

On Dynamics of Functions

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Abstract: The aim of the present paper is to study the dynamics of a class of functions induced by integers. The results are computationally explicit and can be easily implemented on digital computers.

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1. 1. Introduction

The study of behavior of points under iteration of a function f is called the dynamics of the function f . A discrete dynamical system consists of a function and its iterates. The main objective of dynamics is to study the behavior of the sequence $x, f(x), f^2(x), f^3(x), \dots$ for various initial values of x . If f is a function, the set of points $x, f(x), f^2(x), f^3(x), \dots$, is also called the orbit of the point x . Orbits can be quite complicated sets even for simple mappings. However, orbits associated with following types of the points are especially simple and play a central role in the study of entire system.

Definition 1.1^{1,2}: The point x is called a fixed point of a function f if $f(x) = x$. A point x is called a periodic point of period n if $f^n(x) = x$. The least positive integer n for which $f^n(x) = x$ is called the prime period of x . The set of all iterates of a periodic point form a periodic orbit.

Definition 1.2: A point x is defined to be an eventually fixed point of a function f , if there exists an integer N such that $f^{n+1}(x) = f^n(x)$ whenever $n \geq N$. A point x is eventually periodic with period k if there exists $N > 0$, such that $f^{n+k}(x) = f^n(x)$, whenever $n \geq N$.

In this section, we study the dynamics of a function defined on the set of non-negative integers and prove a theorem which states that if a number is

an integral multiple of a given number then it is an eventually fixed point of a mapping induced by the given number.

Let X denote the set of non-negative integers equipped with the Euclidean metric. In the sequel, we shall often write the numbers in X , in the form 10^5p+q or in the form $10^5a+10^4b+10^3c+10^2d+10e+f$,

where p, q, r, a, b, c, d, e and f are non-negative integers such that $q, r, b, c, d, e, f, g < 10$. We observe that a given integer 10^5p+q in X induces a self-mapping f on X under which image of a number

$$10^5a_5+10^4a_4+10^3a_3+10^2a_2+10a_1+a_0$$

in X is given by:

$$(1.1) \quad f(10^5a_5+10^4a_4+\dots+10a_1+a_0) = |(10^4a_4+10^3a_3+\dots+a_0)p - a_5q|.$$

where $p, q, a_0, a_1, \dots, a_4$, and a_5 are non-negative integers with $q, a_4, a_3, a_2, a_1, a_0 < 10$. From this it can be easily shown that

$$(1.2) \quad f(10^5a_5+10^4a_4+\dots+10a_1+a_0) = |(10^4a_4+10^3a_3+\dots+a_0)p - a_5q| \\ < \max \{10^5p+q, 10^5a_5+10^4a_4+\dots+a_0\}$$

The application of fixed point theorems to the dynamical systems was initiated by Pant³. The question of divisibility of one integer by another is an important question.

In this paper, we generalize the results of Pant³ to obtain new results that further generalize to yield conditions of divisibility of the form 10^5p+q .

2. Results

Theorem 2.1: *Let X be the set of non-negative integers and $p, q \geq 1$ be given integers. Let f be the self-mapping of X which is induced by 10^5p+q and defined by (1.1). Then every integral multiple of 10^5p+q is an eventually fixed point of f while every other integer in X is an eventually periodic point of f . Moreover, f possesses fixed points in X .*

Proof: Suppose that $10^5a_5+10^4a_4+10^3a_3+10^2a_2+10a_1+a_0$ is a non-zero integral multiple of 10^5p+q and $a_4, a_3, a_2, a_1, a_0 < 10$. Then there exists an integer $k \geq 1$ such that

$$10^5a_5+10^4a_4+10^3a_3+10^2a_2+10a_1+a_0 = k(10^5p+q).$$

$$\begin{aligned} \text{Then } f(10^5a_5+10^4a_4+\dots+a_0) &= |10^4a_4p+10^3a_3p+\dots+a_0p-a_5q| \\ &= |pk - a_5|(10^5p+q) \end{aligned}$$

That is $f(10^5a_5+10^4a_4+\dots+a_0)$ is an integral multiple of 10^5p+q .
since a_5, p, k are integers. Further, since $k \geq 1$,

$$\max\{10^5p+q, 10^5a_5+10^4a_4+\dots+a_0\} = 10^5a_5+10^4a_4+\dots+a_0$$

By virtue of (1.2) we get

$$f(10^5a_5+10^4a_4+\dots+a_0) < 10^5a_5+10^4a_4+\dots+a_0$$

Therefore the f -image of an integral multiple of 10^5p+q is a smaller integral multiple of 10^5p+q . But there are exactly k integral multiples of 10^5p+q which are smaller than $10^5a_5+10^4a_4+10^3a_3+10^2a_2+10a_1+a_0 = k(10^5p+q)$.

