

Mathematical Model for the Study of Malaria Transmission

B. Rai and Santosh Kumar Singh

Department of Mathematics, University of Allahabad, Allahabad

E-mail: brai@mri.ernet.in ; sant_math@rediffmail.com

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Abstract: in this paper, an ordinary differential equation mathematical model is proposed to study the spread of malaria in human populations due to mosquitoes. Here we have considered compartments for the spread of malaria with a susceptible-infectious-recovered-susceptible (S-I-R-S) pattern for humans and a susceptible-infectious (S-I) pattern for mosquitoes. In addition, we have introduced the effect of applying control measures to reduce the concentration of waste material, household and industrial discharges to control the spread of the disease. It is found that if the control measures are applied then the concentration of waste material, population of infected humans and the population of infected mosquitoes reduce. In view of this a control epidemiological model has been proposed in this paper. We have used stability theory¹⁻⁵ to obtain the criterion to control the spread of malaria. The results are illustrated with the help of a numerical example and its graphical representation.

Key words: Malaria, Epidemic model, Disease free equilibria, Endemic equilibria, Immigration.

2000 AMS Subject Classification Number: 92D40.

1. Introduction

Malaria is a *protozoan* disease caused by infection with parasites of the genus *Plasmodium* and transmitted to man by certain species of female *Anopheles* mosquitoes.

At present, about 100 countries in the world are considered malarial, almost half of which are in sub-Saharan Africa. Although this number is considerably less than it was in mid 1950's., more than 2.6 billion of the world populations are still at risk.

In India, with the implementation of Modified Plan of Operation (MPO) in 1977, the upsurge of malaria cases dropped down from 6.74 million cases

in 1976 to 2.1 million in 1984. Since then, the epidemiological situation did not show any great improvement. It seemed to have reached a plateau which was causing concern. Since 1997, there is a consistently declining trend in the annual malaria incidence in the country. During 2003 about 1.65 million cases were reported with 943 deaths. There were 0.7 million cases of *P. falciparum* malaria⁶. Malaria has been a serious problem in North-eastern states, mainly due to topography and climatic conditions being congenial for perennial malaria transmission. Among North-eastern states, Assam reports maximum malaria cases.

Mathematical modeling of malaria began in 1911 with Ross's model⁷, and major extensions are described in Macdonald's 1957 book⁶. The first model was two-dimensional with one variable representing human and the other representing mosquitoes. Further work on mathematical modeling of malaria has been conducted by Aron⁸ and Bailey⁹, Anderson and May¹⁰. Some recent papers have also included environmental effects^{11,12}. Recently, Nakul Chitnis, J.M. Cushing and J.M. Hymen proposed an ordinary differential equation compartmental model for the spread of malaria with SEIRS pattern for humans and a SEI pattern for mosquitoes. In this paper we extend this model.

The new model divides the human population into three classes: susceptible, S_h ; infectious, I_h ; and recovered, R_h ; and the mosquitoes population into two classes: susceptible, S_m ; and infectious, I_m . In addition we have considered the effect of applying control measures to reduce the concentration of waste material, house hold and industrial discharges (E).

Some of the control measures which can be applied to control the spread of disease are listed as follows:

[a] Residual Spraying: The discovery of DDT in 1940s and followed by other insecticides revolutionized malaria control. The spraying of indoor surfaces of houses with residual insecticides (e.g., DDT, Malathion, and Fenitrothion) is still the most effective measure to kill the adult mosquitoes.

[b] Space Application: This is a major anti-epidemic measure against mosquito-born diseases. It involves the application of pesticides in the form of fog or mist using special equipments. The ultra-low-volume method of pesticides dispersion by air or by ground equipment has proved to be effective and economical.

[c] Individual Protection: Man-vector contact can be reduced by other preventive measures such as the use of repellants, protective clothing, bed nets (preferably impregnated with safe, long-acting repellent insecticides), mosquito coils, screening of houses etc.

[d] Source Reduction: Techniques to reduce mosquito breeding sites which includes drainage or filling, deepening of flushing, management of water level, changing the salt content of water and intermittent irrigation are among the classical methods of malaria control to which attention is being paid again¹³.

2. Model Formulation

We present an ordinary differential equation mathematical model for the study of malaria in human population due to mosquitoes, considering Immigration of humans as susceptible at a constant rate Π . Susceptible humans can be infected when they are bitten by infectious mosquitoes. They progress through the infectious and recovered classes, before reentering the susceptible class. Susceptible mosquito can become infected when they bite infectious humans, and once infected they move to infectious class. Mosquitoes species follow a logistic population model, with humans having disease induced death and immigration.

