

Hypersurface of a Generalized Matsumoto Space

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Abstract: In 1985 Matsumoto¹ discussed the properties of special hypersurface of Randers space with $b_i(x)$ being gradient of a scalar function $b(x)$. He has considered a hypersurface which is given by $b(x) = \text{constant}$. In 2001, Singh and Kumari² have considered the hypersurface of Matsumoto space with the same equation $b(x) = \text{constant}$. In this paper we have considered the hypersurface of generalized Matsumoto space given by $b(x) = \text{constant}$. The conditions under which this hypersurface is a hyperplane of the first or second kind have been obtained. This hypersurface is not a hyperplane of third kind.

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1. Introduction

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space, where M^n is an n -dimensional differentiable manifold and $L(x, y)$ is the fundamental function. The concept of an (α, β) -metric was introduced in 1972 by Matsumoto³. A Finsler metric $L(x, y)$ is called an (α, β) -metric if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$ is one form on M^n . We have some interesting examples of an (α, β) -metric, for instance $L = \alpha + \beta$ (Randers metric)⁴, $L = \alpha^2 / \beta$ (Kropina metric)⁵, $L = (\alpha^2 \beta)^{1/3}$ (a decomposition of cubic metric) and

$L = \alpha^{n+1} / \beta^n$ (Generalized Kropina metric)⁶. In 1989 M. Matsumoto, while studying the slope of mountain, introduced a special (α, β) -metric, given by $L = \alpha^2 / (\alpha - \beta)$ which has been called as Matsumoto metric and the corresponding Finsler space has been called Matsumoto space⁴.

The propose of the present paper is to study the properties of hypersurface of generalized Matsumoto space whose metric is given by

$$(1.1) \quad L(x, y) = \frac{\alpha^{m+1}}{(\alpha - \beta)^m}, \quad m \neq 0, -1.$$

2. Fundamental Quantities of Generalized Matsumoto Space

The derivatives of generalized Matsumoto metric with respect to α and β are given by

$$(2.1a) \quad L_\alpha = \frac{\alpha^{m+1} - (m+1)\beta\alpha^m}{(\alpha - \beta)^{m+1}},$$

$$(2.1b) \quad L_\beta = \frac{m\alpha^{m+1}}{(\alpha - \beta)^{m+1}},$$

$$(2.2) \quad L_{\alpha\alpha} = \frac{m(m+1)\alpha^{m-1}\beta^2}{(\alpha - \beta)^{m+2}},$$

$$(2.3) \quad L_{\beta\beta} = \frac{m(m+1)\alpha^{m+1}}{(\alpha - \beta)^{m+2}},$$

$$(2.4) \quad L_{\alpha\beta} = \frac{-m(m+1)\alpha^m\beta}{(\alpha - \beta)^{m+2}}.$$

The normalized element of support $l_i = \dot{\partial}_i L$ is given by

$$(2.5) \quad l_i = \alpha^{-1}L_\alpha Y_i + L_\beta b_i,$$

where $\dot{\partial}_i$ stand for $\frac{\partial}{\partial y^i}$ and $Y_i = a_{ij}y^j$. The angular metric tensor

$h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$ is given by

$$(2.6) \quad h_{ij} = pa_{ij} + q_0 b_i b_j + q_1 (b_i Y_j + b_j Y_i) + q_2 Y_i Y_j,$$

where

$$(2.7) \quad p = LL_\alpha \alpha^{-1} = \frac{\{\alpha - (m+1)\beta\} \alpha^{2m}}{(\alpha - \beta)^{2m+1}},$$

$$(2.8) \quad q_0 = LL_\beta \beta = \frac{m(m+1)\alpha^{2m+2}}{(\alpha - \beta)^{2m+2}},$$

$$(2.9) \quad q_1 = LL_{\alpha\beta} \alpha^{-1} = \frac{(m+1)\alpha^{2m}\beta}{(\alpha - \beta)^{2m+2}},$$

$$(2.10) \quad q_2 = L(L_{\alpha\alpha} - L_\alpha \alpha^{-1}) \alpha^{-2} \\ = \frac{(m^2 - 1)\alpha^{2m-2}\beta^2 + (m+2)\alpha^{2m-1}\beta - \alpha^{2m}}{(\alpha - \beta)^{2m+2}}.$$

