

On Quarter Symmetric Metric Connection in Almost Contact Manifold

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Abstract: K. Yano¹ studied Semi-symmetric metric connection in a Riemannian manifold. R. S. Mishra and S. N. Pandey² studied affine connection in almost Grayan manifold. S. Golab³ defined semi-symmetric and quarter symmetric linear connection and derived their properties. A. Sharfuddin and S. I. Husain⁴ studied semi-symmetric connection in almost contact manifold. In this paper, some properties of an almost contact and almost Grayan manifold admitting quarter symmetric metric connection have been discussed. Certain identities have also been derived.

Key Words and Phrases: Almost contact manifold, almost Grayan manifold, F-connection, Nijenhuis tensor.

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1. Preliminaries

Let M be an $(2n-1)$ dimensional C^∞ manifold and let there exist in M a vector valued linear function F , a vector field t and a 1- form A such that

$$(1.1) \quad \overline{\overline{X}} + X = A(X)t,$$

where $\overline{\overline{X}}$ stands for $F(X)$ and X is an arbitrary vector field. Such manifold is called an almost contact manifold and the structure (F, t, A) is called an almost contact structure.

From (1.1) we can see that following hold in M

$$(1.2) \quad (a) \ A(\overline{\overline{X}}) = 0, \quad (b) \ A(t) = 1.$$

In addition, if there exist a metric tensor g satisfying

$$g(\overline{X}, \overline{Y}) = -g(\overline{\overline{X}}, Y) \quad \text{and} \quad g(X, t) = A(X),$$

which is equivalent to

$$g(\overline{X}, \overline{Y}) = g(X, Y) - A(X)A(Y),$$

then M is called an almost Grayan manifold and structure (F, t, A, g) is called almost Grayan structure. Putting

$$(1.3) \quad g(\overline{X}, Y) = {}^*F(X, Y),$$

we have

$$(1.4) \quad (a) \quad {}^*F(\overline{X}, \overline{Y}) = {}^*F(X, Y), \quad (b) \quad {}^*F(X, Y) = -{}^*F(Y, X).$$

R. S. Mishra and S. N. Pandey³ has shown that if D is the Riemannian connection in almost Grayan manifold then

$$(1.5) \quad (a) \quad (D_X {}^*F)(\overline{Y}, \overline{\overline{Z}}) = (D_X {}^*F)(\overline{\overline{Y}}, \overline{Z}),$$

$$(b) \quad (D_X {}^*F)(\overline{Y}, \overline{Z}) + (D_X {}^*F)(\overline{\overline{Y}}, \overline{\overline{Z}}) = 0,$$

and also in an almost Grayan manifold the Nijenhuis tensor satisfies

$$(1.6) \quad N(X, Y) = (D_{\overline{X}} F)(Y) - (D_{\overline{Y}} F)(X) - \overline{(D_X F)(Y)} + \overline{(D_Y F)(X)}$$

The tensor ${}^*N(X, Y, Z)$ defined by ${}^*N(X, Y, Z) = g(N(X, Y), Z)$ satisfies

$$(1.7) \quad \begin{aligned} {}^*N(X, Y, Z) &= (D_{\overline{X}} {}^*F)(Y, Z) - (D_{\overline{Y}} {}^*F)(X, Z) \\ &\quad + (D_X {}^*F)(Y, \overline{Z}) - (D_Y {}^*F)(X, \overline{Z}). \end{aligned}$$

If in an almost Grayan manifold

$${}^*F(X, Y) = (D_X A)(Y) - (D_Y A)(X) = dA(X, Y),$$

then M is called almost Sasakian manifold. In an almost Sasakian manifold, we have

$$(1.8) \quad (D_X {}^*F)(Y, Z) + (D_Y {}^*F)(Z, X) + (D_Z {}^*F)(X, Y) = 0.$$

An almost Sasakian manifold is said to be Sasakian manifold if A is a killing vector, i.e.