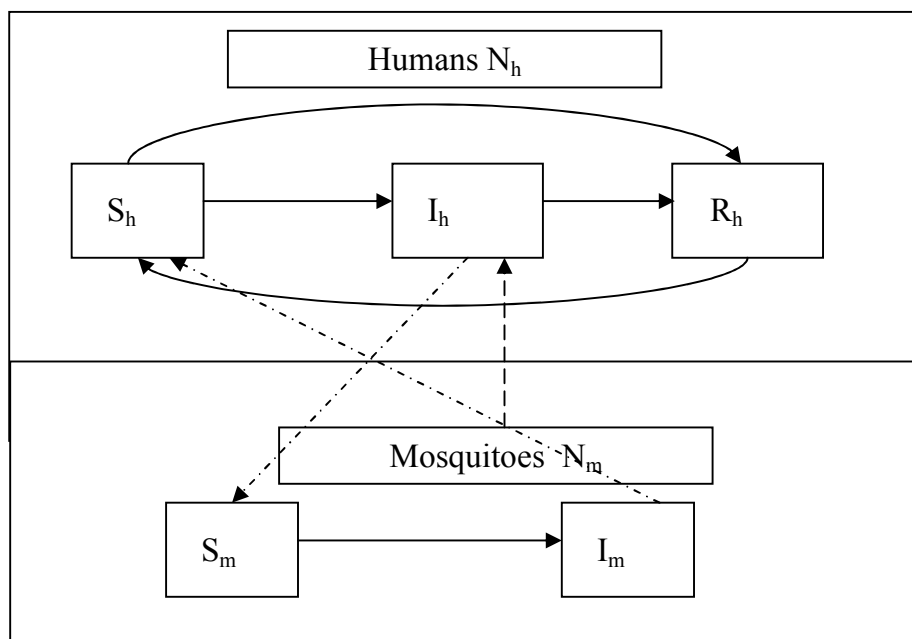


Fig. 1

Fig.1 shows progression of infection as follows:

Susceptible humans (S_h) can be infected when they are bitten by infectious mosquitoes (I_m). They then progress through the infectious class (I_h) and

recovered class (R_h), before reentering the susceptible class. Susceptible mosquitoes (S_m) can become infected when they bite infectious humans. Birth, death and control measures are not shown in the figure.

3. The Model

Here we are considering SIRS dynamics for human population and SI dynamics for mosquitoes population, where the human population density $N_h(t)$ is divided into three classes namely, the susceptible class $S_h(t)$, the infective class $I_h(t)$ and the recovered class $R_h(t)$. The mosquito population density $N_m(t)$ is also divided into susceptible class $S_m(t)$ and the infective class $I_m(t)$.

The proposed mathematical model is as follows:

$$(3.1a) \quad \frac{dS_h}{dt} = \Pi N_h + \mu_1 N_h + \gamma_1 R_h - \beta_1 S_h I_m - d_1 S_h,$$

$$(3.1b) \quad \frac{dI_h}{dt} = \beta_1 S_h I_m - (\nu_1 + \alpha_1 + d_1) I_h,$$

$$(3.1c) \quad \frac{dR_h}{dt} = \nu_1 I_h - (d_1 + \gamma_1) R_h,$$

$$(3.1d) \quad \frac{dN_h}{dt} = \Pi N_h + \mu_1 N_h - \alpha_1 I_h - d_1 N_h,$$

$$(3.1e) \quad \frac{dS_m}{dt} = \mu_2 N_m - \beta_2 S_m I_h - d_2 S_m - \alpha_2 S_m,$$

$$(3.1f) \quad \frac{dI_m}{dt} = \beta_2 S_m I_h - (\alpha_2 + d_2) I_m,$$

$$(3.1g) \quad \frac{dN_m}{dt} = \mu_2 N_m + \beta_3 E N_m - \frac{N_m^2}{K_m} - \alpha_2 N_m - d_2 N_m,$$

$$(3.1h) \quad \frac{dE}{dt} = l N_h - (\delta + g) E,$$

with the initial conditions:

$$S_h(0) = S_{h0} > 0, I_h(0) = I_{h0} > 0,$$

$$R_h(0) = R_{h0} > 0, S_m(0) = S_{m0} > 0,$$

$$I_m(0) = I_{m0} > 0 \text{ \& } E(0) = E_0.$$

and with $S_h + I_h + R_h = N_h$, $S_m + I_m = N_m$, where the state variables and parameters are as follows:

(a) The state variables for the malaria model

S_h : population density of susceptible humans.

I_h : population density of infected humans.

R_h : population density of recovered humans.

N_h : total population density of humans.

S_m : population density of susceptible mosquitoes.

I_m : population density of infected mosquitoes.

N_m : total population density of mosquitoes.

E : cumulative concentration of waste material, household and industrial discharges.

(b) The parameters for the malaria model

Π : Immigration rates of humans, considering immigrants as susceptible.