The fundamental tensor $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2$ is given by

$$(2.11) \quad g_{ij} = pa_{ij} + p_0 b_i b_j + p_1 (b_i Y_j + b_j Y_i) + p_2 Y_i Y_j,$$

where

$$(2.12) \quad p_0 = q_0 + (L_\beta)^2 = \frac{m(2m+1)\alpha^{2m+2}}{(\alpha - \beta)^{2m+2}},$$

$$(2.13) \quad p_1 = q_1 + L^{-1} p L_\beta = \frac{m\alpha^{2m+1} - 2m(m+1)\alpha^{2m}\beta}{(\alpha - \beta)^{2m+2}},$$

$$(2.14) \quad p_2 = q_2 + p^2 L^{-2} = \frac{2m(m+1)\alpha^{2m-2}\beta^2 - m\alpha^{2m-1}\beta}{(\alpha - \beta)^{2m+2}}.$$

The reciprocal tensor g^{ij} of g_{ij} is given by

$$(2.15) \quad g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_1 (b^i y^j + b^j y^i) - s_2 y^i y^j,$$

where

$$\begin{aligned}
(2.16) \quad b^i &= a^{ij} b_j, \quad s_0 = \frac{pp_0 + (p_0 p_2 - p_1^2) \alpha^2}{\tau p}, \\
s_1 &= \frac{pp_1 - (p_0 p_2 - p_1^2) \beta}{\tau p}, \quad s_2 = \frac{pp_2 + (p_0 p_2 - p_1^2) b^2}{\tau p}, \\
b^2 &= a_{ij} b^i b^j, \quad \tau = p(p + p_0 b^2 + p_1 \beta) + (p_0 p_2 - p_1^2)(\alpha^2 b^2 - \beta^2).
\end{aligned}$$

The $h\nu$ -torsion tensor $C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}$ is given by

$$(2.17) \quad 2pC_{ijk} = p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k,$$

where

$$(2.18) \quad \gamma_1 = p \partial p_0 / \partial \beta - 3p_1 q_0, \quad m_i = b_i - \alpha^{-2} \beta Y_i.$$

Clearly the covariant vector m_i is a non-vanishing one and it is orthogonal to the element of support y^i .

Let $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ be the components of Christoffel symbol of the associated

Riemannian space R^n and ∇_k denote the covariant differentiation with respect to x^k relative to Christoffel symbol. We shall consider the following tensors:

$$(2.19) \quad 2E_{ij} = b_{ij} + b_{ji}, \quad 2F_{ij} = b_{ij} - b_{ji},$$

where $b_{ij} = \nabla_j b_i$.

If we denote the Cartan's connection CT as $(\Gamma_{jk}^{*i}, \Gamma_{ok}^{*i}, C_{jk}^i)$, then the difference tensor $D_{jk}^i = \Gamma_{jk}^{*i} - \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$ of generalized Matsumoto space is given by

$$\begin{aligned}
(2.20) \quad D_{jk}^i &= B^i E_{jk} + F_j^i B_k + F_k^i B_j + B_j^i b_{0k} + B_k^i b_{0j} \\
&\quad - b_{0m} g^{im} B_{jk} - C_{jm}^i A_k^m - C_{km}^i A_j^m + C_{jkm} A_s^m g^{is} \\
&\quad + \lambda^s (C_{jm}^i C_{sk}^m + C_{km}^i C_{sj}^m - C_{ms}^i C_{kj}^m),
\end{aligned}$$

where

$$\begin{aligned}
 (2.21) \quad B_i &= p_0 b_i + p_1 Y_i, \quad B^i = g^{ij} B_j, \\
 B_{ij} &= \frac{1}{2} \left\{ p_1 (a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j \right\}, \quad B_j^i = g^{ik} B_{kj}, \\
 A_k^m &= B_k^m E_{00} + B^m E_{k0} + B_k F_0^m + B_0 F_k^m, \\
 \lambda^s &= B^s E_{00} + 2B_0 F_0^s \quad \text{and} \quad B_0 = B_i y^i.
 \end{aligned}$$

In the following we use the subscript 0 for denoting contraction with y^i , except for the quantities p_0 , q_0 and s_0 .