$$(1.9) \quad (D_X A)(Y) + (D_Y A)(X) = 0.$$

Thus in a Sasakian manifold

$${}^*F(X, Y) = 2(D_X A)(Y) = -2(D_Y A)(X).$$

A linear connection ∇ in M whose torsion tensor T satisfies

$$(1.10) \quad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = \pi(Y)F(X) - F(Y)\pi(X),$$

where π is 1-form and F is a tensor field of type (1,1) then the connection ∇ is said to be quarter symmetric. Further if this connection satisfies $\nabla_X g(Y, Z) = 0$, then ∇ is called quarter symmetric metric connection.

2. Quarter symmetric connection is almost contact manifold

From the definition of an almost contact manifold we have $F(X) = \bar{X}$ which convert equation (1.10) in the form

$$(2.1) \quad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = A(Y)\bar{X} - A(X)\bar{Y},$$

where A is a 1-form. By using (1.1), (1.2) we deduce the following from (2.1)

$$\begin{aligned} (a) \quad & T(\bar{X}, \bar{Y}) = 0, \\ (b) \quad & T(X, t) = \bar{X}, \\ (c) \quad & T(\bar{X}, t) = \bar{\bar{X}} = A(X)t - X, \\ (2.2) \quad (d) \quad & \bar{\bar{T}}(\bar{X}, t) + T(X, t) = 0, \\ (e) \quad & T(\bar{X}, Y) = A(X)A(Y)t - A(Y)X, \\ (f) \quad & \bar{\bar{T}}(\bar{X}, Y) + T(X, \bar{Y}) = T(X, Y), \\ (g) \quad & \bar{\bar{\bar{T}}}(\bar{X}, Y) + T(X, Y) = 0. \end{aligned}$$

Theorem 2.1: For any pair of vector field X and Y , in an almost contact manifold with quarter symmetric connection there exists a vector field Z such that $T(X, Y) = \bar{Z}$, and further $\bar{\bar{Z}} + Z = 0$.

Proof: If we take $Z = \bar{\bar{T}}(Y, \bar{X})$ then from equation (2.2 g), we have

$$\begin{aligned} \bar{Z} &= \bar{\bar{\bar{T}}}(Y, \bar{X}) = -T(Y, X) \\ \bar{Z} &= T(X, Y) \end{aligned}$$

and further

$$\bar{\bar{Z}} = \bar{\bar{\bar{\bar{T}}}}(Y, \bar{X}) = T(X, Y).$$

Also

$$Z + \bar{Z} = \overline{T(Y, X)} + \overline{T(X, Y)} = 0.$$

3. Quarter Symmetric connection is almost Grayan manifold

Let D be a Riemannian connection and ∇ be a quarter symmetric metric connection in almost Grayan Manifold M . Let

$$(3.1) \quad \nabla_X Y = D_X Y + H(X, Y),$$

then it can be seen that

$$H(X, Y) = \frac{1}{2} \{T(X, Y) + P(X, Y) + P(Y, X)\},$$

where $P(X, Y)$ is given by

$$g(T(Z, X)Y) = g(P(X, Y)Z),$$

then we have

$$(3.2) \quad H(X, Y) = A(Y)\bar{X} - g(\bar{X}, Y)t,$$

and

$$(3.3) \quad H(X, Y) = P(Y, X).$$

Putting

$$(3.4a) \quad g(T(X, Y), Z) = T(X, Y, Z),$$

and

$$(3.4b) \quad g((H(X, Y), Z) = {}^*H(X, Y, Z),$$

then we can write

$$(3.5a) \quad T(X, Y, Z) = - \begin{vmatrix} g(X, t) & g(Y, t) \\ g(\bar{X}, Z) & g(\bar{Y}, Z) \end{vmatrix},$$

and

$$(3.5b) \quad {}^*H(X, Y, Z) = \begin{vmatrix} g(t, Y) & g(t, Z) \\ g(\bar{X}, Y) & g(\bar{X}, Z) \end{vmatrix}.$$