μ_1 : natural birth rate of human population and it is assumed that all the new born humans are susceptible.

μ_2 : natural birth rate of mosquito population and it is assumed that all the new born mosquito are susceptible.

β_1 : disease transmission coefficient involving susceptible humans and infected mosquito population.

β_2 : interaction coefficient of susceptible mosquitoes with the infected human class.

β_3 : growth rate coefficient of mosquito population due to concentration of waste material, household and industrial discharges.

d_1 : natural death rate of human population, assuming same for each classes.

d_2 : natural death rate of mosquito population, assuming same for each classes.

γ_1 : recovery rate coefficient of recovered human class joining the susceptible human class.

v_1 : transfer rate coefficient of infected human class to recovered human class.

α_1 : disease related death rate constant of human infected class.

α_2 : death rate of mosquito population due to control measures.

l : cumulative growth rate coefficient of environmental discharges and waste material, which is dependent on human population.

δ : rate of application of control measure for removing the waste material, effluent, household and industrial discharges.

g : cumulative depletion rate of waste material, house hold and industrial discharges.

K_m : carrying capacity of mosquito population in natural environment.

All the above parameters Π , μ_1 , μ_2 , β_1 , β_2 , β_3 , d_1 , d_2 , γ_1 , ν_1 , α_1 , α_2 , l , δ , g & K_m are assumed positive.

Since $S_h + I_h + R_h = N_h$ & $S_m + I_m = N_m$, above system of equations (1.1) to (1.8) reduces to the following system of equations

$$(3.2a) \quad \frac{dI_h}{dt} = \beta_1(N_h - I_h - R_h)I_m - (\nu_1 + \alpha_1 + d_1)I_h,$$

$$(3.2b) \quad \frac{dR_h}{dt} = \nu_1 I_h - (d_1 + \gamma_1)R_h,$$

$$(3.2c) \quad \frac{dN_h}{dt} = \Pi N_h + \mu_1 N_h - \alpha_1 I_h - d_1 N_h,$$

$$(3.2d) \quad \frac{dI_m}{dt} = \beta_2(N_m - I_m)I_h - (\alpha_2 + d_2)I_m,$$

$$(3.2e) \quad \frac{dN_m}{dt} = \mu_2 N_m + \beta_3 E N_m - \frac{N_m^2}{K_m} - \alpha_2 N_m - d_2 N_m,$$

$$(3.2f) \quad \frac{dE}{dt} = l N_h - (\delta + g)E.$$

4. Equilibrium States and Their Stability Analysis

Equilibrium states are the positive solutions of the following system of equations

$$\frac{dI_h}{dt} = 0, \frac{dR_h}{dt} = 0, \frac{dN_h}{dt} = 0, \frac{dI_m}{dt} = 0, \frac{dN_m}{dt} = 0 \text{ \& } \frac{dE}{dt} = 0.$$

Thus for equilibria we need to solve

$$(4.1a) \quad \beta_1(N_h - I_h - R_h)I_m - (\nu_1 + \alpha_1 + d_1)I_h = 0,$$

$$(4.1b) \quad \nu_1 I_h - (d_1 + \gamma_1)R_h = 0,$$

$$(4.1c) \quad \Pi N_h + \mu_1 N_h - \alpha_1 I_h - d_1 N_h = 0,$$

$$(4.1d) \quad \beta_2(N_m - I_m)I_h - (\alpha_2 + d_2)I_m = 0,$$

$$(4.1e) \quad \mu_2 N_m + \beta_3 E N_m - \frac{N_m^2}{K_m} - \alpha_2 N_m - d_2 N_m = 0,$$

$$(4.1f) \quad l N_h - (\delta + g) E = 0.$$

4.1. Disease Free Equilibrium States

The disease free equilibrium states, that is $I_h = 0$, $I_m = 0$ for the above system of equations (4.1a) to (4.1f) are given by :

$$R_h = 0, N_h = 0, N_m = 0, \text{ or } N_m = K_m(\mu_2 - \alpha_2 - d_2) \text{ \& } E = 0.$$

Thus for existence of disease free equilibrium states, required condition is

$$[\mathbf{C-1}] \quad 0 < (\mu_2 - \alpha_2 - d_2) < 1, \text{ since } N_m < K_m.$$

i.e., disease free equilibrium states will exist if; the natural birth rate of mosquitoes is less than the sum of natural death rate of mosquitoes and death rate of mosquitoes due to control measures $\{\mu_2 < (1 + \alpha_2 + d_2)\}$.

Hence the two disease free equilibrium states are

$$E_{10}^* : \{0, 0, 0, 0, 0, 0\} \text{ \& } E_{20}^* : \{0, 0, 0, 0, K_m(\mu_2 - \alpha_2 - d_2), 0\}$$

4.2. Stability Analysis of Disease Free Equilibrium States (E_{10}^* \& E_{20}^*)

The analysis of a model to judge the stability of its equilibrium states depending upon whether the differential equations are assumed to apply over all conceivable combination of population size or only in the neighborhood of the equilibrium states, is referred to as global or local, respectively.