3. Induced Cartan Connection

Let F^{n-1} be a hypersurface of F^n , given by the equation $x^i = x^i(u^\alpha)$, $\alpha = 1, 2, \dots, n-1$. Suppose that the matrix of the projection factor $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$ is of rank $n-1$. Then the $B_\alpha^i(u)$ may be regarded as $n-1$ linearly independent vectors tangential to F^{n-1} at the point (u^α) and the vector X^i tangential to F^{n-1} at the point may be expressed in the form $X^i = B_\alpha^i X^\alpha$, where X^α are the components of vector with respect to coordinate system (u^α) .

The element of support y^i of F^n is taken tangential to F^{n-1} , i.e.

$$(3.1) \quad y^i = B_\alpha^i(u) v^\alpha.$$

Thus v^α is the element of support of F^{n-1} at the point (u^α) . The metric tensor $g_{\alpha\beta}$ and $h\nu$ -torsion tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$(3.2) \quad g_{\alpha\beta} = g_{ij} B_\alpha^i B_\beta^j, \quad C_{\alpha\beta\gamma} = C_{ijk} B_\alpha^i B_\beta^j B_\gamma^k.$$

At each point (u^α) of F^{n-1} , the unit normal vector $N^i(u, v)$ is defined by

$$(3.3) \quad g_{ij}(x(u), y(u, v)) B_\alpha^i N^j = 0, \quad g_{ij}(x(u), y(u, v)) N^i N^j = 1.$$

As for the angular metric tensor $h_{ij} = g_{ij} - l_i l_j$, we have $h_{\alpha\beta} = h_{ij} B_\alpha^i B_\beta^j$,

$$(3.4) \quad h_{ij} B_\alpha^i N^j = 0, \quad h_{ij} N^i N^j = 1.$$

If (B_i^α, N_i) is the inverse matrix of (B_α^i, N^i) , we have

$$(3.5) \quad B_i^\alpha = g^{\alpha\beta} g_{ij} B_\beta^j, \quad B_\alpha^i B_i^\beta = \delta_\alpha^\beta, \quad B_\alpha^i N_i = 0, \\ N^i N_i = 1 \quad \text{and} \quad B_\alpha^i B_j^\alpha + N^i N_j = \delta_j^i.$$

The induced Cartan's connection $ICT = (\Gamma_{\beta\gamma}^{*\alpha}, G_\beta^\alpha, C_{\alpha\beta}^\alpha)$ of the hypersurface F^{n-1} , induced from the Cartan's connection $CT = (\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, C_{jk}^i)$, is given by¹

$$(3.6) \quad \Gamma_{\beta\gamma}^{*\alpha} = B_i^\alpha (B_{\beta\gamma}^i + \Gamma_{jk}^{*i} B_\beta^j B_\gamma^k) + M_\beta^\alpha H_\gamma,$$

$$(3.7) \quad G_\beta^\alpha = B_i^\alpha (B_{0\beta}^i + \Gamma_{0j}^{*i} B_\beta^j),$$

$$(3.8) \quad C_{\beta\gamma}^\alpha = B_i^\alpha C_{jk}^i B_\beta^j B_\gamma^k,$$

where

$$(3.9) \quad M_{\alpha\beta} = N_i C_{jk}^i B_\alpha^j B_\beta^k, \quad M_\beta^\alpha = g^{\alpha\gamma} M_{\gamma\beta},$$

$$(3.10) \quad H_\alpha = N_i (B_{0\alpha}^i + \Gamma_{0j}^{*i} B_\alpha^j),$$

$$(3.11) \quad B_{\alpha\beta}^i = \frac{\partial B_\alpha^i}{\partial u^\beta}, \quad B_{0\beta}^i = B_{\alpha\beta}^i v^\alpha.$$

The quantities $M_{\alpha\beta}$ and H_α are called second fundamental v -tensor and normal curvature vector respectively¹. The second fundamental h -tensor $H_{\alpha\beta}$ is defined as¹

$$(3.12) \quad H_{\alpha\beta} = N_k (B_{\alpha\beta}^k + \Gamma_{ij}^{*k} B_\alpha^i B_\beta^j) + M_\alpha H_\beta,$$

where

$$(3.13) \quad M_\beta = N_i C_{jk}^i B_\beta^j N^k.$$

The relative h - and v -covariant derivatives of projection factor B_α^i with respect to ICT are given by

$$(3.14) \quad B_{\alpha|\beta}^i = H_{\alpha\beta}N^i, \quad B_\alpha^i|_\beta = M_{\alpha\beta}N^i.$$

