

Some Results on Jakimovski and Leviatan Type Operators

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Abstract: In the present chapter we will modify a sequence of Favard – Szasz¹ type operators, which are introduced by A. Jakimovski and D. Leviatan² and study some approximation results

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1. Introduction

Let $g(u) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic function in the disc $|z| < R$, $R > 1$

and suppose that $g(1) \neq 0$. Define the Appell polynomials $p_k(x)$, $k \geq 0$ by :

$$(1.1) \quad g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k.$$

To each function f defined in $[0, \infty)$ associate the operators:

$$(1.2) \quad (L_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right)$$

we suppose $p_k(x) \geq 0$ for $x \in [0, \infty)$, $k = 0, 1, 2, \dots$

If $a_n \geq 0$, $n \in \mathbb{N}$, this supposition is satisfied, because

$$(1.3) \quad p_k(x) = \sum_{i=0}^k a_i \frac{x_{k-i}}{(k-i)!}.$$

A. Jakimovski² and B. Wood³ have studied the properties of this sequence of operators.

For $f \in L_1[0, \infty)$, we replace $f\left(\frac{k}{n}\right)$ by $\frac{n^{\lambda+k+1}}{\lambda+k+1} \int_0^\infty e^{-nt} t^{\lambda+k} (S_n f)(t) dt$

into L_n' and thus we define the following class of positive linear operators:

$$(1.4) \quad (M_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\lambda+k+1} \int_0^\infty e^{-nt} t^{\lambda+k} (S_n f)(t) dt,$$

$$\text{where } (S_n f)(t) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right).$$

Next, we denote by E the class of function of exponential type, which have the property that $|f(t)| \leq e^{At}$, for each $t \geq 0$ and some finite number A .

2. Basic Results

The following results are essential to study the convergence of the sequence $M_n f$.

$$(2.1) \quad (M_n 1)(x) = 1,$$

$$(2.2) \quad (M_n t)(x) = x + \frac{1}{n} \frac{g'(1)}{g(1)} \frac{(\lambda+1)}{n},$$

$$(2.3) \quad (M_n t^2)(x) = \left\{ x^2 + \left[(2\lambda+5) + \frac{2g'(1)}{g(1)} \right] \frac{x}{n} + \frac{1}{n^2} \left[\frac{g''(1) + (2\lambda+5)g'(1)}{g(1)} + (\lambda+1)(\lambda+3) \right] \right\}.$$

In particular, we have

$$(2.4) \quad (M_n(t-x))(x) = \frac{1}{n} \left[\frac{g'(1)}{g(1)} + (\lambda+1) \right] \text{ and}$$

$$(2.5) \quad (M_n(t-x)^2)(x) = \frac{3x}{n} + \frac{1}{n^2} \left[\frac{g''(1) + (2\lambda+5)g'(1)}{g(1)} + (\lambda+1)(\lambda+3) \right].$$

Proof: We use the following results

$$(2.6) \quad (S_n 1)(x) = 1, (L_n 1)(x) = 1,$$

$$(2.7) \quad (S_n t)(x) = x, (L_n t)(x) = x + \frac{1}{n} \frac{g'(1)}{g(1)},$$

$$(2.8) \quad (S_n t^2)(x) = x^2 + \frac{x}{n},$$

$$(L_n t^2)(x) = \left[x^2 + \frac{x}{n} \left(1 + \frac{2g'(1)}{g(1)} \right) \right] + \frac{1}{n^2} \left(\frac{g''(1) + g'(1)}{g(1)} \right).$$

Using the definition of Gamma-function, we have

$$\int_0^{\infty} e^{-t} t^{-n} dt = \frac{1}{n-1}.$$

Using (2.6), we have

$$\begin{aligned} (M_n 1)(x) &= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} (S_n 1)(t) dt \\ &= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} dt \\ &= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) = (L_n 1)(x) = 1. \end{aligned}$$

Now

$$\begin{aligned} (M_n t)(x) &= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} t dt \\ &= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \left(\frac{\lambda+k+1}{n} \right), \text{ by Gamma function} \\ &= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \cdot \frac{k}{n} + \frac{\lambda+1}{n} \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \\ &= (L_n t)(x) + \frac{\lambda+1}{n} (L_n 1)(x) \end{aligned}$$

$$= x + \frac{1}{n} \frac{g'(1)}{g(1)} + \frac{\lambda+1}{n}.$$

Next $(M_n t^2)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} \left(t^2 + \frac{t}{n}\right) dt.$

using (2.7)

$$\begin{aligned} &= \left[\frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k+2} dt \right. \\ &\quad \left. + \frac{1}{n} \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k+1} dt \right] \\ &= \left[\frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{(\lambda+k+2)(\lambda+k+1)}{n^2} \right. \\ &\quad \left. + \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{(\lambda+k+1)}{n^2} \right] \\ &= \left[\frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{k^2}{n^2} + \frac{(2\lambda+4)}{n} \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{k}{n} \right. \\ &\quad \left. + \frac{(\lambda+1)(\lambda+3)}{n^2} \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \right] \\ &= \left[(L_n t^2)(x) + \frac{(2\lambda+4)}{n} (L_n t)(x) \right. \\ &\quad \left. + \frac{(\lambda+1)(\lambda+3)}{n^2} (L_n 1)(x) \right] \\ &= \left\{ x^2 + \left[\frac{2g'(1)}{g(1)} + (2\lambda+5) \right] \frac{x}{n} \right. \\ &\quad \left. + \frac{1}{n^2} \left[\frac{g''(1) + (2\lambda+5)g'(1)}{g(1)} + (\lambda+1)(\lambda+3) \right] \right\}, \end{aligned}$$

using (2.6) to (2.8).