Theorem 3.1: *In an almost Grayan manifold with quarter symmetric metric connection the following identities hold:*

- (i) $\nabla H(X, \bar{Y}, \bar{Z}) = 0 = \nabla T(\bar{X}, \bar{Y}, Z),$
- (ii) $\nabla T(\bar{X}, Y, \bar{Z}) - \nabla T(\bar{Y}, X, \bar{Z}) = \nabla T(X, Y, Z),$
- (iii) $\nabla H(\bar{X}, Y, \bar{Z}) - \nabla H(\bar{Y}, X, \bar{Z}) = \nabla T(X, Y, Z),$
- (iv) $\nabla H(X, Y, Z) = \nabla T(Z, Y, \bar{\bar{X}}),$
- (v) $\nabla T(X, Y, Z) + \nabla T(Y, X, Z) = 0,$
- (vi) $\nabla H(X, Y, Z) + \nabla H(X, Z, Y) = 0,$
- (vii) $\nabla H(\bar{\bar{X}}, Y, Z) + \nabla H(X, Y, Z) = 0,$
- (viii) $\nabla T(X, Y, \bar{\bar{Z}}) + \nabla T(X, Y, Z) = 0,$
- (ix) $\nabla H(X, Y, Z) + \nabla H(Y, Z, X) + \nabla H(Z, X, Y) = 0,$
- (x) $\nabla T(X, Y, Z) + \nabla T(Y, Z, X) + \nabla T(Z, X, Y) = 0,$
- (xi) $\nabla H(\bar{X}, Y, Z) + \nabla H(X, \bar{Y}, Z) + \nabla H(X, Y, \bar{Z}) = 0,$
- (xii) $\nabla T(\bar{X}, Y, Z) + \nabla T(X, \bar{Y}, Z) + \nabla T(X, Y, \bar{Z}) = 0,$
- (xiii) $\nabla H(\bar{\bar{X}}, Y, \bar{\bar{Z}}) - \nabla H(\bar{\bar{X}}, Z, \bar{\bar{Y}}) = \nabla H(X, Y, Z) = \nabla T(\bar{\bar{X}}, Y, \bar{\bar{Z}}) - \nabla T(\bar{\bar{X}}, Z, \bar{\bar{Y}}),$
- (xiv) $\nabla H(\bar{X}, Y, \bar{Z}) - \nabla H(\bar{X}, Z, \bar{Y}) = \nabla H(X, Y, Z),$
- (xv) $A(\nabla H(\bar{X}, \bar{Y}) - A(H(X, Y))) = 0.$

Proof: of all above relations follow by calculation using the properties previously obtained.

Corollary 3.1: *In an almost Grayan manifold with quarter symmetric metric connection the following identities hold:*

- (a) $\nabla H(\bar{X}, t, \bar{Y}) = -g(X, \bar{Y}) = \nabla H(\bar{\bar{X}}, t, \bar{\bar{Y}}),$
- (b) $\nabla T(\bar{X}, t, \bar{Y}) = -g(X, \bar{Y}) = \nabla T(\bar{\bar{X}}, t, \bar{\bar{Y}}),$
- (c) $\nabla H(\bar{\bar{X}}, t, Y) = -F(X, Y) = \nabla H(\bar{\bar{X}}, t, \bar{Y}),$
- (d) $\nabla T(\bar{\bar{X}}, t, Y) = -F(X, Y) = \nabla T(\bar{\bar{X}}, t, \bar{Y}),$

$$(e) \quad \overline{H}(\overline{X}, Y, \overline{Z}) - \overline{H}(\overline{X}, Y, \overline{Z}) = 0,$$

$$(f) \quad \overline{T}(\overline{X}, Y, \overline{Z}) - \overline{T}(\overline{X}, Y, \overline{Z}) = 0.$$