Here we are considering only local stability analysis of disease free equilibrium states. The local stability analysis can be done by computing the eigenvalues of the variational matrix at the equilibrium states.

The variational matrix for system of equations (3.2a) to (3.2f) is given by-

$$(4.2a) \quad M(I_h, R_h, N_h, I_m, N_m, E) =$$

$$\begin{bmatrix} -(v_1 + \alpha_1 + d_1) - \beta_1 I_m & -\beta_1 I_m & \beta_1 I_m & \beta_1 (N_h - I_h - R_h) & 0 & 0 \\ v_1 & -(d_1 + \gamma_1) & 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & (\pi + \mu_1 - d_1) & 0 & 0 & 0 \\ \beta_2 (N_m - I_m) & 0 & 0 & -(d_2 + \alpha_2) - \beta_2 I_h & \beta_2 I_h & 0 \\ 0 & 0 & 0 & 0 & (\mu_2 - \alpha_2 - d_2 + \beta_3 E - \frac{2N_m}{K_m}) & \beta_3 N_m \\ 0 & 0 & l & 0 & 0 & -(\delta + g) \end{bmatrix}$$

Now we consider both the disease free equilibrium states E_{10}^* & E_{20}^* separately.

The characteristic equation for the equilibrium states $E_{10}^* : \{0, 0, 0, 0, 0, 0\}$, is given by:

$$\begin{vmatrix} -(\nu_1 + \alpha_1 + d_1) - \lambda & 0 & 0 & 0 & 0 & 0 \\ \nu_1 & -(d_1 + \gamma_1) - \lambda & 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & (\pi + \mu_1 - d_1) - \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -(d_2 + \alpha_2) - \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & (\mu_2 - \alpha_2 - d_2) - \lambda & 0 \\ 0 & 0 & l & 0 & 0 & -(\delta + g) - \lambda \end{vmatrix} = 0$$

The eigenvalues are

$-(\nu_1 + \alpha_1 + d_1)$, $-(d_1 + \gamma_1)$, $(\pi + \mu_1 - d_1)$, $-(d_2 + \alpha_2)$, $(\mu_2 - \alpha_2 - d_2)$ & $-(\delta + g)$.

Thus if $(\pi + \mu_1 - d_1) < 0$ & $(\mu_2 - \alpha_2 - d_2) < 0$ then E_{10}^* will be asymptotically stable. But from condition [C-1] we have $(\mu_2 - \alpha_2 - d_2) > 0$, so if E_{20}^* will exist then E_{10}^* will be unstable.

The characteristic equation for the equilibrium state

$E_{20}^* : \{0, 0, 0, 0, K_m(\mu_2 - \alpha_2 - d_2), 0\}$, is given by-

$$\begin{vmatrix} -(\nu_1 + \alpha_1 + d_1) - \lambda & 0 & 0 & 0 & 0 & 0 \\ \nu_1 & -(d_1 + \gamma_1) - \lambda & 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & (\pi + \mu_1 - d_1) - \lambda & 0 & 0 & 0 \\ \beta_2 K_m(\mu_2 - d_2 + \alpha_2) & 0 & 0 & -(d_2 + \alpha_2) - \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & (\mu_2 - \alpha_2 - d_2) - \lambda & -\lambda \beta_3 K_m(\mu_2 - \alpha_2 - d_2) - \lambda \\ 0 & 0 & l & 0 & 0 & -(\delta + g) - \lambda \end{vmatrix} = 0$$

The eigenvalues are

$-(\nu_1 + \alpha_1 + d_1)$, $-(d_1 + \gamma_1)$, $(\pi + \mu_1 - d_1)$, $-(\alpha_2 + d_2)$, $-(\mu_2 - \alpha_2 - d_2)$ & $-(\delta + g)$.

Clearly, all the four eigenvalues excepting $(\pi + \mu_1 - d_1)$ & $-(\mu_2 - \alpha_2 - d_2)$ are negative because we have taken all the parameters positive. Since from condition (C-1) $(\mu_2 - \alpha_2 - d_2) > 0$, so fifth eigenvalue $\{-(\mu_2 - \alpha_2 - d_2)\}$ is also negative. Hence we have, E_{20}^* will be

asymptotically stable if the third eigenvalue $(\pi + \mu_1 - d_1)$ is negative, then we have the following theorem:

Theorem 4.2.1: *An N-S condition for the existence and linear stability of the disease free equilibrium state E_{20}^* is that the following conditions must be satisfied:*

$$(C-1) \quad (\mu_2 - \alpha_2 - d_2) > 0,$$

$$(C-2) \quad (\pi + \mu_1 - d_1) < 0.$$

5. Endemic Equilibrium State

The endemic equilibrium state $\bar{E}\{I_h^*, R_h^*, N_h^*, I_m^*, N_m^*, E^*\}$ for the system of equations (4.1a) to (4.1f) is given by

$$R_h^* = \left(\frac{\nu_1}{d_1 + \gamma_1}\right) I_h^*, N_h^* = \left(\frac{\alpha_1}{\pi + \mu_1 - d_1}\right) I_h^*,$$

$$N_m^* = K_m \left[\mu_2 - \alpha_2 - d_2 + \frac{\beta_3 l \alpha_1}{(\delta + g)(\pi + \mu_1 - d_1)} I_h^* \right],$$

$$E^* = \left(\frac{l}{\delta + g}\right) \left(\frac{\alpha_1}{\pi + \mu_1 - d_1}\right) I_h^*,$$

$$I_m^* = \frac{(\nu_1 + \alpha_1 + d_1)}{\beta_1 \left[\frac{\alpha_1}{(\pi + \mu_1 - d_1)} - \frac{\nu_1}{(d_1 + \gamma_1)} - 1 \right]} \quad \text{and} \quad I_h^* = \frac{(\alpha_2 + d_2) I_m^*}{\beta_2 (N_m^* - I_m^*)}.$$

For non-negative values of $I_h^*, R_h^*, N_h^*, I_m^*, N_m^*$ & E^* following three conditions should be satisfied

$$(D-1) \quad (\pi + \mu_1 - d_1) > 0,$$

$$(D-2) \quad (\mu_2 - \alpha_2 - d_2) > 0 \text{ and}$$

$$(D-3) \quad \left\{ \frac{\alpha_1}{(\pi + \mu_1 - d_1)} - \frac{\nu_1}{(d_1 + \gamma_1)} - 1 \right\} > 0.$$

Here we have reduced every state variable at endemic equilibrium state in terms of I_h^* . Thus if we establish the existence and uniqueness of I_h^* (>0), then the existence and uniqueness of endemic equilibrium is established.

Substituting values of I_m^* & N_m^* in $I_h^* = \frac{(\alpha_2 + d_2)I_m^*}{\beta_2(N_m^* - I_m^*)}$, we get

$$I_h^* = \frac{(d_2 + \alpha_2) \frac{(\nu_1 + \alpha_1 + d_1)}{\beta_1 \left[\frac{\alpha_1}{(\pi + \mu_1 - d_1)} - \frac{\nu_1}{(d_1 + \gamma_1)} - 1 \right]}}{\beta_2 \left[K_m \left\{ \mu_2 - \alpha_2 - d_2 + \frac{\beta_3 l \alpha_1}{(\delta + g)(\pi + \mu_1 - d_1)} I_h^* \right\} - \frac{(\nu_1 + \alpha_1 + d_1)}{\beta_1 \left\{ \frac{\alpha_1}{(\pi + \mu_1 - d_1)} - \frac{\nu_1}{(d_1 + \gamma_1)} - 1 \right\}} \right]}$$

$$I_h^* = \frac{(d_2 + \alpha_2)A}{\beta_2 [K_m \{B + CI_h^*\} - A]},$$

where

$$A = \frac{(\nu_1 + \alpha_1 + d_1)}{\beta_1 \left[\frac{\alpha_1}{(\pi + \mu_1 - d_1)} - \frac{\nu_1}{(d_1 + \gamma_1)} - 1 \right]}, \quad (A > 0, \text{ from condition (D-3)})$$

$$B = \mu_2 - \alpha_2 - d_2, \quad (B > 0, \text{ from condition (D-2)})$$

$$C = \frac{\beta_3 l \alpha_1}{(\delta + g)(\pi + \mu_1 - d_1)}, \quad (C > 0, \text{ from condition (D-1)})$$

$$(5.1a) \quad \beta_2 C (I_h^*)^2 + \beta_2 (K_m B - A) (I_h^*) - (d_2 + \alpha_2) A = 0.$$

The equation (5.1a) can be written as

$$(5.1b) \quad P I_h^{*2} + Q I_h^* - R = 0$$

where

$$P = \beta_2 \frac{\beta_3 l \alpha_1}{(\delta + g)(\pi + \mu_1 - d_1)},$$

$$Q = \beta_2 \left\{ K_m (\mu_2 - \alpha_2 - d_2) - \frac{(\nu_1 + \alpha_1 + d_1)}{\beta_1 \left[\frac{\alpha_1}{(\pi + \mu_1 - d_1)} - \frac{\nu_1}{(d_1 + \gamma_1)} - 1 \right]} \right\},$$

$$R = \frac{(\alpha_2 + d_2)(\nu_1 + \alpha_1 + d_1)}{\beta_1 \left[\frac{\alpha_1}{(\pi + \mu_1 - d_1)} - \frac{\nu_1}{(d_1 + \gamma_1)} - 1 \right]}.$$

This is of the form $ax^2 + bx + c = 0$.