Now

$$\begin{aligned} M_n(t-x)(x) &= M_n(t)(x) - x(M_n 1)(x) \\ &= \frac{1}{n} \frac{g'(1)}{g(1)} + \frac{\lambda+1}{n} \end{aligned}$$

and

$$\begin{aligned} M_n(t-x)^2(x) &= (M_n t^2)(x) - 2x(M_n t)(x) + x^2(M_n 1)(x) \\ &= \left\{ x^2 + \frac{x}{n} \left(2\lambda+5 + \frac{2g'(1)}{g(1)} \right) + \frac{1}{n^2} \left[\frac{g''(1) + (2\lambda+5)g'(1)}{g(1)} \right. \right. \\ &\quad \left. \left. + (\lambda+1)(\lambda+3) \right] - 2x \left(x + \frac{1}{n} \frac{g'(1)}{g(1)} + \frac{\lambda+1}{n} \right) + x^2 \right\} \\ &= \frac{3x}{n} + \frac{1}{n^2} \left(\frac{g''(1) + (2\lambda+5)g'(1)}{g(1)} + (\lambda+1)(\lambda+3) \right). \end{aligned}$$

3. Main Results

Theorem 1: If $f \in L_1[0, \infty) \cap E$, then $\lim_{n \rightarrow \infty} (L_n f)(x) = f(x)$, the convergence being uniform in each compact $[0, l]$.

Proof: From (2.1), (2.2) and (2.3) of the basic results one results that

$$\lim_{n \rightarrow \infty} (L_n f)(x) = f(x), \text{ for } f(x) = 1, x \text{ and } x^2$$

In accordance with the Bohman - Korovkin theorem, we obtain the desired result.

Theorem 2: If $f \in C[0, b)$, then for any $x \in [0, b)$, we have

$$|(M_n f)(x) - f(x)| \leq \frac{u}{b} \|f\| + \frac{3}{4} w_2(f; u) \left(3 + \frac{b}{u} \right)$$

where $u = \sqrt{M_n(t-x)^2(x)}$.

Proof: Let f_k be the Steklov function attached to the function f . Let us use the following result of V.V. Juk⁵. If $f \in C[0, b)$ and $u \in \left(0, \frac{b-a}{2}\right)$,

then $\|f - f_k\| \leq \frac{3}{4} w_2(f; u)$ and $\|f''\| \leq \frac{3}{2} \frac{1}{u^2} w_2(f; u)$.

Since $(M_n 1)(x) = 1$, we can write

$$\begin{aligned}
 | (M_n f)(x) - f(x) | &= | (M_n f)(x) - (M_n f_k)(x) \\
 &\quad + (M_n f_k)(x) - f_k(x) + f_k(x) - f(x) | \\
 &\leq \left[| M_n(f - f_k)(x) | + | (M_n f_k)(x) - f_k(x) | + | f_k(x) - f(x) | \right] \\
 (3.1) \quad &\leq 2 \|f - f_k\| + | (M_n f_k)(x) - f_k(x) |
 \end{aligned}$$

Because $\|M_n f\| \leq M_n \sup |f(x)| = M_n \|f\| = \|f\| M_n 1 = \|f\|$.

By using the relation

$$\begin{aligned}
 | (M_n g)(x) - g(x) | &\leq \|g'\| \sqrt{M_n(t-x)^2(x)} + \frac{1}{2} \|g''\| x \left(M_n(t-x)^2 \right)(x), \\
 &\text{For } g \in C^2[0, b],
 \end{aligned}$$

We have

$$\begin{aligned}
 | (M_n f_k)(x) - f_k(x) | &\leq \left[\|f_k'\| \sqrt{M_n(t-x)^2(x)} + \frac{1}{2} \|f_k''\| x \right. \\
 &\quad \left. \left(M_n(t-x)^2 \right)(x) \right]. \quad \text{for } f_k \in C^2[0, b]
 \end{aligned}$$

In accordance with a result from³⁻⁴, we obtain

$$\|f_k'\| \leq \frac{1}{b} \|f_k\| + \frac{b}{2} \|f_k''\| \leq \frac{1}{b} \|f_k\| + \frac{3b}{4} \frac{1}{u^2} w_2(f; u).$$

Thus we have

$$\begin{aligned}
 | (M_n f_k)(x) - f_k(x) | &\leq \left\{ \left[\frac{1}{b} \|f_k\| + \frac{3b}{4} \frac{1}{u^2} w_2(f; u) \right] \right. \\
 &\quad \left. \sqrt{M_n(t-x)^2(x)} + \left[\frac{1}{2} \frac{3}{2} \cdot \frac{1}{u^2} w_2(f; u) M_n(t-x)^2(x) \right] \right\}
 \end{aligned}$$

Using $u = \sqrt{M_n(t-x)^2(x)}$, we have

$$| (M_n f_k)(x) - f_k(x) | \leq \left[\frac{u}{b} \|f_k\| + \frac{3b}{4} \frac{1}{u} w_2(f; u) + \frac{3}{4} w_2(f; u) \right].$$

From (3.1), we can write

$$\begin{aligned}
 | (M_n f)(x) - f(x) | &\leq \left[\frac{3}{2} w_2(f; u) + \frac{u}{b} \|f\| \right. \\
 &\quad \left. + \frac{3b}{4} \frac{1}{u} w_2(f; u) + \frac{3}{4} w_2(f; u) \right] \\
 &= \frac{u}{b} \|f\| + \frac{9}{4} w_2(f; u) + \frac{3b}{4} \frac{1}{u} w_2(f; u) \\
 &= \frac{u}{b} \|f\| + \frac{3}{4} w_2(f; u) \left(3 + \frac{b}{u} \right).
 \end{aligned}$$

Hence the proof.

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