Moreover

$$f(10^5p+q) = |(pq - qp)| = 0$$

and

$$f(0) = f(10^5 \cdot 0 + 10^4 \cdot 0 + \dots + 10 \cdot 0 + 0) = 0.$$

Thus if

$$10^5a_5+10^4a_4+\dots+10a_1+a_0 = k(10^5p+q)$$

then

$$f^n(10^5a_5+10^4a_4+\dots+a_0) = 0 \quad \text{for some } n \leq k,$$

$$f^m(10^5a_5+10^4a_4+\dots+a_0) = 0,$$

for each $m \geq n$, and $f(0) = 0$.

In other words, 0 is a fixed point of f and every other integral multiple of (10^5p+q) is an eventually fixed point of f . If an integer $10^5a_5+10^4a_4+\dots+10a_1+a_0$ is a non-multiple of (10^5p+q) then, by virtue of (1.2), there exists $n > 0$, such that

$$0 \leq f^n(10^5a_5+10^4a_4+\dots+a_0) < (10^5p+q).$$

Moreover, for each $i > 0$ we get,

$$0 \leq f^{i+n}(10^5a_5+10^4a_4+\dots+a_0) < (10^5p+q).$$

If $f^n(10^5a_5+10^4a_4+\dots+a_0) = 0$ then $10^5a_5+10^4a_4+\dots+a_0$ is an eventually fixed point of f , otherwise the points $\{f^{i+n}(10^5a_5+10^4a_4+\dots+a_0) : i \geq 0\}$ form a periodic cycle and $10^5a_5+10^4a_4+\dots+a_0$ is an eventually periodic point of f .

This completes the proof.

Now we give a number theoretic application of the above fixed point theorem to obtain a test of divisibility of an integer by another integer. In proving our theorem we shall use the well known elementary result that if

the integers a and b are relatively prime and a divides the product of the integer b and c then a divides c . In sequel, we shall use the notation $x \mid y$ to show that x divides y .

Theorem 2.2. *If $a_0, a_1, \dots, a_5, p, q$ are non-negative integers and p and q are relatively prime then*

$$(i) (10^5 p + q) \mid 10^5 a_5 + 10^4 a_4 + \dots + a_0 \Leftrightarrow (10^5 p + q) \mid (10^4 a_4 p + 10^3 a_3 p + \dots + a_0 p - a_5 q),$$

$$(ii) (10^5 p - q) \mid 10^5 a_5 + 10^4 a_4 + \dots + a_0 \Leftrightarrow (10^5 p - q) \mid (10^4 a_4 p + \dots + a_0 p + a_5 q).$$

Proof: Let $(10^5 p + q) \mid 10^4 a_4 p + 10^3 a_3 p + \dots + a_0 p - a_5 q$. Then there exists some integer k such that

$$10^4 a_4 p + 10^3 a_3 p + \dots + a_0 p - a_5 q = k (10^5 p + q).$$

Then $q (a_5 + k) = p (10^4 a_4 + 10^3 a_3 + \dots + a_0 - 10^5 k)$. This implies that

$$p \mid (a_5 + k),$$

as p and q are relatively prime.

We further get $10^5 a_5 + 10^4 a_4 + \dots + a_0 = \frac{(a_5 + k)}{p} (10^5 p + q)$.

This means that $(10^5 p + q)$ divides $10^5 a_5 + 10^4 a_4 + \dots + a_0$ since $p \mid (a_5 + k)$ and $\left(\frac{a_5 + k}{p} \right)$ is an integer.

Conversely, let $(10^5 p + q) \mid 10^5 a_5 + 10^4 a_4 + \dots + a_0$.

Then $10^5 a_5 + 10^4 a_4 + \dots + a_0 = k(10^5 p + q)$ for some integer k .

This yields $(10^4 a_4 p + 10^3 a_3 p + \dots + a_0 p - a_5 q) = (pk - a_5) (10^5 p + q)$.

That is $(10^5 p + q) \mid (10^4 a_4 p + 10^3 a_3 p + \dots + a_0 p - a_5 q)$.

This proves the first part of the theorem. The second part follows on similar lines.

