)} (\tilde{g}(\tilde{Y}, \tilde{Z}) \tilde{X} - \tilde{g}(\tilde{X}, \tilde{Z}) \tilde{Y}),$$

$$\begin{aligned} \tilde{W}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) &= \tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) - \frac{\tilde{K}}{m(m-1)} (\tilde{g}(\tilde{Y}, \tilde{Z}) \tilde{g}(\tilde{X}, \tilde{W}) \\ &\quad - \tilde{g}(\tilde{X}, \tilde{Z}) \tilde{g}(\tilde{Y}, \tilde{W})) \end{aligned}$$

and

$$\begin{aligned} \tilde{W}(\tilde{X}, \tilde{Y}, \tilde{\tilde{Z}}, \tilde{\tilde{W}}) &= -\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) \\ &\quad - \frac{\tilde{K}}{m(m-1)} (\tilde{F}(\tilde{Z}, \tilde{Y}) \tilde{F}(\tilde{W}, \tilde{X}) - \tilde{F}(\tilde{Z}, \tilde{X}) \tilde{F}(\tilde{W}, \tilde{Y})). \end{aligned}$$

Therefore

$$\begin{aligned} (2.3) \quad \tilde{W}(\tilde{X}, \tilde{Y}, \tilde{\tilde{Z}}, \tilde{\tilde{W}}) + \tilde{W}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) \\ &= -\frac{\tilde{K}}{m(m-1)} [\tilde{g}(\tilde{Z}, \tilde{Y}) \tilde{g}(\tilde{X}, \tilde{W}) + \tilde{F}(\tilde{Z}, \tilde{Y}) \\ &\quad \tilde{F}(\tilde{W}, \tilde{X}) - \tilde{g}(\tilde{X}, \tilde{Z}) \tilde{g}(\tilde{Y}, \tilde{W}) - \tilde{F}(\tilde{Z}, \tilde{X}) \tilde{F}(\tilde{W}, \tilde{Y})] \\ &= -\frac{\tilde{K}}{m(m-1)} [\tilde{g}(\tilde{Y}, \tilde{Z}) \tilde{g}(\tilde{X}, \tilde{W}) + \tilde{F}(\tilde{Y}, \tilde{Z}) \tilde{F}(\tilde{X}, \tilde{W}) \\ &\quad - \tilde{g}(\tilde{X}, \tilde{Z}) \tilde{g}(\tilde{Y}, \tilde{W}) - \tilde{F}(\tilde{X}, \tilde{Z}) \tilde{F}(\tilde{Y}, \tilde{W})]. \end{aligned}$$

Theorem 2.2: In a para Kähler manifold of dimension greater than 2, if $\tilde{W}$ stands for Weyl projective curvature tensor given by (2.2), then $\tilde{W}(\tilde{X}, \tilde{Y}, \tilde{\tilde{Z}}, \tilde{\tilde{W}}) + \tilde{W}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W})$ can not be zero.

Proof: If the left hand side of (2.3) vanishes then contracting over Y & Z and next on X & W, we get $m(m-2)=0$, which is not possible.

Let us define on $\tilde{M}$

$$(2.4) \quad \tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = a [\tilde{g}(\tilde{X}, \tilde{W}) \tilde{g}(\tilde{Y}, \tilde{Z}) - \tilde{g}(\tilde{X}, \tilde{Z}) \tilde{g}(\tilde{Y}, \tilde{W})]$$

$$\begin{aligned}
& + \tilde{g}(\tilde{F} \tilde{X}, \tilde{W}) \tilde{g}(\tilde{F} \tilde{Y}, \tilde{Z}) \tilde{g}(\tilde{F} \tilde{X}, \tilde{Z}) \tilde{g}(\tilde{F} \tilde{Y}, \tilde{W}) \\
& + b \left[\tilde{g}(\tilde{F} \tilde{X}, \tilde{W}) \tilde{g}(\tilde{Y}, \tilde{Z}) - \tilde{g}(\tilde{F} \tilde{X}, \tilde{Z}) \tilde{g}(\tilde{Y}, \tilde{W}) \right. \\
& \left. + \tilde{g}(\tilde{X}, \tilde{W}) \tilde{g}(\tilde{F} \tilde{Y}, \tilde{Z}) - \tilde{g}(\tilde{X}, \tilde{Z}) \tilde{g}(\tilde{F} \tilde{Y}, \tilde{W}) \right],
\end{aligned}$$

then $\tilde{M}$ is called a para Kähler manifold with almost constant curvature.

Let us write (2.4) in index notation

$$\begin{aligned}
R_{ijkl} &= a \left[g_{il} g_{ij} - g_{ik} g_{jl} + F_{il} F_{jk} - F_{ik} F_{jl} \right] \\
&+ b \left[F_{il} g_{jk} - F_{ik} g_{jl} + g_{il} F_{jk} - g_{ik} F_{jl} \right].
\end{aligned}$$

From this we may derive

$$\begin{aligned}
R_{jk} &= a \left[m g_{jk} - g_{jk} - F_k^l F_{jl} \right] + b \left[-F_k^l g_{jl} + m F_{jk} - g_k^l F_{jl} \right] \\
&= a \left[(m-2) g_{kj} \right] + b \left[-F_{jk} + m F_{jk} - F_{jk} \right] \\
R_{jk} &= a(m-2) g_{kj} - b(m-2) F_{kj},
\end{aligned}$$

which implies $a = K / m(m-2)$ where $K = R_{jk} g^{jk}$.

Theorem 2.3: Constant 'a' in the equation (2.4) is $\frac{K}{m(m-2)}$.