From (3.12) it is clear that the second fundamental h -tensor $H_{\alpha\beta}$ is not symmetric and

$$(3.15) \quad H_{\alpha\beta} - H_{\beta\alpha} = M_\alpha H_\beta - M_\beta H_\alpha.$$

From (3.10), (3.12) and (3.13) it is clear that

$$(3.16) \quad H_{0\alpha} = H_\alpha, \quad H_{\alpha 0} = H_\alpha + M_\alpha H_0.$$

We quote the following lemmas due to Matsumoto¹.

Lemma 1: *The normal curvature $H_0 = H_\alpha v^\alpha$ vanishes iff the normal curvature vector H_α vanishes.*

Lemma 2: *A hypersurface F^{n-1} is a hyperplane of the first kind iff $H_\alpha = 0$.*

Lemma 3: *A hypersurface F^{n-1} is a hyperplane of the second kind iff $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

Lemma 4: *A hypersurface F^{n-1} is a hyperplane of the third kind with respect to the connection CT iff $H_\alpha = 0$, $M_{\alpha\beta} = 0$ and $H_{\alpha\beta} = 0$.*

4. Hypersurface $F^{n-1}(c)$ of Matsumoto Space

Let us consider a special Matsumoto metric with a gradient $b_i(x) = \frac{\partial b}{\partial x^i}$,

for a scalar function $b(x)$ and consider a hypersurface $F^{n-1}(c)$ which is given by $b(x) = c$ (constant). From parametric equations $x^i = x^i(u^\alpha)$ of F^{n-1} we get $\frac{\partial b(x(u))}{\partial u^\alpha} = 0$, which implies $b_i B_\alpha^i = 0$. This shows that

$b_i(x)$ are covariant components of a normal vector field of $F^{n-1}(c)$. Therefore along the $F^{n-1}(c)$, we have

$$(4.1) \quad b_i B_\alpha^i = 0 \quad \text{and} \quad b_i y^i = 0.$$

In general the induced metric $L(u, v)$ from the generalized Matsumoto metric, is given by

$$L(u, v) = \frac{\left[a_{ij}(x(u)) B_\alpha^i B_\beta^j v^\alpha v^\beta \right]^{\frac{m+1}{2}}}{\left[\sqrt{a_{ij}(x(u)) B_\alpha^i B_\beta^j v^\alpha v^\beta} - b_i(x(u)) B_\alpha^i v^\alpha \right]^m}.$$

Therefore the induced metric of the $F^{n-1}(c)$ becomes

$$(4.2) \quad L(u, v) = \sqrt{a_{\alpha\beta}(u) v^\alpha v^\beta},$$

which is a Riemannian metric. At the points of $F^{n-1}(c)$, the quantities given in equations (2.7), (2.8), (2.9) and (2.10) become

$$(4.3) \quad p = 1, \quad q_0 = m(m+1), \quad q_1 = 0, \quad q_2 = -\alpha^{-2},$$

the quantities given in (2.12), (2.13) and (2.14) reduce to

$$(4.4) \quad p_0 = m(2m+1), \quad p_1 = m/\alpha, \quad p_2 = 0,$$

whereas (2.16) reduces to

$$(4.5) \quad s_0 = \frac{m(m+1)}{1+mb^2+m^2b^2}, \quad s_1 = \frac{m}{\alpha(1+mb^2+m^2b^2)},$$

$$s_2 = \frac{-m^2b^2}{\alpha^2(1+mb^2+m^2b^2)}, \quad \tau = 1+mb^2+m^2b^2.$$

Therefore from (2.15) we have

$$(4.6) \quad g^{ij} = a^{ij} - \left(\frac{m}{1+mb^2+m^2b^2} \right) \left[(m+1)b^i b^j + \frac{b^i y^j + b^j y^i}{\alpha} - \frac{mb^2 y^i y^j}{\alpha^2} \right].$$

