

On Fixed and Coincidence Points for Fibre - Preserving Maps

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(Received January 23, 2009)

Abstract: We give conditions under which fibre-preserving maps can be deformed into fixed point free maps and coincidence point free maps.

Keywords: Fixed and coincidence points, obstruction class, fundamental group

2000 Mathematics Subject Classification: 47H10, 54H25, 55M20, 55S35.

1. Introduction

In 1954, Fuller¹ employed classical obstruction theory to study the problem whether two maps f and g from a complex to a manifold can be deformed to be coincidence free. Later in 1981, Fadell and Husseini² investigated this further as to when fibre preserving maps deform into fixed point free maps. Recently, Dold and Goncalves³ have given interesting results on self-coincidences of fibre maps.

Section 3 of this paper deals with fibre-preserving maps where the fibre is S^n , n odd, and S^1 . When the dimension of the fibre is greater than or equal to three, the obstruction $o(g, f)$ to such deformations is given in terms of cohomology classes where f and g are fibre-preserving maps, when the fibre is S^1 , the criterion can be given in terms of the fundamental group.

We show in Section 4 that the obstruction class $o(g, f)$ is non zero if the obstruction class $o(\bar{g}, \bar{f})$ for the maps $\bar{f}$ and $\bar{g}$ in the corresponding covering space is non zero. We get known results as corollaries.

2. Preliminaries

The following material is taken from². We consider the fibration $Y \xrightarrow{i} M \xrightarrow{p} B$, where M , B and Y are the total space, base space

and fibre respectively, i is the inclusion and p the projection maps. We assume that M , B and Y are smooth, connected and compact manifolds without boundary. Consider the fibre square

$$\begin{array}{ccc} M \times_B M & \xrightarrow{j_2} & M \\ j_1 \downarrow & & \downarrow p \\ M & \xrightarrow{p} & B \end{array}$$

where $j_1(x, y) = x$, $j_2(x, y) = y$ and $M \times_B M = \{(x, y) \in M \times M: p(x) = p(y)\}$.

Then E. Fadell and S.Husseini² states that

Proposition 2.1: $(M \times_B M, M \times_B M - \Delta M, j_1, M)$ is a locally trivial fibred pair with fibre $(Y, Y - y_0)$, $y_0 \in Y$, ΔM is the diagonal of $M \times M$.

The inclusion map $M \times_B M - \Delta M \rightarrow M \times_B M$ can be replaced by a fibre map $q: E(M) \rightarrow M \times_B M$. Let F denote the fibre of q . We have from F.B.Fuller¹.

Proposition 2.2: The inclusion $(Y, Y - y_0) \rightarrow (M \times_B M, M \times_B M - \Delta M)$ induces an isomorphism (for all n)

$$\pi_n(Y, Y - y_0) \approx \pi_n(M \times_B M, M \times_B M - \Delta M) \approx \pi_{n-1}(F).$$

Let $\dim Y = k \geq 3$, then $(Y, Y - y_0)$ is $(k-1)$ -connected. Hence F is $(k-2)$ -connected. From E. Fadell and S.Husseini², we have

Proposition 2.3: The fibration $F \rightarrow E(M) \xrightarrow{q} M \times_B M$ has $(k-2)$ -connected fibre and when $k \geq 3$, the fibre F is simply connected so that $\pi_{k-1}(F)$ generates a local system $\mathcal{B}_B$ on $M \times_B M$ with local group isomorphic to $Z[\pi']$ where $\pi' = \pi_1(y)$.

Suppose, we are given two fibre preserving maps f and g (over B), that is $fp = p$ and $gp = p$

$$\begin{array}{ccc} M & \xrightarrow{(g,f)} & M \\ & \searrow p \quad \swarrow p & \\ & B & \end{array}$$

Then by E. Fadell and S.Husseini², we have:

Proposition 2.4: *Maps f and g can be deformed over B to the coincidence free maps if and only if there exists a lifting $h: M \rightarrow E(M)$ which makes the diagram below homotopy commutative*

$$\begin{array}{ccc}
 & & E(M) \\
 & \nearrow h & \downarrow q \\
 M & \xrightarrow{(g, f)} & M \times_B M
 \end{array}$$

where $(g, f)(x) = (g(x), f(x))$. Let us take the pullback of q by (g, f) .

$$\begin{array}{ccc}
 E(f) & \xrightarrow{\quad} & E(M) \\
 \downarrow q' & & \downarrow q \\
 M & \xrightarrow{(g, f)} & M \times_B M
 \end{array}$$

Then f and g are fibre homotopy equivalent to coincidence free maps if and only if q' admits a cross section.

We denote by $o(g, f) \in H^k(M, \pi_{k-1}(F))$ the obstruction for finding a cross section for the map q' . Thus if f, g can be deformed to coincidence free maps then $o(g, f) = 0$.