Theorem 3.2: Let ∇ be quarter symmetric metric connection in an almost Grayan manifold M with a Riemannian connection D then

$$(3.6a) \quad (\nabla_X \overline{F})(Y, Z) = (D_X \overline{F})(Y, Z) - \overline{H}(\overline{X}, Y, Z),$$

$$(3.6b) \quad (\nabla_X \overline{F})(\overline{Y}, \overline{Z}) = (D'_X \overline{F})(\overline{Y}, \overline{Z}).$$

Proof : We have

$$XF(Y, Z) = (D_X \overline{F})(Y, Z) + \overline{F}(D_X Y, Z) + \overline{F}(Y, D_X Z)$$

$$XF(Y, Z) = (\nabla_X \overline{F})(Y, Z) + \overline{F}(\nabla_X Y, Z) + \overline{F}(Y, \nabla_X Z)$$

Subtracting we get

$$\begin{aligned} (D_X \overline{F})(Y, Z) - (\nabla_X \overline{F})(Y, Z) &= \overline{F}(-D_X Y + \nabla_X Y, Z) + \overline{F}(Y, \nabla_X Z - D_X Z) \\ &= \overline{F}(H(X, Y), Z) + \overline{F}(Y, H(X, Z)) \\ &= -g(H(X, Y), \overline{Z}) + g(\overline{Y}, H(X, Z)) \\ &= \overline{H}(X, \overline{Z}, Y) + \overline{H}(X, Z, \overline{Y}) \\ &= \overline{H}(\overline{X}, Y, Z) \end{aligned}$$

$$(D_X \overline{F})(Y, Z) - (\nabla_X \overline{F})(Y, Z) = \overline{H}(\overline{X}, Y, Z)$$

and further barring Y, Z in (3.6a)

$$(\nabla_X \overline{F})(\overline{Y}, \overline{Z}) = (D_X \overline{F})(\overline{Y}, \overline{Z}) - \overline{H}(\overline{X}, \overline{Y}, \overline{Z}),$$

$$(\nabla_X \overline{F})(\overline{Y}, \overline{Z}) = (D_X \overline{F})(\overline{Y}, \overline{Z})$$

Corollary 3.2a: In an almost Grayan manifold with a quarter symmetric metric connection ∇ , we have

$$(3.7) \quad (D_{\overline{X}} \overline{F})(Y, Z) = (\nabla_{\overline{X}} \overline{F})(Y, Z) - \overline{H}(X, Y, Z)$$

Proof: Barring X in (3.6a), and using identity (viii) of theorem 3.1, we have

$$\begin{aligned} (\nabla_{\overline{X}} \overline{F})(Y, Z) &= (D_{\overline{X}} \overline{F})(Y, Z) - \overline{H}(\overline{X}, Y, Z) \\ &= (D_{\overline{X}} \overline{F})(Y, Z) + \overline{H}(X, Y, Z), \end{aligned}$$

$$(\nabla_{\bar{X}} F)(Y, Z) - {}^{\circ}H(X, Y, Z) = (D_{\bar{X}} F)(Y, Z).$$

Corollary 3.2b: *In an almost Grayan manifold M with a quarter symmetric metric connection ∇ , we have $(\nabla_X F)(Y, Z) = (D_X F)(Y, Z)$,*

if and only if ${}^{\circ}H(\bar{X}, Y, Z) = 0$, or equivalently $T(\bar{X}, Y, Z) = T(\bar{X}, Z, Y)$.