Thus along $F^{n-1}(c)$, (4.1) and (4.6) give

$$(4.7) \quad g^{ij} b_i b_j = \frac{b^2}{1+mb^2+m^2b^2}.$$

Hence we have

$$(4.8) \quad b_i(x(u)) = N_i \sqrt{\frac{b^2}{1 + mb^2 + m^2 b^2}}, \quad b^2 = a^{ij} b_i b_j.$$

Also from (4.6) and (4.8) we get

$$(4.9) \quad b^i = a^{ij} b_j = N^i \sqrt{b^2(1 + mb^2 + m^2 b^2)} + mb^2 \alpha^{-1} y^i.$$

Theorem 4.1: Let $F^n = (M^n, L)$ be a generalized Matsumoto space with gradient $b_i(x) = \frac{\partial b}{\partial x^i}$ and let $F^{n-1}(c)$ be a hypersurface of F^n given by $b(x) = c$ (constant). Suppose that the Riemannian metric $a_{ij}(x) dx^i dx^j$ is positive definite and b_i are components of a non-zero vector field, then the induced metric on $F^{n-1}(c)$ is a Riemannian metric given by (4.2) and relations (4.8) and (4.9) hold.

Using (4.3) in (2.6) we get angular metric tensor along $F^{n-1}(c)$, given by

$$(4.10) \quad h_{ij} = a_{ij} + m(m+1)b_i b_j - \alpha^{-2} Y_i Y_j.$$

Again, using (4.3) and (4.4) in (2.11) we get metric tensor along $F^{n-1}(c)$, given by

$$(4.11) \quad g_{ij} = a_{ij} + m(2m+1)b_i b_j + m\alpha^{-1}(b_i Y_j + b_j Y_i).$$

If $h_{\alpha\beta}^{(a)}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$, then using (4.1) and (4.10), we have along $F^{n-1}(c)$,

$$(4.12) \quad h_{\alpha\beta} = h_{\alpha\beta}^{(a)}.$$

From (2.12), we have

$$\frac{\partial p_0}{\partial \beta} = m(2m+1)(2m+2) \frac{\alpha^{2m+2}}{(\alpha - \beta)^{2m+3}}.$$

Thus along $F^{n-1}(c)$, we have

$$(4.13) \quad \frac{\partial p_0}{\partial \beta} = \frac{m(2m+1)(2m+2)}{\alpha}.$$

Therefore, (2.18) gives

$$(4.14) \quad \gamma_1 = m(m+1)(m+2), \quad \text{and} \quad m_i = b_i.$$

In view of (4.13) and (4.14) the $h\nu$ -torsion tensor given by (2.17) reduces to

$$(4.15) \quad C_{ijk} = \frac{m}{2\alpha} \{ (h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + (m+1)(m+2)b_ib_jb_k \}.$$

Hence from (3.9), (4.1), (4.8) and (4.15) we have

$$(4.16) \quad M_{\alpha\beta} = \frac{m}{2\alpha} \sqrt{\frac{b^2}{1+mb^2+m^2b^2}} h_{\alpha\beta},$$

and from (3.4), (3.13), (4.1) and (4.15) we have

$$(4.17) \quad M_\alpha = 0.$$

On using (4.17) in (3.15) we have

$$(4.18) \quad H_{\alpha\beta} = H_{\beta\alpha}.$$

Theorem 4.2: *The second fundamental ν -tensor $M_{\alpha\beta}$ of $F^{n-1}(c)$ is given by (4.16) and the second fundamental h -tensor $H_{\alpha\beta}$ is symmetric.*

Now from (4.1) we get $b_{i|\beta} B_\alpha^i + b_i B_{\alpha|\beta}^i = 0$. Using (3.14) and the fact that $b_{i|\beta} = b_{i|j} B_\beta^j + b_i |_j N^j H_\beta$, we have

$$(4.19) \quad [b_{i|j} B_\beta^j + b_i |_j N^j H_\beta] B_\alpha^i + b_i H_{\alpha\beta} N^i = 0.$$

Since $b_i |_j = -b_h C_{ij}^h$, from (3.13), (4.8) and (4.17) we have

$$(4.20) \quad b_{i|j} B_\alpha^i B_\beta^j + H_{\alpha\beta} \sqrt{\frac{b^2}{1+mb^2+m^2b^2}} = 0.$$