The condition for a unique positive root is $\frac{C}{a} < 0$, which is same as $-\frac{R}{P} < 0$,
i.e., $\frac{R}{P} > 0$.

Now we consider

$$\frac{R}{P} = \frac{\frac{(d_2 + \alpha_2)(v_1 + \alpha_1 + d_1)}{\beta_1[(\frac{\alpha_1 - \Pi}{(\mu_1 - d_1)}) - (\frac{v_1}{(d_1 + \gamma_1)}) - 1]}}{\frac{\beta_2 \beta_3 l \alpha_1}{(\delta + g)(\pi + \mu_1 - d_1)}}$$

$$\text{or } \frac{R}{P} = \frac{(d_2 + \alpha_2)(v_1 + \alpha_1 + d_1)(\delta + g)(\pi + \mu_1 - d_1)}{\beta_1 \beta_2 \beta_3 l \alpha_1 [(\frac{\alpha_1 - \Pi}{(\mu_1 - d_1)}) - (\frac{v_1}{(d_1 + \gamma_1)}) - 1]}.$$

From conditions (D-1) & (D-3) it is clear that $\frac{R}{P} > 0$, and therefore we have the following theorem:

Theorem 5.1 : *There will exist a unique endemic equilibrium state $\bar{E} \{I_h^*, R_h^*, N_h^*, I_m^*, N_m^*, E^*\}$ for the above considered system of non-linear ordinary differential equations (3.1a)-(3.1h) if the conditions (D1), (D2) and (D3) hold.*

6. Linear stability analysis of endemic equilibrium state ($\bar{E}$)

Taking the perturbations from the endemic equilibrium state $\bar{E} \{I_h^*, R_h^*, N_h^*, I_m^*, N_m^*, E^*\}$ as

$$(6.1a) \quad I_h(t) = I_h^* + U_1(t),$$

$$(6.1b) \quad R_h(t) = R_h^* + U_2(t),$$

$$(6.1c) \quad N_h(t) = N_h^* + U_3(t),$$

$$(6.1d) \quad I_m(t) = I_m^* + U_4(t),$$

$$(6.1e) \quad N_m(t) = N_m^* + U_5(t),$$

$$(6.1f) \quad E(t) = E^* + U_6(t).$$

The systems of differential equations (3.2a) to (3.2f) in terms of the new state variables become

$$(6.2a) \quad \frac{dU_1(t)}{dt} = -(\beta_1 I_m^* + \nu_1 + \alpha_1 + d_1)U_1 - \beta_1 I_m^* U_2 + \beta_1 I_m^* U_3 + \beta_1 (N_h^* - I_h^* - R_h^*)U_4,$$

$$(6.2b) \quad \frac{dU_2(t)}{dt} = \nu_1 U_1 - (d_1 + \gamma_1)U_2,$$

$$(6.2c) \quad \frac{dU_3(t)}{dt} = -\alpha_1 U_1 + (\pi + \mu_1 - d_1)U_3,$$

$$(6.2d) \quad \frac{dU_4(t)}{dt} = \beta_2 (N_m^* - I_m^*)U_1 - (\beta_2 I_h^* + d_2 + \alpha_2)U_4 + \beta_2 I_h^* U_5,$$

$$(6.2e) \quad \frac{dU_5(t)}{dt} = (\beta_3 E^* - \frac{2N_m^*}{K_m} + \mu_2 - \alpha_2 - d_2)U_5 + \beta_3 N_m^* U_6,$$

$$(6.2f) \quad \frac{dU_6(t)}{dt} = lU_3 - (\delta + g)U_6.$$

Now we consider positive definite function

$$(6.3) \quad V(t) = \frac{1}{2}U_1^2 + \frac{1}{2}U_2^2 + \frac{1}{2}U_3^2 + \frac{1}{2}U_4^2 + \frac{1}{2}U_5^2 + \frac{1}{2}U_6^2.$$

The time derivative of $V(t)$ is given by

$$(6.4) \quad \frac{dV(t)}{dt} = U_1 \frac{dU_1}{dt} + U_2 \frac{dU_2}{dt} + U_3 \frac{dU_3}{dt} + U_4 \frac{dU_4}{dt} + U_5 \frac{dU_5}{dt} + U_6 \frac{dU_6}{dt}$$