Theorem 3.3: *In an almost Grayan manifold M with a quarter symmetric metric connection ∇ and a Riemannian connection D*

$$(\nabla_X \bar{Y}) = (D_X \bar{Y}) + F(X, Y)t$$

Proof: From (3.1), we get

$$\nabla_X Y = D_X Y + H(X, Y),$$

and from (3.2), we get

$$A(Y)\bar{X} - g(\bar{X}, Y)t = H(X, Y)$$

then we have

$$(\nabla_X Y) = D_X Y + A(Y)\bar{X} - g(\bar{X}, Y)t$$

Barring Y

$$(\nabla_X \bar{Y}) = D_X \bar{Y} + A(\bar{Y})\bar{X} - g(\bar{X}, \bar{Y})t$$

$$\nabla_X \bar{Y} = D_X \bar{Y} - g(\bar{X}, \bar{Y})t$$

Barring Y again

$$\begin{aligned} \nabla_X \bar{\bar{Y}} &= D_X \bar{\bar{Y}} - g(\bar{X}, \bar{\bar{Y}})t = D_X \bar{\bar{Y}} + g(\bar{X}, Y)t \\ &= D_X \bar{\bar{Y}} + F(X, Y)t \end{aligned}$$

Theorem 3.4: *In an almost Grayan manifold M with a quarter symmetric metric connection, the Nijenhuis tensor satisfies the following relation.*

$$N(X, Y) = (\nabla_{\bar{X}} F)(Y) - (\nabla_{\bar{Y}} F)(X) - \overline{(\nabla_X F)(Y)} + \overline{(\nabla_Y F)(X)} + T(X, Y).$$

Proof: In consequence of (1.2a) and

$$(\nabla_X Y) = D_X Y + A(Y)\bar{X} - g(\bar{X}, Y)t,$$

and also from the fact

$$(D_X F)(Y) = D_X \bar{Y} - \overline{D_X Y},$$

the equation

$$N(X, Y) = (D_{\bar{X}} F)(Y) - (D_{\bar{Y}} F)(X) - \overline{(D_X F)(Y)} + \overline{(D_Y F)(X)}$$

takes the form

$$\begin{aligned} N(X, Y) &= D_{\bar{X}} \bar{Y} - \overline{D_X Y} - D_{\bar{Y}} \bar{X} + \overline{D_Y X} - D_X \bar{Y} + \overline{D_X Y} + \overline{D_Y X} - \overline{D_Y X} \\ &= \nabla_{\bar{X}} \bar{Y} + g(\bar{X}, \bar{Y})t - \overline{\nabla_X Y} + A(Y)\bar{X} - \nabla_{\bar{Y}} \bar{X} - g(\bar{X}, Y)t + \overline{\nabla_Y X} - \overline{A(X)\bar{Y}} \\ &\quad - \overline{\nabla_X Y} + \overline{\nabla_X Y} - A(Y)\bar{X} + \overline{\nabla_Y X} - \overline{\nabla_Y X} - A(X)\bar{Y} \\ &= \nabla_{\bar{X}} \bar{Y} - \overline{\nabla_X Y} - \nabla_{\bar{Y}} \bar{X} + \overline{\nabla_Y X} + A(Y)\bar{X} \\ &\quad - \overline{\nabla_X Y} + \overline{\nabla_X Y} + \overline{\nabla_Y X} - \overline{\nabla_Y X} - A(X)\bar{Y} \\ &= (\nabla_{\bar{X}} F)(Y) - (\nabla_{\bar{Y}} F)(X) - \overline{(\nabla_X F)(Y)} + \overline{(\nabla_Y F)(X)} + T(X, Y). \end{aligned}$$

Corollary 3.4: *In an almost Grayan manifold M with a quarter symmetric metric connection, the Nijenhuis tensor satisfies the following relation*

$$N(\bar{X}, \bar{Y}) = (\nabla_{\bar{X}} F)(\bar{Y}) - (\nabla_{\bar{Y}} F)(\bar{X}) - \overline{(\nabla_X F)(\bar{Y})} + \overline{(\nabla_Y F)(\bar{X})}$$

Proof: Baring X and Y in theorem 3.4 and using $T(\bar{X}, \bar{Y}) = 0$, we get the result.

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