Since $b_{i|j}$ is symmetric tensor, contracting (4.20) with respect to ν^β and using (3.16), we get

$$(4.21) \quad b_{i|j} B_{\alpha}^i y^j + H_{\alpha} \sqrt{\frac{b^2}{1+mb^2+m^2b^2}} = 0.$$

Further contracting (4.21) with v^{α} , we have

$$(4.22) \quad b_{i|j} y^i y^j + H_0 \sqrt{\frac{b^2}{1+mb^2+m^2b^2}} = 0.$$

From Lemmas 1 and 2, it is clear that hypersurface $F^{n-1}(c)$ is a hyperplane of first kind iff $H_0 = 0$. Thus from (4.22), it is obvious that hypersurface $F^{n-1}(c)$ is a hyperplane of first kind iff $b_{i|j} y^i y^j = 0$. This $b_{i|j}$ being covariant derivative with respect to CT of F^n , may depend on y^i . But $b_{ij} = \nabla_j b_i$ is covariant derivative with respect to Riemannian connection $\left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\}$, constructed from $a_{ij}(x)$. Hence b_{ij} does not depend on y^i .

Since b_i is gradient vector, from (2.19) we have

$$(4.23) \quad E_{ij} = b_{ij}, \quad F_{ij} = 0 \quad \text{and} \quad F_j^i = 0.$$

Using (4.23) in (2.20) we get

$$(4.24) \quad D_{jk}^i = B^i b_{jk} + B_j^i b_{0k} + B_k^i b_{0j} - b_{0m} g^{im} B_{jk} - C_{jm}^i A_k^m - C_{km}^i A_j^m \\ + C_{jkm} A_s^m g^{is} + \lambda^s (C_{jm}^i C_{sk}^m + C_{km}^i C_{sj}^m - C_{ms}^i C_{kj}^m).$$

Now using (4.3), (4.4) and (4.6) in (2.21) we get

$$(4.25) \quad B_i = m(2m+1)b_i + \frac{m}{\alpha} Y_i, \\ B^i = \frac{m}{1+mb^2+m^2b^2} [(m+1)b_i + \alpha^{-1} y^i], \\ B_{ij} = \frac{m}{2\alpha} [a_{ij} - \alpha^{-2} Y_i Y_j + (2m+1)(2m+2)b_i b_j],$$

$$B_j^i = \frac{m}{2\alpha} \left(\delta_j^i - \alpha^{-2} y^i Y_j \right) + \frac{m(m+1)(3m+2)}{2\alpha(1+mb^2+m^2b^2)} b^i b_j$$

$$- \frac{m^2 \left[1 + (2m+1)(2m+2)b^2 \right]}{2\alpha^2(1+mb^2+m^2b^2)} b_j y^i,$$

$$A_i^r = B_i^r b_{00} + B^r b_{i0} \quad \text{and} \quad \lambda^r = B^r b_{00}.$$

In view of (4.1) we have $B_0^i = 0$ and $B_{i0} = 0$, which on using (4.25) gives $A_0^r = B^r b_{00}$. Therefore the contraction of (4.24) with y^k gives

$$(4.26) \quad D_{j0}^i = B^i b_{j0} + B_j^i b_{00} - B^r C_{jr}^i b_{00}.$$

Again contracting (4.26) with y^j , we get

$$(4.27) \quad D_{00}^i = B^i b_{00} = \frac{m}{1+mb^2+m^2b^2} \left[(m+1)b^i + \alpha^{-1} y^i \right] b_{00}.$$

Paying attention to (4.1), along $F^{n-1}(c)$, we finally get

$$(4.28) \quad b_i D_{j0}^i = \frac{m(m+1)b^2}{1+mb^2+m^2b^2} b_{j0} + \frac{m(1+2b^2+6mb^2+4m^2b^2)}{2\alpha(1+mb^2+m^2b^2)} b_j b_{00}$$

$$- b_i B^r C_{jr}^i b_{00},$$

$$(4.29) \quad b_i D_{00}^i = \frac{m(m+1)b^2}{1+mb^2+m^2b^2} b_{00}.$$

From (3.9), (4.8), (4.9), (4.16) and (4.17) we have $b^r b_i C_{jr}^i B_\alpha^j = b^2 M_\alpha = 0$.