In view of equations (6.2a) to (6.2f), above equation reduces to

$$(6.5) \quad \begin{aligned} \frac{dV(t)}{dt} = & -[b_{11}U_1^2 + b_{12}U_1U_2 + b_{13}U_1U_3 + b_{14}U_1U_4 + b_{22}U_2^2 + b_{33}U_3^2 \\ & + b_{36}U_3U_6 + b_{44}U_4^2 + b_{45}U_4U_5 + b_{55}U_5^2 + b_{56}U_5U_6 + b_{66}U_6^2] \end{aligned}$$

where

$$b_{11} = (\beta_1 I_m^* + \nu_1 + \alpha_1 + d_1),$$

$$b_{12} = (\beta_1 I_m^* - \nu_1),$$

$$b_{13} = -(\beta_1 I_m^* + \alpha_1),$$

$$\begin{aligned}
b_{14} &= -\{\beta_1(N_h^* - I_h^* - R_h^*) + \beta_2(N_m^* - I_m^*)\}, \\
b_{22} &= (d_1 + \gamma_1), \\
b_{33} &= (\pi + \mu_1 - d_1), \\
b_{36} &= -l, \\
b_{44} &= (\beta_2 I_h^* + \alpha_2 + d_2), \\
b_{45} &= -\beta_2 I_h^*, \\
b_{55} &= (-\beta_3 E^* + \frac{2N_m^*}{K_m} + \alpha_2 + d_2 - \mu_2) = \frac{N_m^*}{K_m}, \\
b_{56} &= -\beta_3 N_m^*, \\
b_{66} &= (\delta + g).
\end{aligned}$$

Using the Sylvester criteria on the right hand side of the equation (6.5), it can be shown that $\frac{dV(t)}{dt}$ is negative definite at the endemic equilibrium state ($\bar{E}$) if the following inequalities hold

$$[E1] \quad b_{11} > 0, b_{22} > 0, b_{33} > 0, b_{44} > 0, b_{55} > 0, b_{66} > 0 \&$$

$$[E2](a) \quad 4b_{11}b_{22} > 3b_{12}^2, \quad (b) \quad 2b_{11}b_{33} > 3b_{13}^2,$$

$$(c) \quad 2b_{11}b_{44} > 3b_{14}^2, \quad (d) \quad b_{33}b_{66} > b_{36}^2,$$

$$(e) \quad b_{44}b_{55} > b_{45}^2, \quad (f) \quad b_{55}b_{66} > 3b_{56}^2.$$

Thus we have the following theorem:

Theorem 6.1: The endemic equilibrium state $\bar{E}\{I_h^*, R_h^*, N_h^*, I_m^*, N_m^*, E^*\}$ is asymptotically stable if the inequalities (E1) and (E2) are satisfied.

7. Example

In order to illustrate the above analysis we consider the following example of the model given by system of equations (3.2a) to (3.2f). All the parameters and variables are taken for mathematical convenience, not exactly from a real epidemiological system.

$$(7.1a) \quad \frac{dI_h}{dt} = 0.03755(N_h - I_h - R_h)I_m - 0.164212I_h,$$

$$(7.1b) \quad \frac{dR_h}{dt} = 0.13100I_h - 0.026712R_h,$$

$$(7.1c) \quad \frac{dN_h}{dt} = 0.00374N_h - 0.0290N_h,$$

$$(7.1d) \quad \frac{dI_m}{dt} = 0.03550(N_m - I_m - R_m)I_h - 0.059I_m,$$

$$(7.1e) \quad \frac{dN_m}{dt} = 0.1160N_m + 0.075EN_m - \frac{N_m^2}{50000},$$

$$(7.1f) \quad \frac{dE}{dt} = 0.0315N_h - 0.1850E.$$

The following values of the parameters have been taken into consideration

$\Pi=0.002286$, $\mu_1=0.005666$, $\mu_2=0.1750$, $\beta_1=0.03755$, $\beta_2=0.03550$,
 $\beta_3=0.075$, $d_1=0.004212$, $d_2=0.0250$, $\gamma_1=0.02250$, $\nu_1=0.1310$, $\alpha_1=0.0290$,
 $\alpha_2=0.0340$, $l=0.0315$, $\delta=0.075$, $g=0.11$, $K_m=50000$.