3. Fixed point and coincidence point free maps

In this section we give criterions for fibre - preserving maps to be fixed point free and coincidence point free when the fibre is S^n , n being a positive odd integer and S^1 respectively. Let B be as in section 2.

Proposition 3.1: *Let $f: B \times S^n \rightarrow B \times S^n$ be a fibre - preserving map (n odd), where p_1 is the projection onto the first factor. Then f can be deformed over B to a fixed point free map if $f = (l, g)$, where $g: B \times S^n \rightarrow S^n$ is homotopic to the projection onto the second factor $p_2: B \times S^n \rightarrow S^n$.*

$$\begin{array}{ccc}
 & f & \\
 B \times S^n & \xrightarrow{\quad} & B \times S^n \\
 p_1 \searrow & & \swarrow p_1 \\
 & B &
 \end{array}$$

Proof: For $b \in B$ and $z \in S^n$ we write $f(b, z) = (1, g(b, z))$ since the map f is fibre - preserving and it is the identity map on the fibre S^n (n odd), f is homotopic (deformable) to the antipodal map. Hence f can be deformed over B to a fixed point free map.

We consider the fibration $S^1 \rightarrow M \rightarrow B$. Let $x_0 \in M$ be a base point of M and let f, g and h be the maps as in section 2. We assume that $f(x_0) \neq g(x_0)$ and denote by $(g(x_0), f(x_0))$ the base point of $E(M) = M \times_B M - \Delta M$ and $M \times_B M$.

Proposition 3.2: *The lift h exists if and only if*

$$(g, f)_\# \pi_1(M, x_0) = q_\# \pi_1(E(M); (g(x_0), f(x_0))).$$

Proof: Let $\bar{M}$ be the covering space of $M \times_B M$ which belongs to the subgroup $(g, f)_\# \pi_1(M, x_0)$ of $\pi_1(M \times_B M)$, that is

$$(3.1) \quad p_\# \pi_1(\bar{M}) = (g, f)_\# \pi_1(M, x_0)$$

Consider the following commutative diagram

$$\begin{array}{ccccc}
 & & & \bar{M} & \\
 & & \nearrow \bar{f} & & \downarrow p \\
 & E(M) & \nearrow j & & \\
 M & \xrightarrow{h} & (g, f) & \xrightarrow{q} & M \times_B M
 \end{array}$$

Since p is a covering space, there exists a lifting $\bar{f}$ of (g, f) . Suppose that

$$(3.2) \quad (g, f)_\# \pi_1(M, x_0) = q_\# \pi_1(E(M); (g(x_0), f(x_0))).$$

This implies that there is a lifting $j: E(M) \rightarrow \bar{M}$ of the map q . By $\pi_1(M \times_B M, E(M)) \approx \pi_1(S^1, S^1 - y_0) = 0$ for $i > 1$, $y_0 \in S^1$. It follows from the homotopy exact sequence of pairs that for $i > 1$, $q_\#: \pi_i(E(M)) \rightarrow \pi_i(M \times_B M)$ is an isomorphism. Also, it is well known that for $i > 1$, $p_\#: \pi_i(\bar{M}) \rightarrow \pi_i(M \times_B M)$

$(M \times_B M)$ is an isomorphism. Now $p_{\#}: \pi_1(\bar{M}) \rightarrow \pi_1(M \times_B M)$ being a monomorphism, the construction of the covering space implies that

$$j_{\#}: \pi_1(E(M)) \rightarrow \pi_1(\bar{M})$$

is an isomorphism. So j induces isomorphism in homotopy groups of all dimensions. $E(M)$ and $\bar{M}$ being CW - complexes there exists a homotopy inverse of j , say $\ell: \bar{M} \rightarrow E(M)$. We take $h = \ell \circ \bar{f}$.

Next suppose that the lift h exists. Since $q \circ h \simeq (g, f)$ we get

$$(3.3) \quad (g, f)_{\#} \pi_1(M, x_0) \subset q_{\#} \pi_1(E(M); (g(x_0), f(x_0))).$$

Let $p_1: E(M) \rightarrow M$ be the first projection. Then we get the fibration

$$S^1 - \{g(x_0)\} \rightarrow E(M) \xrightarrow{p_1} M$$

Hence $(p_1)_{\#}: \pi_1(E(M)) \rightarrow \pi_1(M)$ is an isomorphism. Since $p_1 \circ h \simeq \text{identity}$ we get $h_{\#}: \pi_1(M) \rightarrow \pi_1(E(M))$ is an isomorphism implying the equality,

$$(3.4) \quad (g, f)_{\#} \pi_1(M, x_0) = q_{\#}(\pi_1(E(M)), (g(x_0), f(x_0))).$$

Theorem 3.3: *The maps f, g can be deformed over B to become coincidence free if and only if*

$$(g, f)_{\#} \pi_1(M, x_0) = q_{\#} \pi_1(E(M); (g(x_0), f(x_0))).$$

Proof: This follows from Propositions (2.4) and (3.2).