Thus the relation $b_{i|j} = b_{ij} - b_r D_{ij}^r$ and the equations (4.28) and (4.29) give

$$(4.30) \quad b_{i|j} y^i y^j = \frac{1}{1+mb^2+m^2b^2} b_{00}.$$

Hence (4.21) and (4.22) respectively reduce to

$$(4.31) \quad H_\alpha \sqrt{b^2} + \frac{1}{\sqrt{1+mb^2+m^2b^2}} b_{i0} B_\alpha^i = 0,$$

$$(4.32) \quad H_0 \sqrt{b^2} + \frac{1}{\sqrt{1+mb^2+m^2b^2}} b_{00} = 0.$$

From equation (4.32) and Lemmas 1 and 2, it is clear that the necessary and sufficient condition for $F^{n-1}(c)$ to be hyperplane of first kind is that $b_{00} = 0$. Since b_{ij} does not depend on y^i satisfying (4.1), this condition may be written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j) = 0$ for some $c_j(x)$. Therefore

$$(4.33) \quad 2b_{ij} = b_i c_j + b_j c_i.$$

From (4.1) and (4.33) it follows that

$$(4.34) \quad b_{00} = 0, \quad b_{ij} B_\alpha^i B_\beta^j = 0, \quad b_{ij} B_\alpha^i y^j = 0.$$

Hence (4.32) gives $H_0 = 0$. Again from (4.33), (4.24) and (4.25) we have

$$(4.35) \quad b_{i0} b^i = \frac{1}{2} c_0 b^2, \quad \lambda^r = 0, \quad A_j^i B_\beta^j = 0, \quad B_{ij} B_\alpha^i B_\beta^j = \frac{m}{2\alpha} h_{\alpha\beta}.$$

Thus from (4.23) we have

$$(4.36) \quad b_r D_{ij}^r B_\alpha^i B_\beta^j = -\left(g^{rs} b_r b_s\right) \frac{m c_0}{4\alpha} h_{\alpha\beta} + b_r C_{ijm} A_s^m g^{rs} B_\alpha^i B_\beta^j.$$

Also from (4.6) we find

$$(4.37) \quad g^{rs} b_r b_s = \frac{b^2}{1+mb^2+m^2b^2}.$$

With the help of (3.9), (4.9), (4.16), (4.24) and (4.25) we get

$$(4.38) \quad b_r C_{ijm} A_s^m g^{rs} B_\alpha^i B_\beta^j = \left(b_r b_s g^{rs}\right) \frac{m^2(m+1)c_0 b^2}{4\alpha(1+mb^2+m^2b^2)} h_{\alpha\beta}.$$

Substituting (4.37) and (4.38) in (4.36), we get

$$(4.39) \quad b_r D_{ij}^r B_\alpha^i B_\beta^j = \frac{-m c_0 b^2}{4\alpha(1+mb^2+m^2b^2)} h_{\alpha\beta}.$$

Therefore (4.20) reduces to

$$(4.40) \quad H_{\alpha\beta} \sqrt{\frac{b^2}{1+mb^2+m^2b^2}} + \frac{mc_0b^2}{4\alpha(1+mb^2+m^2b^2)} h_{\alpha\beta} = 0.$$

Hence the hypersurface $F^{n-1}(c)$ is umbilic.

Theorem 4.3: *The necessary and sufficient condition for hypersurface $F^{n-1}(c)$ to be a hyperplane of the first kind is (4.33) and in this case the second fundamental tensor is proportional to its angular metric tensor.*

Now from Lemma 3, the hypersurface $F^{n-1}(c)$ is a hyperplane of the second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus (4.40) gives

$c_0 = c_i(x)y^i = 0$. Therefore there exists a function $\phi(x)$ such that

$$(4.41) \quad c_i(x) = \phi(x)b_i(x).$$

Hence (4.33) reduces to $b_{ij} = \phi b_i b_j$.

Theorem 4.4: *The necessary and sufficient condition for hypersurface $F^{n-1}(c)$ to be a hyperplane of the second kind is $b_{ij} = \phi b_i b_j$.*

Again Lemma 4 together with (4.16) and (4.17) shows that $F^{n-1}(c)$ does not become a hyperplane of third kind.

Theorem 4.5: *The hypersurface $F^{n-1}(c)$ is not a hyperplane of third kind.*

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