Note: If $\tilde{M}$ is para Kähler, then $\tilde{\nabla}_X \tilde{R} = 0$, i.e. $\tilde{M}$ is symmetric space.

3. Submanifold

Let $\tilde{M}$ be an m-dimensional para complex Riemannian manifold with structure $(\tilde{F}, \tilde{g})$ and M be an n-dimensional ($n \leq m$) Riemannian manifold isometrically immersed in $\tilde{M}$. We denote the metric tensor of $\tilde{M}$ by $\tilde{g}$ as well as that of M. The Gauss and Weingarten formulae for M are given by

$$(3.1) \quad \tilde{\nabla}_X Y = \tilde{\nabla}_X Y + \sigma(X, Y)$$

$$(3.2) \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$$

for any tangent vectors X, Y in M and any vector field ξ normal to M. where σ denotes the second fundamental form to the normal section ξ and

D is the linear connection induced in the normal bundle $T_M^\perp$. σ and A_ξ are related as

$$(3.3) \quad g(\sigma(X, Y)\xi) = g(A_\xi X, Y)$$

The covariant differentiation of second fundamental form σ , denoted by $\bar{\nabla}$ in $T(M) \oplus \perp^\perp M$, is given by

$$(3.4) \quad (\bar{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

for arbitrary vector fields X, Y and Z on $T(M)$, where $T(M)$ is tangent bundle of M and $T^\perp(M)$ is normal bundle of M . The equation of Gauss and Codazzi are given by

$$(3.5) \quad R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W))$$

and

$$(3.6) \quad (R(X, Y, Z))^\perp = (\nabla_X \sigma)(Y, Z) - (\nabla_Y \sigma)(X, Z)$$

for arbitrary vector fields X, Y, Z and W tangent to M . $(R(X, Y, Z))^\perp$ denotes the normal part of $R(X, Y, Z)$.

For any vector field X tangent to M , we put

$$(3.7) \quad \tilde{F}X = PX + QX$$

where PX and QX denote the tangential part and normal part of $\tilde{F}X$ respectively. Also for any vector field ξ normal to M , we put

$$(3.8) \quad \tilde{F}\xi = t\xi + q\xi$$

where $t\xi$ and $q\xi$ respectively denote the tangential part and the normal part of $\tilde{F}\xi$. Then by virtue of (3.7) and (3.8) we respectively have

$$(3.9) \quad P^2X + tQX = X, \quad QPX + qQX = 0,$$

$$(3.10) \quad Pt\xi + tq\xi = 0, \quad Qt\xi + q^2\xi = \xi,$$

for any vector field X tangent to M and any vector field ξ normal to M .

Then Q is related to t by

$$(3.11) \quad g(QX, \xi) = -g(X, t\xi).$$

Theorem 3.1: *In a submanifold M of an almost para complex manifold $\tilde{M}$, we have*

$$(3.12) \quad PtQX = 0, \quad qQ t\xi = 0,$$

for any vector field X tangent to M and ξ normal to M .

Proof: Let U be tangent to M , put $U = U_1 + U_2$, where $\tilde{F}U_1$ is tangent to M and $\tilde{F}U_2$ is normal to M . Then from (3.7), we have

$$\begin{aligned} g(PtQX, U) &= g((\tilde{F} - Q)tQX, U) \\ &= g(\tilde{F}tQX, U) \\ &= -g(tQX, \tilde{F}U) \\ &= g(tQX, \tilde{F}U_1) \\ &= g((\tilde{F} - q)QX, \tilde{F}U_1) \\ &= g(\tilde{F}QX, \tilde{F}U_1) = g(QX, \tilde{F}U_1) = 0, \end{aligned}$$

i.e. $PtQX=0$. We can prove second part of (3.12) similarly.

From (3.9), (3.10) and from Theorem 3.1 we have

$$P^3X - PX = 0 \text{ and } q^3\xi - q\xi = 0$$

Thus, we have:

Theorem 3.2: *In a submanifold M of an almost para complex manifold, P and q act as f -structure tangent bundle on M (i.e. $T(M)$) and on normal bundle of M (i.e. $T(M)^\perp$).*

4. Para invariant submanifold of almost para complex manifold

Definition 4.1: Let M be an n -dimensional submanifold of m -dimensional almost para complex manifold $\tilde{M}$ with structure $(\tilde{F}, g)$. The submanifold M is called Para-Invariant submanifold of $\tilde{M}$ if the image of P makes a differentiable distribution D on M and image $P = \text{Ker}Q$ at each point of M .

For para invariant submanifold (PI-submanifold) M we have

$$(4.1) \quad \tilde{F}(D_x) = D_x, \quad \tilde{F}(D_x^\perp) \subset T_x(M)^\perp$$

where $D^\perp$ denotes the complex orthogonal distribution of D .

Definition 4.2: A PI – submanifold M of an almost para complex manifold $\tilde{M}$ is said to be invariant (respectively anti-invariant) if the distribution in D satisfies

$$(4.2) \quad D_x = T_x(M) \text{ (resp. } D_x = 0 \text{ for each } x \text{ of } M \text{)}.$$

Following facts can be easily proved:

Fact 1: From (3.7), a PI-submanifold M of an almost para complex manifold $\tilde{M}$ is invariant if and only if $Q=0$ (or equivalently $t=0$), that is

$$(4.3) \quad \tilde{F}U = PU \text{ (or equivalently } \tilde{F}\xi = q\xi \text{)}$$

for any vector field U tangent to M and any vector field ξ normal to M .

Fact 2: A PI-submanifold M of an almost para complex manifold $\tilde{M}$ is anti-invariant if and only if $P=0$ identically.

References

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