Corollary 3.4: *Taking g to be the identity map on M , we get Proposition (1.3) of D.L.Goncalves⁴.*

Corollary 3.5: *Let f, g be two fibre - preserving maps over CP^n , the complex projective space for the Hopf fibration $S^1 \rightarrow S^{2n+1} \rightarrow CP^n$, that is*

$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{(g, f)} & S^{2n+1} \\ & \searrow p & \swarrow p \\ & CP^n & \end{array}$$

then f and g can be deformed to become coincidence free maps.

Proof: Applying theorem 3.3 for $M=S^{2n+1}$ we get that f and g can be deformed into coincidence free maps if and only if

$$(3.5) \quad (g, f)_{\#} \pi_1 (S^{2n+1}, x_0) = q_{\#} \pi_1 (E(S^{2n+1}), (g(x_0), f(x_0)))$$

Since S^{2n+1} , $n \geq 1$ is simply connected and $(p_1)_{\#}: \pi_1 E(S^{2n+1}) \rightarrow \pi_1 (S^{2n+1})$ is an isomorphism (see proof of Prop. 3.2), both $\pi_1 (S^{2n+1})$ and $\pi_1 E(S^{2n+1})$ are trivial and the above equality holds.

Corollary 3.6: *Let $S^l \xrightarrow{i} K \xrightarrow{p} S^l$ be the S^l -fibration where K is the Klein bottle. Then if both f and g are the identity maps on K then by Theorem 3.3, f and g cannot be deformed to become coincidence free maps⁴.*

4. The obstruction class $o(g, f)$

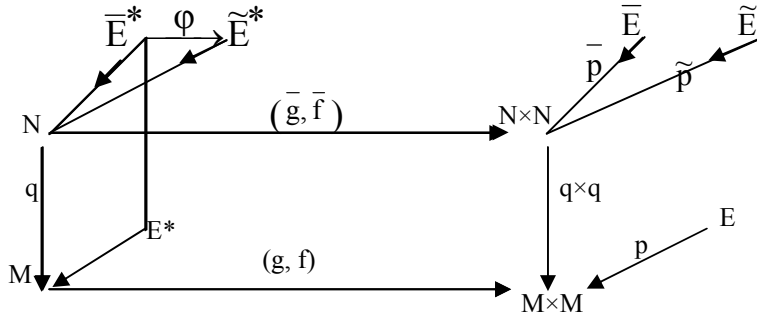
We assume that all manifolds are compact, triangulable and of dimension ≥ 3 . Let $q: N \rightarrow M$ be a covering space of M . We show that $o(g, f) \neq 0$ if $o(\bar{g}, \bar{f}) \neq 0$ where $\bar{f}$ and $\bar{g}$ are suitable maps: $N \rightarrow N$.

Proposition 4.1: *Consider the commutative diagram below:*

$$\begin{array}{ccc} N & \xrightarrow{(\bar{g}, \bar{f})} & N \\ q \downarrow & & \downarrow q \\ M & \xrightarrow{(g, f)} & M \end{array}$$

If $o(\bar{g}, \bar{f}) \neq 0$ then $o(g, f) \neq 0$.

Proof: Let $A = (q \times q)^{-1}(\Delta M)$ with $(q \times q)(N \times N - A) \subset M \times M - \Delta M$. Also let $p: E \rightarrow M \times M$ denote the fibre map replacing the inclusion $M \times M - \Delta M \subset M \times M$. Further, let $\bar{p}: \bar{E} \rightarrow N \times N$ replace the inclusion map $N \times N - A \subset N \times N$. Let E^* and $\bar{E}^*$ denote the pullbacks of p and $\bar{p}$ by (g, f) and $(\bar{g}, \bar{f})$ respectively. Now $\Delta N \subset A$. We replace the inclusion map $N \times N - \Delta N \subset N \times N$ by $\tilde{p}: \tilde{E} \rightarrow N \times N$. Let $\tilde{E}^*$ denote the pullback of $\tilde{p}$ by $(\bar{g}, \bar{f})$. Thus we get the following diagram:



We obtain a fibre preserving map ϕ . We also get the (local) coefficient homomorphism.

$$(4.1) \quad \phi^* : H^k(N; \pi_{k-1}(\bar{F})) \rightarrow H^k(N; \pi_{k-1}(\tilde{F}))$$

and hence by naturality we get

$$(4.2) \quad \phi^* q^* [o(g, f)] = o[(\bar{g}, \bar{f})]$$

Corollary 4.2: Taking g and $\bar{g}$ to be the identity maps on M and N respectively we obtain proposition 2.7 of Fadell and S.Husseini⁵.

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