We note that the above values of parameters satisfy the conditions as given in the Theorem 5.1, which are necessary for the existence of a unique endemic equilibrium state $\bar{E}$ as

$$[D1] \quad (\pi + \mu_1 - d_1) = 0.00374 > 0,$$

$$[D2] \quad (\mu_2 - \alpha_2 - d_2) = 0.11600 > 0,$$

$$[D3] \quad \left\{ \frac{\alpha_1}{(\pi + \mu_1 - d_1)} - \frac{\nu_1}{(d_1 + \gamma_1)} - 1 \right\} = 1.85132 > 0$$

and endemic equilibrium state is given by

$$\bar{E} : \{ I_h^* = 0.487057, R_h^* = 2.38861, N_h^* = 3.77664, \\ I_m^* = 2.36218, N_m^* = 11541.6, E^* = 0.643048 \}$$

We can also check that the above values of parameters satisfy the conditions given in the Theorem 6.1. We have shown the dynamics of the model considered for above example with the help of following two figures [2] and [3] for the set of corresponding values of parameters of control measures as given in the following table:

Figure No.	ν_1	α_2	d_2
[2]	0.1310	0.0340	0.250
[3]	0.3200	0.0320	0.0200

Table 1

8. Conclusion

A mathematical model which incorporates many of the essential epidemiological aspects of malaria transmission is designed and used to assess the potential impact of control measures to control the spread of disease through different ways mentioned above. Some of the main mathematical and biological findings of this study include the following:

8.1. For the model considered, we have for the existence of disease free equilibrium states the required condition is, the natural birth rate of human population (μ_1) should be greater than , natural death rate of human population (d_1) i.e., $(\mu_1 - d_1) > 0$.

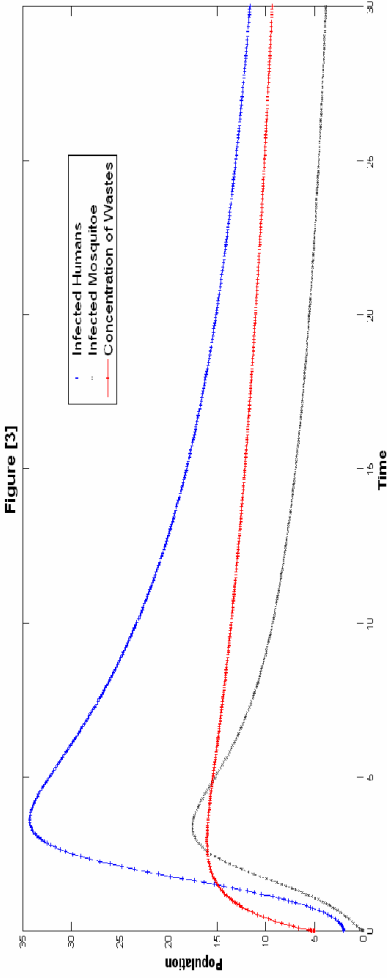
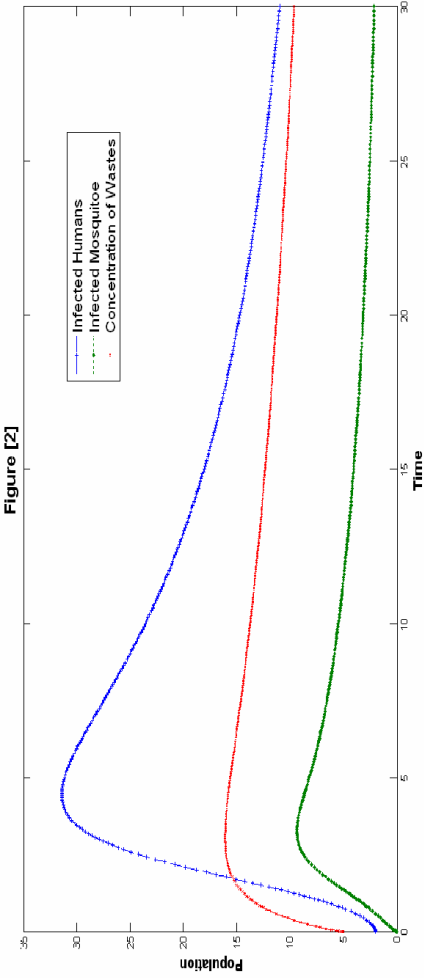
8.2. The dynamics of the model are almost completely determined by the control reproduction number R_V , where

$$R_V = \left[\left(\frac{\alpha_1 - \Pi}{(\mu_1 - d_1)} \right) - \left(\frac{\nu_1}{(d_1 + \gamma_1)} \right) \right].$$

The model has a unique asymptotically stable endemic equilibrium state ($\bar{E}$) whenever $R_V < 1$. This shows that malaria can be eliminated from the community if the vaccination and control measures have sufficient infection-blocking efficacy and coverage rate to bring (and maintain) R_V to a value greater than unity.

8.3. In the last we have shown by the figures [2] and [3], how the populations of infected human (I_h), infected mosquito (I_m) and concentration of waste materials (E) changes with corresponding changes in the parameters- ν_1 , α_2 & d_2 as given in the table 1.

Overall, this study shows that a future malaria transmission in the community can be effectively controlled by applying control measures in such a way so that it maintains the control reproduction number (R_V) greater than unity. It should be mentioned that this study is based on a relatively simple model. The model needs to be refined as further data (and knowledge) become available.



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