Theorem 2.3. *If $a_0, a_1, a_2, a_3, a_4, a_5, a_6, p, q$ are non-negative integers and p and q are relatively prime then*

$$(i) (10^5 p + q) \mid 10^6 a_6 + 10^5 a_5 + \dots + 10 a_1 + a_0 \\ \Leftrightarrow (10^5 p + q) \mid 10^4 a_4 p + 10^3 a_3 p + \dots + 10(a_1 p - a_6 q) + (a_0 p - a_5 q),$$

$$(ii) (10^5 p - q) \mid 10^6 a_6 + 10^5 a_5 + \dots + 10 a_1 + a_0 \\ \Leftrightarrow (10^5 p - q) \mid 10^4 a_4 p + 10^3 a_3 p + \dots + 10(a_1 p + a_6 q) + (a_0 p + a_5 q).$$

Proof: Since p and q are relatively prime, using Theorem 2.2, we get

$$\begin{aligned}
 (10^5 p + q) & \mid 10^6 a_6 + 10^5 a_5 + \dots + a_0 \\
 \Leftrightarrow (10^5 p + q) & \mid 10^5 (10 a_6 + a_5) + 10^4 a_4 + \dots + 10 a_1 + a_0 \\
 \Leftrightarrow (10^5 p + q) & \mid 10^4 a_4 p + 10^3 a_3 p \dots + a_0 p - q (10 a_6 + a_5) \\
 \Leftrightarrow (10^5 p + q) & \mid 10^4 a_4 p + 10^3 a_3 p \dots + 10(a_1 p - a_6 q) + (a_0 p - a_5 q).
 \end{aligned}$$

For the prove of (ii) part replacing q by $(-q)$ in the above equation we get

$$\begin{aligned}
 (10^5 p - q) & \mid 10^6 a_6 + 10^5 a_5 + \dots + 10 a_1 + a_0 \\
 \Leftrightarrow (10^5 p - q) & \mid 10^4 a_4 p + 10^3 a_3 p + \dots + 10(a_1 p + a_6 q) + (a_0 p + a_5 q).
 \end{aligned}$$

This completes the proof.

The proofs of the next two theorems follow along the lines of Theorem 2.2, 2.3; therefore we state the result but omit the proof.

Theorem 2.4. *If $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, p, q$ are non-negative integers and p, q are relatively prime then*

$$\begin{aligned}
 (i) \quad (10^5 p + q) & \mid 10^7 a_7 + 10^6 a_6 + 10^5 a_5 + \dots + 10 a_1 + a_0 \\
 & \Leftrightarrow (10^5 p + q) \mid 10^4 a_4 p + 10^3 a_3 p + 10^2 (a_2 p - a_7 q) + 10(a_1 p - a_6 q) + (a_0 p - a_5 q), \\
 (ii) \quad (10^5 p - q) & \mid 10^7 a_7 + 10^6 a_6 + 10^5 a_5 + \dots + 10 a_1 + a_0 \\
 & \Leftrightarrow (10^5 p - q) \mid 10^4 a_4 p + 10^3 a_3 p + 10^2 (a_2 p + a_7 q) + 10(a_1 p + a_6 q) + (a_0 p + a_5 q).
 \end{aligned}$$

Similarly, using mathematical induction, the above theorems can be generalized to obtain the following theorems.

Theorem 2.5. *If $p > 0, q > 0, a_0, a_1, \dots, a_{5n+4}$, are non-negative integers and p, q are relatively prime then*

$$\begin{aligned}
 (i) \quad (10^5 p + q) & \mid 10^{(5n+4)} a_{(5n+4)} + 10^{(5n+3)} a_{(5n+3)} + \dots + 10 a_1 + a_0 \\
 \Leftrightarrow (10^5 p + q) & \mid 10^4 \left[\sum_{i=0}^n (-1)^i a_{(5i+4)} p^{(n-i)} q^i \right] + 10^3 \left[\sum_{i=0}^n (-1)^i a_{(5i+3)} p^{(n-i)} q^i \right] + \dots + \\
 & \left[\sum_{i=0}^n (-1)^i a_{5i} p^{(n-i)} q^i \right], \\
 (ii) \quad (10^5 p - q) & \mid 10^{(5n+4)} a_{(5n+4)} + 10^{(5n+3)} a_{(5n+3)} + \dots + 10 a_1 + a_0
 \end{aligned}$$

$$\Leftrightarrow (10^5 p - q) \mid 10^4 \left[\sum_{i=0}^n a_{(5i+4)} p^{(n-i)} q^i \right] + 10^3 \left[\sum_{i=0}^n a_{(5i+3)} p^{(n-i)} q^i \right] + \dots + \left[\sum_{i=0}^n a_{5i} p^{(n-i)} q^i \right].$$

Theorems 2.5 can be considered as the generalized divisibility theorem.

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Reference

1. R.L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Benjamin / Cummings Publishing Co., California, 1986.
2. R.A. Holmgren, *A First Course in Discrete Dynamical Systems*, Springer-Verlag, New York, 1994
3. R.P. Pant, Meir-Keeler type fixed point theorems and dynamics of functions, *Demonstratio Math.*, **35**(2003)199-206.
4. R.P. Pant, Fixed point theorems and dynamics of functions, *J. Indian Math. Soc.*, **70**(2003)135-143.